Juvitop: Spectral Sequences

Michael Andrews
Department of Mathematics
MIT

November 5, 2013

Abstract

The goal of this talk is to present the way that I think about spectral sequences and to convince the audience that they are a beautiful linear algebra device to be enjoyed rather than feared.

1 Introduction

Everyone in the audience probably has at least a vague idea of what a spectral sequence is. A few characteristics might come to mind and depending on who you are a differing array of emotions will be associated with each of these characteristics.

1. Many, many groups all written out in a plane or worse (better) a group with many gradings.
2. Pages, i.e. one of these collections for each natural number.
3. Fear (or excitement) at the prospect of the potentially vast array of differentials appearing as a mess (or beautiful display) of lines on each page.

I would like for everyone to feel the emotions in parentheses by the end of this talk.

I have spent the summer writing up a computation which makes use of many spectral sequences. In this write up I rarely refer to pages other than the E_1 or E_2-page. It is my feeling that a spectral sequences is a beautiful linear algebra device and that in time this device can also begin to feel simple. All a SS really consists of is a (multigraded) group or vector space together with a collection of correspondences satisfying a few properties.

2 A perspective on spectral sequences

2.1 Basic definitions and lemmas

The reader is probably familiar with the notion of an exact couple which is one of the most common ways in which a spectral sequence arises.

Definition. An exact couple consists of abelian groups A and E together with homomorphisms i, j and k such that the following triangle is exact:

$$\begin{tikzcd} A \arrow[r, "i"] & A \arrow[d, "j"] \\ & E \arrow[ul, "k"] \end{tikzcd}$$

Given an exact couple one can form the associated derived exact couple; iterating this process gives rise to a spectral sequence. Experience has led me to conclude that although this inductive definition is slick, it disguises some of the important features that SSs have and which are familiar to those who work with them on a daily basis. Various properties become buried in the induction and I feel that first time users should not have to struggle for long periods of time to discover these properties however rewarding that process might be.

An alternative approach exploits correspondences. We will find that the picture becomes clearer, especially once gradings are introduced, when we 'spread out' the exact couple:

$$\begin{tikzcd} \dots \arrow[r] & A \arrow[r, "i"] \arrow[d, "j"] & \dots \arrow[r, "i"] & A \arrow[r] & A \arrow[r] \arrow[d] & \dots \\ & E & & & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $\pi : E \times A \times A \times E \longrightarrow E \times E$ be the projection map. Then we make the following definition.

Definition. For each $r \geq 1$ let $\tilde{d_r} = \{(x, \tilde{x}, \tilde{y}, y) \in E \times A \times A \times E : kx = \tilde{x} = i^{r-1}\tilde{y} \text{ and } j\tilde{y} = y\}$ and $d_r = \pi(\tilde{d_r})$.

$$\begin{tikzcd} \tilde{y} \arrow[r, mapsto, "i"] \arrow[d, mapsto, "j"] & \dots \arrow[r, mapsto, "i"] & \tilde{x} & \\ y & & & x \arrow[ul, mapsto, "k"] \end{tikzcd}$$

Since i, j, k and π are homomorphisms of abelian groups $\tilde{d_r}$ and d_r are subgroups of $E \times A \times A \times E$ and $E \times E$, respectively.

Notation. We write $d_r x = y$ if $(x, y) \in d_r$.

We see that d_1 is the function jk. We also have the following useful observations.

