18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Adem Relations (Lecture 4)

Remark 1. Throughout this lecture, we will work over the field $\mathbf{F}_2$ with two elements. If X is a topological space, we will simply write $\mathrm{H}_*(X)$ and $\mathrm{H}^*(X)$ to denote the homology and cohomology of X with coefficients in $\mathbf{F}_2$. Similarly, we let $C_*(X)$ and $C^*(X)$ denote the chain and cochain complexes of X, respectively.

Our goal in this lecture is to prove the Adem relations. We begin by describing our context. For any chain complex V, we have defined the nth extended power $D_n(V) = V^{\otimes n}_{h\Sigma_n}$. We now observe that there is a canonical map

$$\phi : D_m(D_n(V)) \to D_{mn}(V).$$

More concretely, the left hand side is given by

$$(V^{\otimes n}_{h\Sigma_n})^{\otimes m}_{h\Sigma_m} \simeq V^{\otimes mn}_{hG},$$

where G denotes the wreath product $\Sigma_n^m \rtimes \Sigma_m$. The right hand side is simply given by $V^{\otimes mn}_{h\Sigma_{mn}}$. The map ϕ is induced by the inclusion of finite groups $G \hookrightarrow \Sigma_{mn}$.

Definition 2. Let V be a complex equipped with a symmetric multiplication $m : D_2(V) \to V$. We will say that m is *good* if there exists a map $m' : D_4(V) \to V$ such that the diagram

$$\begin{array}{ccc} D_2(D_2(V)) & \xrightarrow{D_2(m)} & D_2(V) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle m} \\ D_4(V) & \xrightarrow{m'} & V. \end{array}$$

Example 3. Let V be an E_∞-algebra over the field $\mathbf{F}_2$. Then the symmetric multiplication on V is good. In particular, if X is a topological space then the cochain complex $C^*(X)$ has a good symmetric multiplication.

Notation 4. Let i and j be integers. We let

$$(i, j) = \begin{cases} \binom{i+j}{i} = \binom{i+j}{j} = \frac{(i+j)!}{i!j!} & \text{if } i, j \geq 0 \\ 0 & \text{otherwise.} \end{cases}.$$

We will regard (i, j) as taking values in the finite field $\mathbf{F}_2$. We observe that if $i, j \geq 0$, then (i, j) is equal to 1 if the sum of i and j in base 2 can be computed "without carrying", and equal to zero otherwise.

Our goal in this lecture is to prove the following:

Proposition 5 (Adem Relations)**.** *Let V be a complex equipped with a good symmetric multiplication, and let $v \in \mathrm{H}^n(V)$. For any pair of integers $a < 2b$, we have*

$$\mathrm{Sq}^a \mathrm{Sq}^b(v) = \sum_k (2k - a, b - k - 1)\, \mathrm{Sq}^{b+k} \mathrm{Sq}^{a-k}(v).$$

Actually, we will not give a complete proof in this lecture. We will instead show how to reduce the statement of Proposition 5 from a calculation in the homology of groups (Lemma **??**). This calculation will be carried out in the next lecture.

Remark 6. The sum appearing in Proposition 5 is actually finite, since $(2k - a, b - k - 1)$ vanishes unless $a \leq 2k < 2b$.

Definition 7. Let $\mathbf{F}_2\{\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \mathrm{Sq}^1, \ldots\}$ denote the free associative $\mathbf{F}_2$-algebra generated by the symbols $\{\mathrm{Sq}^i\}_{i \in \mathbf{Z}}$. The *big Steenrod algebra* $\mathcal{A}^{\mathrm{Big}}$ is defined to be the quotient of $\mathbf{F}_2\{\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \mathrm{Sq}^1, \ldots\}$ by imposing the Adem relations

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_k (2k - a, b - k - 1) \mathrm{Sq}^{b+k} \mathrm{Sq}^{a-k}.$$

for every $a < 2b$.

We observe that $\mathcal{A}^{\mathrm{Big}}$ the structure of a graded algebra, where each generator Sq^i is given degree i. A *module* over the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$ is a graded vector space V over the field $\mathbf{F}_2$, equipped with an action $\mathcal{A}^{\mathrm{Big}} \otimes V \to V$ which respects the grading: if $v \in V$ is homogeneous of degree n, then $\mathrm{Sq}^k(v)$ is homogeneous of degree $n + k$. We will say that V is *unstable* if, whenever $\mathrm{Sq}^k(v)$ vanishes whenever v is homogeneous of degree $< k$.

