18.917 Topics in Algebraic Topology: The Sullivan Conjecture Fall 2007

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The Sullivan Conjecture (Lecture 30)

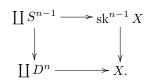
In this lecture we will combine some of our previous results to deduce a version of the Sullivan conjecture.

Theorem 1. Let X be a finite-dimensional CW complex, X^{\vee} its p-profinite completion, and K a connected p-profinite space. Then the diagonal map

$$X^{\vee} \to (X^{\vee})^K$$

is an equivalence of p-profinite spaces.

Proof. Let us say that a space X is good if $X^{\vee} \to (X^{\vee})^K$ is an equivalence. Since p-profinite completion preserves homotopy pushout squares (being a left adjoint) and K is atomic in the p-profinite category, the collection of good spaces is stable under the formation of homotopy pushouts. We now show that every space X of finite dimension n is good, using induction on n. We have a homotopy pushout diagram



The inductive hypothesis guarantees that $\operatorname{sk}^{n-1} X$ and $\coprod S^{n-1}$ are good. It will therefore suffice to show that $\coprod D^n$ is good. But this coproduct is homotopy equivalent to a discrete topological space, which is obviously good.

Corollary 2. Let X be a finite dimensional CW complex, and K a connected p-profinite space. Then every map $K \to X^{\vee}$ in the p-profinite category is homotopic to a constant map.

In the special case where K = BG, where G is a finite p-group, we can identify $(X^{\vee})^K$ with the homotopy fixed point set $(X^{\vee})^{hG}$, where G acts trivially on X. There is a more general form of Theorem 1 where we do not assume that the action of G is trivial.

Lemma 3. Let G be a finite p-group, and let $\mathfrak{S}_p^{\vee}(G)$ denote the category of p-profinite spaces with an action of G. Then the functor

$$\mathfrak{S}_p^{\vee}(G) \to \mathfrak{S}_p^{\vee}$$
$$X \mapsto X^{hG}$$

preserves finite homotopy colimits.

Proof. We can identify $\mathfrak{S}_p^{\vee}(G)$ with $\mathfrak{S}_{p,/BG}^{\vee}$, and the formation of homotopy fixed points with the pushforward functor f_* , where $f: BG \to *$ is the projection. The desired result now follows from the observation that BG is atomic.

Theorem 4. Let G be a finite p-group, X a finite-dimensional G-CW complex, and X^G the subcomplex of G-fixed points. Then the composite map

$$\phi: (X^G)^{\vee} \to (X^{hG})^{\vee} \to (X^{\vee})^{hG}$$

is a homotopy equivalence of p-profinite spaces.

Proof. The space X admits a filtration

$$X^G = Y_{-1} \subseteq Y_0 \subseteq \ldots \subseteq Y_n = X,$$

where Y_j denotes the union of X^G with the *j*-skeleton of X. We will prove by induction on *j* that the conclusion of the theorem is valid for Y_j . The case j = -1 follows from Theorem 1. In the general case, we have a homotopy pushout diagram

where each H_{α} is a proper subgroup of G. Since p-profinite completion and passage to homotopy fixed points with respect to G preserve homotopy pushout squares, we get a homotopy pushout square

of *p*-profinite spaces. By the inductive hypothesis, the upper right corner is equivalent to the *p*-profinite completion of X^G . It will therefore suffice to show that the *p*-profinite spaces in the left column are empty.

We will show that $Z = ((\coprod_{\alpha} S^{j-1} \times G/H_{\alpha})^{\vee})^{hG}$ is empty; the same argument will show that $((\coprod_{\alpha} D^{j} \times G/H_{\alpha})^{\vee})^{hG}$ is empty as well. The group G has only finitely many proper subgroups H. We can therefore decompose Z as a coproduct of spaces of the form

$$Z_H = ((\coprod_{H_{\alpha}=H} S^{j-1} \times G/H)^{\vee})^{hG}$$

It will therefore suffice to show that each Z_H is empty. But Z_H can be identified with

$$((\coprod S^{j-1})^{\vee} \times G/H)^{hG}$$

We therefore have a map from Z_H to the homotopy fixed set $(G/H)^{hG}$, which is empty because H is a proper subgroup of G.

Remark 5. We can formulate Theorem ?? as follows: the map ϕ identifies the homotopy fixed set $(X^{\vee})^{hG}$ with the *p*-profinite completion of the actual fixed set X^G .