Lemma (*)**.**

1. *For $r > 1$, $d_r x$ is defined if and only if $d_{r-1}x = 0$, i.e.*

$$(x, 0) \in d_{r-1} \iff \exists y : (x, y) \in d_r.$$

2. *For $r > 1$, $d_r 0 = y$ if and only if there exists an x with $d_{r-1}x = y$, i.e.*

$$(0, y) \in d_r \iff \exists x : (x, y) \in d_{r-1}.$$

Proof.

$$\begin{tikzcd}
\tilde{y} \arrow[r, mapsto, "i"] \arrow[d, mapsto, "j"'] & \tilde{y}' \arrow[r, mapsto, "i^{r-2}"] \arrow[d, mapsto, "j"'] & \tilde{x} & \\
y & 0 & & x \arrow[ul, mapsto, "k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{y} \arrow[r, mapsto, "i^{r-2}"] \arrow[d, mapsto, "j"'] & \tilde{x} \arrow[r, mapsto, "i"] & 0 & \\
y & & x \arrow[ul, mapsto, "k"'] & 0 \arrow[ul, mapsto, "k"']
\end{tikzcd}$$

1. If $d_r x$ is defined then there exists a $\tilde{y}$ with $i^{r-1}\tilde{y} = kx$; let $\tilde{y}' = i\tilde{y}$, then $i^{r-2}\tilde{y}' = kx$ and exactness gives $j\tilde{y}' = 0$ so $d_{r-1}x = 0$. If $d_{r-1}x = 0$ then there exists $\tilde{y}'$ with $i^{r-2}\tilde{y}' = kx$ and $j\tilde{y}' = 0$; by exactness there exists $\tilde{y}$ with $i\tilde{y} = \tilde{y}'$ so $i^{r-1}\tilde{y} = kx$ and $d_r x$ is defined.

2. If $d_r 0 = y$ then there exists a $\tilde{y}$ with $j\tilde{y} = y$ and $i^{r-1}\tilde{y} = 0$; let $\tilde{x} = i^{r-2}\tilde{y}$, then $i\tilde{x} = 0$ and so by exactness there exists an x with $kx = \tilde{x}$ which gives $d_{r-1}x = y$. If $d_{r-1}x = y$ then there exists a $\tilde{y}$ with $i^{r-2}\tilde{y} = kx$ and $j\tilde{y} = y$; by exactness $i^{r-1}\tilde{y} = ikx = 0$ and so $d_r 0 = y$.

□

Corollary ((*) to the second part of the lemma)**.** *For $r > 1$, the following conditions are equivalent:*

1. *$d_r x = y$ and $d_r x = y'$;*
2. *$d_r x = y$ and there exists an x' with $d_{r-1}x' = y' - y$.*

Proof. $d_r x = y$ and $d_r x = y'$ implies $d_r 0 = y' - y$; $d_r x = y$ and $d_r 0 = y' - y$ implies $d_r x = y'$. □

Lemma. *Let $r \geq 1$. Then $d_r x = y \implies d_s y = 0$ for any $s \geq 1$.*

Proof. Suppose $d_r x = y$. Then there exists $\tilde{y}$ with $j\tilde{y} = y$. By exactness $ky = 0$. □

In general, d_r is a correspondence. We can define the domain and image of a correspondence.

Definition. A correspondence $f : G_1 \longrightarrow G_2$ is a subgroup $f \subset G_1 \times G_2$. If $\pi_1 : G_1 \times G_2 \longrightarrow G_1$ and $\pi_2 : G_1 \times G_2 \longrightarrow G_2$ denote the projections then $\pi_1(f)$ and $\pi_2(f)$ are called the domain and image of f, respectively. We write $\mathrm{dom}(f)$ and $\mathrm{im}(f)$, respectively.

We can also define the kernel of a correspondence, a subgroup of the domain.

Definition. Give a correspondence $f : G_1 \longrightarrow G_2$. The kernel of f is the abelian group $\ker(f) = \{x \in G_1 : (x, 0) \in f\} \subset \mathrm{dom}(f)$.

2.2 The content of these definitions and lemmas

Part 1 of the starred lemma shows that for $r > 1$, $\mathrm{dom}(d_r) = \ker(d_{r-1})$. Although it will mean that we have to deal with the cases $r = 1$ and $r > 1$ separately we will prefer $\ker(d_{r-1})$ to $\mathrm{dom}(d_r)$ since that way our formulae appear like those appearing in a classical account of spectral sequences. In fact, if one takes the conventions that $\ker(d_0) = E$ and $\mathrm{im}(d_0) = 0$ then all of the following statements are also valid when $r = 1$.