Example 8. Let V be a complex equipped with a good symmetric multiplication. Then Proposition 5 implies that the cohomology $\mathrm{H}^*(V)$ has the structure of a unstable $\mathcal{A}^{\mathrm{Big}}$-module.

Definition 9. The *Steenrod algebra* $\mathcal{A}$ is defined to be the quotient of $\mathcal{A}^{\mathrm{Big}}$ by the (two-sided) ideal generated by the element $1 - \mathrm{Sq}^0$. We will say that a (graded) $\mathcal{A}$-module is *unstable* if it is unstable when regarded as an $\mathcal{A}^{\mathrm{Big}}$-module.

Example 10. Let X be a topological space. Since Sq^0 acts by the identity on the cohomology $\mathrm{H}^*(X)$, we conclude that $\mathrm{H}^*(X)$ has the structure of an unstable module over the Steenrod algebra.

Remark 11. In the last lecture, we saw another feature of the action of Steenrod operations on the cohomology of spaces: the operations Sq^{-a} vanish for $a > 0$. In fact, this is a formal consequence of Adem relations and the fact that Sq^0 acts by the identity. In other words, for $a > 0$ the element Sq^{-a} is equal to zero in the Steenrod algebra $\mathcal{A}$. We will prove this by induction on a. For this, we invoke the Adem relations to deduce

$$\mathrm{Sq}^{-a} = \mathrm{Sq}^{-a} \mathrm{Sq}^0 = \sum_k (2k + a, -k - 1) \mathrm{Sq}^k \mathrm{Sq}^{-a-k}.$$

If $k \geq 0$ or $-\frac{a}{2} < k$, then the coefficient $(2k + a, -k - 1)$ vanishes. But if $-\frac{a}{2} \leq k < 0$, then Sq^{-a-k} is equal to zero in $\mathcal{A}$ by the inductive hypothesis.

We now turn to the proof of Proposition 5. We begin with the following observation:

Remark 12. Recall that if V is a complex equipped with a symmetric multiplication, then ΩV inherits a symmetric multiplication, and the isomorphism

$$\mathrm{H}^*(V) \simeq \mathrm{H}^{*+1}(\Omega V)$$

is compatible with the action of the Steenrod operations. The same argument shows that if V has a *good* symmetric multiplication, then the induced symmetric multiplication is also good. Consequently, in proving Proposition 5 we are free to replace V by any shift $\Omega^{n'}(V)$. In other words, we are free to enlarge the degree n of the cohomology class v.

The formula of Proposition 5 looks very assymetric: the left hand side has only one term, while the right hand side has many terms. We will deduce Proposition 5 from the following more symmetric looking assertion:

Lemma 13. *Let p and q be positive integers, let V be a complex with a good symmetric multiplication, and let $v \in \mathrm{H}^n(V)$. Then we have an equality*

$$\sum_l (p-2l, l) \,\mathrm{Sq}^{2n-q-l}\,\mathrm{Sq}^{n-p+l}(v) = \sum_{l'} (q-2l', l')\,\mathrm{Sq}^{2n-p-l'}\,\mathrm{Sq}^{n-q-l'}(v)$$

in $\mathrm{H}^{4n-p-q}(V)$.

Assuming Lemma 13, we can now prove Proposition 5.

Proof. Choose an integer $m \gg 0$. According to Remark 12, we are free to enlarge n as much as we like; in particular, we can choose $n = 2^m - 1 + b$. We will now apply Lemma 13 with $p = 2^m - 1$ and $q = 2n - a$. Let us now evaluate both sides of the expression appearing in Lemma 13. The left hand side is given by

$$\sum_l (2^m - 1 - 2l, l)\,\mathrm{Sq}^{a-l}\,\mathrm{Sq}^{b+l}(v).$$

The coefficient $(2^m - 1 - 2l, l)$ obviously vanishes if $l < 0$, or if $l \geq 2^{m-1}$. If $0 < l < 2^{m-1}$, then we can write $l = 2^x + 2^{x+1}y$, where $0 \leq x \leq m-2$. We now observe that 2^x appears in the base 2 expansion of both $2^m - 1 - 2l$ and l, so the coefficient $(2^m - 1 - 2l, l)$ vanishes. It follows that the left hand side consists of only one nonzero term, given by the expression $\mathrm{Sq}^a\,\mathrm{Sq}^b(v)$.