Proposition. *The lemmas of the last subsection show that for $r > 1$, d_r defines a homomorphism*

$$\ker(d_{r-1}) / \bigcup_s \mathrm{im}(d_s) \longrightarrow \bigcap_s \ker(d_s) / \mathrm{im}(d_{r-1}).$$

Definition. Let $E^1 = E$ and for $r > 1$, $E^r = \ker(d_{r-1})/\operatorname{im}(d_{r-1})$.

Corollary. *Precomposing the homomomorphism of the proposition by $E^r \to \ker(d_{r-1})/\bigcup_s \operatorname{im}(d_s)$ and postcomposing by $\bigcap_s \ker(d_s)/im(d_{r-1}) \to E^r$ gives a homomorphism $E^r \to E^r$. This is usually how d_r is defined.*

We now highlight something that is clear from our definitions and which is somewhat buried in the derived exact couple construction. View d_r as a homomorphism $E^r \longrightarrow E^r$ and suppose that we have $d_r\overline{x} = \overline{y}$. Suppose that x and y are any representatives of $\overline{x}$ and $\overline{y}$. By the starred corollary we have a zig-zag $(x, \tilde{x}, \tilde{y}, y)$. In particular, we can hit any resprentative of $\overline{y}$ on the nose, we don't have to choose a particular respresentative. Contrast this with

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "(0,\mathrm{id})"] \arrow[d, "0"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\mathrm{id},0)"] \arrow[d, "\mathrm{id}\oplus 0"'] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "(0,\mathrm{id})"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(\mathrm{id},0)"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the last vertical map is surjective but the middle vertical map is not.

We will use the following terminology.

Definition. Suppose $d_r x = y$; then x is a said to support a d_r. If, in addition, $y \notin \operatorname{im}(d_{r-1})$, x is said to support a nontrivial differential. Elements of $\bigcap_s \ker(d_s)$ will be called permanent cycles.

Definition. We write E^∞ for $\bigcap_s \ker(d_s)/\bigcup_s \operatorname{im}(d_s)$.

3 Convergence of SSs

In this seminar we will be focussing on the EHP SS. We are less concerned about what the EHP SS converges to since this can be computed more effectively with other methods but we still need convergence of truncated EHP SSs. We go about discussing what the convergence of a SS means since it is not all that hard!

With the set up above it is most often the case that A and E are graded objects. There is often more than one grading but one usually stands out and warrants being called the filtration degree. So in this section we make some assumptions.

Assumption. Suppose that A and E have a $\mathbb{Z}$-grading s, that $i : A_s \longrightarrow A_{s+1}$, $j : A_s \longrightarrow E_s$ and that $k : E_{s+1} \longrightarrow A_s$.

Under this assumption we can redraw the exact couple as

$$\begin{tikzcd}
\ldots \arrow[r] & A_s \arrow[r, "i"] \arrow[d, "j"'] & \ldots \arrow[r, "i"] & A_{s+r-1} \arrow[r] & A_{s+r} \arrow[r] \arrow[d] & \ldots \\
 & E_s & & & E_{s+r} \arrow[ul, "k"] &
\end{tikzcd}$$

We see that d_r has degree $-r$ so E^∞ becomes $\mathbb{Z}$-graded too. Thus we have a natural homomorphism which we define presently.

Definition. Let $F_s = \mathrm{im}(A_s \longrightarrow \mathrm{colim}_s A_s)$. Then we have a natural homomorphism $F_s \longrightarrow E_s^\infty$ given by the following procedure:

1. Suppose given an element $z \in F_s$.
2. z is the image of some element $\tilde{y} \in A_s$.
3. $y = j\tilde{y} \in \bigcap_r \ker(d_r)$ defines an element $\overline{y}$ in E_s^∞.
4. The homomorphism should take z to $\overline{y}$.

Lemma. *The above homomorphism is well-defined.*

Proof. Suppose that we choose a different $\tilde{y}$, call it $\tilde{y}'$. The image of $\tilde{y} - \tilde{y}'$ in $\mathrm{colim}_s A_s$ is zero and so the image of $\tilde{y} - \tilde{y}'$ is zero in A_{s+r} for large r and we see that $y - y' = j\tilde{y} - j\tilde{y}'$ is the image of a differential.

$$\begin{tikzcd}
A_s \arrow[r, "i^{r-1}"] \arrow[d, "j"'] & A_{s+r-1} \arrow[r, "i"] & A_{s+r} \\
E_s & & E_{s+r} \arrow[ul, "k"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{y} - \tilde{y}' \arrow[r, mapsto, "i^{r-1}"] \arrow[d, mapsto, "j"'] & \bullet \arrow[r, mapsto, "i"] & 0 \\
y - y' & & \bullet \arrow[ul, mapsto, "k"]
\end{tikzcd}$$

$\square$

Lemma. *The kernel of the above homomorphism is F_{s-1}.*

Proof. If $z \in F_{s-1}$ then we can choose our $\tilde{y}$ to be in the image of i so that by exactness $y = j\tilde{y} = 0$. If y is the target of a differential $d_r x = y$ then there exists $\tilde{y}' \in A_s$ mapping to y in E_s and zero in $\mathrm{colim}_s A_s$. Thus $\tilde{y} - \tilde{y}'$ maps to zero in E_s and z in $\mathrm{colim}_s A_s$. By exactness $\tilde{y} - \tilde{y}'$ lifts to A_{s-1} and so we conclude that $z \in F_{s-1}$.

$$\begin{tikzcd}
A_s \arrow[r, "i^{r-1}"] \arrow[d, "j"'] & A_{s+r-1} \arrow[r, "i"] & A_{s+r} \\
E_s & & E_{s+r} \arrow[ul, "k"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{y}' \arrow[r, mapsto, "i^{r-1}"] \arrow[d, mapsto, "j"'] & \tilde{x} \arrow[r, mapsto, "i"] & 0 \\
y & & x \arrow[ul, mapsto, "k"]
\end{tikzcd}$$

$\square$

Corollary. *We have a natural injection $F_s/F_{s-1} \longrightarrow E_s^\infty$.*

There are many conditions which naturally arise and allow one to prove that a SS converges. Since we are dealing with the EHP sequence we'll automatically make the following assumptions.

Assumption. $A_s = E_s = 0$ for $s < 0$.

Lemma. $F_s/F_{s-1} \longrightarrow E_s^\infty$ *is an isomorphism.*

Proof. If $y \in \bigcap_r \ker(d_r) \subset E_s$ then $ky = 0$ because it must necessarily have 0 as a lift. By exactness y lifts to $\tilde{y} \in A_s$. Let $\overline{z}$ be the image of $\tilde{y}$ in F_s/F_{s-1}. Then $\overline{z}$ maps to $\overline{y}$ proving surjectivity.

$$\begin{tikzcd}
A_{-1} \arrow[r, "i^s"] \arrow[d, "j"'] & A_{s-1} & A_s \arrow[d, "j"] \\
E_{-1} & & E_s \arrow[ul, "k"]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, mapsto, "i^s"] \arrow[d, mapsto, "j"'] & ky & \tilde{y} \arrow[d, mapsto] \\
0 & & y \arrow[ul, mapsto, "k"]
\end{tikzcd}$$

$\square$

4 Example: The EHP SS and its truncations

In the last talk, Michael Donovan showed that working in the category of 2-local spaces we have a fibration sequence

$$S^n \xrightarrow{e} \Omega S^{n+1} \xrightarrow{h} \Omega S^{2n+1}$$

for each $n \geq 1$. We can loop and splice the fibration sequences to obtain

$$\begin{tikzcd}
 & 0 & 1 & & s-1 & s & & \\
* \arrow[r] & \Omega S^1 \arrow[r] \arrow[d] & \Omega^2 S^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^s S^s \arrow[r, "e"] \arrow[d] & \Omega^{s+1} S^{s+1} \arrow[r] \arrow[d, "h"] & \cdots \arrow[r] & QS^0 \\
 & \Omega S^1 & \Omega^2 S^3 \arrow[ul, dotted] & & \Omega^s S^{2s-1} \arrow[ul, dotted] & \Omega^{s+1} S^{2s+1} \arrow[ul, dotted, "p"'] & &
\end{tikzcd}$$

Define

$$A_{s,t}(\mathrm{EHP}) = \begin{cases} \pi_{s+t}(\Omega^{s+1} S^{s+1}) & \text{when } s \geq 0 \text{ and } s+t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E^1_{s,t}(\mathrm{EHP}) = \begin{cases} \pi_{s+t}(\Omega^{s+1} S^{2s+1}) & \text{when } s \geq 0 \text{ and } s+t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

[Notice that $E^1_{s,t} = 0$ when either $s < 0$ or $t < 0$. In $E^1_{s,t}$ t is the stem; in $A_{s,t}$ $(s+t)$ is the stem.]

Applying π_* to a fibration sequence gives a long exact sequence. Thus applying π_* to the interlocking fibration sequences above gives an exact couple. The zig-zag defining the correspondence d_r takes the form of the first diagram displayed below. We draw the case when $s \geq 0$ and $s+t \geq 0$ in the second diagram below.

$$\begin{tikzcd}
A_{s,t} \arrow[r] \arrow[d] & A_{s+1,t-1} \arrow[r] & \ldots \arrow[r] & A_{s+r-1,t-r+1} & \\
E^1_{s,t} & & & & E^1_{s+r,t-r+1} \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{s+t}(\Omega^{s+1} S^{s+1}) \arrow[r, "e"] \arrow[d, "h"'] & \pi_{s+t}(\Omega^{s+2} S^{s+2}) \arrow[r] & \ldots \arrow[r, "e"] & \pi_{s+t}(\Omega^{s+r} S^{s+r}) & \\
\pi_{s+t}(\Omega^{s+1} S^{2s+1}) & & & & \pi_{s+t+1}(\Omega^{s+r+1} S^{2(s+r)+1}) \arrow[ul, "p"']
\end{tikzcd}$$

Proposition. *We have constructed a spectral sequence converging to $\pi_*(QS^0)$ in which d_r has degree $(-r, r-1)$ and the filtration degree is given by s, i.e.*

$$E^1_{s,t}(\mathrm{EHP}) \overset{s}{\Longrightarrow} \pi_{s+t}(QS^0).$$

In particular, we have an identification

$$E^\infty_{s,t}(\mathrm{EHP}) = F_s \pi_{s+t}(QS^0) / F_{s-1} \pi_{s+t}(QS^0)$$

where $F_s\pi_*(QS^0) = \mathrm{im}(\pi_*(\Omega^{s+1}S^{s+1}) \longrightarrow \pi_*(QS^0))$ for $s \geq 0$.

The identification is given by writing an element of $F_s\pi_{s+t}(QS^0)$ as the image of an element in $\pi_{s+t}(\Omega^{s+1}S^{s+1})$ and shooting this this element down by h to $\pi_{s+t}(\Omega^{s+1}S^{2s+1}) = E^1_{s,t}$ to give a permanent cycle.

Finally, we remark that the SS is a first quadrant spectral sequence and so for $r > \max\{s,t\}+1$ we have $E^r_{s,t}(\mathrm{EHP}) = E^\infty_{s,t}(\mathrm{EHP})$.

The purpose of the EHP SS is not to compute the stable homotopy groups of spheres. We have other more efficient tools for doing this, namely the Adams SS and the Adams-Novikov SS. The purpose of the EHP SS is to compute the unstable homotopy groups of spheres. In order to do this we use a truncation technique together with an inductive algorithm. We will describe the truncation technique now and leave the computations for the next talk.

Suppose we splice finitely many of the EHP fibrations together to give the diagram below.

$$\begin{tikzcd}
 & 0 & 1 & & k-1 & k & & \\
{*} \arrow[r] & \Omega S^1 \arrow[r] \arrow[d] & \Omega^2 S^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^k S^k \arrow[r] \arrow[d] & \Omega^k S^k \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^k S^k \\
 & \Omega S^1 & \Omega^2 S^3 \arrow[ul, dotted] & & \Omega^k S^{2k-1} \arrow[ul, dotted] & {*} \arrow[ul, dotted] & &
\end{tikzcd}$$

Define

$$A_{s,t}(\text{EHP-}k) = \begin{cases} \pi_{s+t}(\Omega^{s+1}S^{s+1}) & \text{when } 0 \leq s < k \text{ and } s+t \geq 0 \\ \pi_{s+t}(\Omega^k S^k) & \text{when } s \geq k \text{ and } s+t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E^1_{s,t}(\text{EHP-}k) = \begin{cases} \pi_{s+t}(\Omega^{s+1}S^{2s+1}) & \text{when } 0 \leq s < k \text{ and } s+t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition. *We have constructed a spectral sequence converging to $\pi_*(S^k)$ in which d_r has degree $(-r, r-1)$ and the filtration degree is given by s, i.e.*

$$E^1_{s,t}(\text{EHP-}k) \overset{s}{\Longrightarrow} \pi_{s+t}(\Omega^k S^k).$$

In particular, we have an identification

$$E^\infty_{s,t}(EHP\text{-}k) = F_s\pi_{s+t}(\Omega^k S^k)/F_{s-1}\pi_{s+t}(\Omega^k S^k)$$

where $F_s\pi_*(\Omega^k S^k) = \mathrm{im}(\pi_*(\Omega^{s+1}S^{s+1}) \longrightarrow \pi_*(\Omega^k S^k))$ for $0 \leq s < k$.

The identification is given by writing an element of $F_s\pi_{s+t}(\Omega^k S^k)$ as the image of an element in $\pi_{s+t}(\Omega^{s+1}S^{s+1})$ and shooting this this element down by h to $\pi_{s+t}(\Omega^{s+1}S^{2s+1}) = E^1_{s,t}$ to give a permanent cycle.

Finally, we remark that the SS is a first quadrant spectral sequence and so for $r > \max\{s,t\}+1$ we have $E^r_{s,t}(\text{EHP-}k) = E^\infty_{s,t}(\text{EHP-}k)$.

Since we have a "map of towers"

$$\begin{tikzcd}
* \arrow[r] & \Omega S^1 \arrow[r] \arrow[d] & \Omega^2 S^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^k S^k \arrow[r] \arrow[d] & \Omega^k S^k \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^k S^k \\
& \Omega S^1 & \Omega^2 S^3 \arrow[ul, dotted] & & \Omega^k S^{2k-1} \arrow[ul, dotted] & * \arrow[ul, dotted] & &
\end{tikzcd}$$

$$\Big\downarrow$$

$$\begin{tikzcd}
* \arrow[r] & \Omega S^1 \arrow[r] \arrow[d] & \Omega^2 S^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^s S^s \arrow[r, "e"] \arrow[d] & \Omega^{s+1} S^{s+1} \arrow[r] \arrow[d, "h"] & \cdots \arrow[r] & QS^0 \\
& \Omega S^1 & \Omega^2 S^3 \arrow[ul, dotted] & & \Omega^s S^{2s-1} \arrow[ul, dotted] & \Omega^{s+1} S^{2s+1} \arrow[ul, dotted, "p"] & &
\end{tikzcd}$$

we obtain a map of SSs $E^*_{*,*}(\text{EHP-}k) \longrightarrow E^*_{*,*}(\text{EHP})$. A map of spectral sequences consists of a map at the level of E^1-pages commuting with the correspondences d_r. In our set up, a zig-zag $(x, \tilde{x}, \tilde{y}, y)$ in the source SS gives rise to a zig-zag in the target spectral sequence and so the map is apparent. Because the E^1-page map commutes with the correspondences the corresponding $\ker d_{r-1}$'s and $\operatorname{im} d_{r-1}$'s are mapped into one another and so one obtains a map on each page of the SS. Viewing d_r as a differential on the E_r-page we see that each of these maps is a chain map.

The map $E^1_{*,*}(\text{EHP-}k) \longrightarrow E^1_{*,*}(\text{EHP})$ is an inclusion but the target SS has more differentials to account for the fact that $\pi_*(\Omega^k S^k) \longrightarrow \pi_*(QS^0)$ is not injective.

4.1 The flavour of this SS

I hope that the account I have given here should enable anyone to answer questions about what differentials really mean for themselves. However, let's point out a few things for the EHP SS.

First, consider the following paragraph.

Suppose we have an element $\tilde{y} \in \pi_{s+t}(\Omega^{s+1} S^{s+1})$ with non-zero Hopf invariant, i.e. $y = h\tilde{y}$ is non-zero in $\pi_{s+t}(\Omega^{s+1} S^{2s+1})$; then by exactness $\tilde{y}$ does not desuspend. If $\tilde{y}$ defines a non-zero element $z \in \pi_{s+t}(QS^0)$ then z is detected by y.

$$\begin{tikzcd}
\pi_{s+t}(\Omega^s S^s) \arrow[r, "e"] & \pi_{s+t}(\Omega^{s+1} S^{s+1}) \arrow[r] \arrow[d, "h"] & \pi_{s+t}(QS^0) \\
& \pi_{s+t}(\Omega^{s+1} S^{2s+1}) &
\end{tikzcd}$$

$$\begin{tikzcd}
\nexists \bullet \arrow[r, mapsto, "e"] & \tilde{y} \arrow[r, mapsto] \arrow[d, mapsto, "h"] & z \\
& y \neq 0 &
\end{tikzcd}$$

This last statement is false. If z was zero then one would not expect y to detect anything. Since y is a permanent cycle the only way this can be the case is for it to be be hit by a differential. The argument just carried out is just the extreme case when $z \in F_{-1}$. In the above paragraph we have

$z \in F_s$ by construction but z might lie in lower filtration: we could have $z \neq 0$ but $z \in F_{s-1}$. Then there exists $\tilde{y}' \in \pi_{s+t}(\Omega^s S^s)$ mapping to z. Since $\tilde{y} - e\tilde{y}'$ has image zero in $\pi_*(QS^0)$,

$$y = h\tilde{y} = h(\tilde{y} - e\tilde{y}')$$

is the target of a differential.

I propose the following way of thinking about a differential in the EHP sequence.

1. Suppse $d_r x = y$.

2. Then we have an element $\tilde{y}$ (part of the zig-zag $(x, \tilde{x}, \tilde{y}, y)$) lifting y which eventually becomes zero.

3. On the other hand, if we have another element $\tilde{y}'$ lifting y we know that $\tilde{y}' - \tilde{y}$ desuspends and has the same image as $\tilde{y}'$ eventually.

4. At some point $\tilde{y}'$ becomes homotopic to $\tilde{y}' - \tilde{y}$; this is encoded by a nullhomotopy of $\tilde{y}$ which, in turn, is encoded by x.

5. x gives an eventual homotopy of any lift of y to an element which desuspends.