We now evaluate the right hand side. Let $k = 2^m + b - l' - 1$, so that the left hand sum can be written as

$$\sum_k (2k - a, 2^m + b - k - 1)\,\mathrm{Sq}^{b+k}\,\mathrm{Sq}^{a-k}(v).$$

To complete the proof, it will suffice to show that for every integer k, either

$$(2k - a, 2^m + b - k - 1) = (2k - a, b - k - 1)$$

or $\mathrm{Sq}^{b+k}\,\mathrm{Sq}^{a-k}(v)$ vanishes. We consider four cases:

(i) $2k < a$: In this case, we have

$$(2k - a, 2^m + b - k - 1) = (2k - a, b - k - 1) = 0.$$

(ii) $a \leq 2k < 2b$: In this case, $2k - a < 2b - a \leq 2^m$. It follows that $(2k - a, z) = (2k - a, z + 2^m)$ for every nonnegative integer x (see Notation 4).

(iii) $2b \leq 2k < a + 2^m$: The expression $(2k - a, b - k - 1)$ vanishes in this case. Moreover, we have $2k - a \geq 2b - a > 0$, so we can choose a nonnegative integer y such that $2^y \leq 2k - a \leq 2^{y+1} - 1$. Our assumption implies that $y < m$. Since $2k \leq 2^{y+1} + a - 1 \leq 2^{y+1} + 2b - 2$, we deduce that $k - b + 1 \leq 2^y$. We now observe that 2^y appears in the base 2 expansion of both $2k - a$ and $2^m - (k - b + 1)$, so the expression $(2k - a, 2^m + b - k - 1)$ vanishes.

(iv) $a + 2^m \leq 2k$: In this case, we have

$$\deg(\mathrm{Sq}^{a-k}(v)) = (a - k) + n = (a - k) + (2^m + b - 1).$$

Since $a + 2^m \leq 2k$, we get $\deg(\mathrm{Sq}^{a-k}(v)) \leq k + b - 1 < k + b$. Thus $\mathrm{Sq}^{k+b}\,\mathrm{Sq}^{a-k}(v)$ vanishes for reasons of degree.

□

We now turn to the proof of Lemma 13. As usual, the equation among Steenrod operations on a complex V with a symmetric multiplication is an immediate consequence of the following more universal relation, which holds for any complex V:

Lemma 14. *Let V be a complex, let p and q be positive integers, and let $v \in \mathrm{H}^n(V)$. Then the sums*

$$\sum_l (p-2l, l)\,\overline{\mathrm{Sq}}^{2n-q-l}\,\overline{\mathrm{Sq}}^{n-p+l}(v) \in \mathrm{H}^{4n-p-q}(D_2(D_2(V))$$

$$\sum_{l'} (q-2l', l')\,\overline{\mathrm{Sq}}^{2n-p-l'}\,\overline{\mathrm{Sq}}^{n-p+l'}(v) \in \mathrm{H}^{4n-p-q}(D_2(D_2(V)))$$

have the same image in $\mathrm{H}^{4n-p-q}(D_4(V))$ under the map $\phi : D_2(D_2(V)) \to D_4(V)$.

To prove Lemma 14, we may assume that $V \simeq \mathbf{F}_2[-n]$ is generated by the cohomology class v. In this case, $D_4(V) \simeq V^{\otimes 4}_{h\Sigma_4}$ can be identified with a $(4n)$-fold shift of the chain complex $C_*(B\Sigma_4)$. Similarly,

$$D_2(D_2(V)) \simeq D_2(C_*(B\Sigma_2)[-2n]) \simeq D_2(C_*(B\Sigma_2))[-4n]$$

can be identified with a shift of the chain complex $C_*(BG)$, where G is the semidirect product $\Sigma_2 \times \Sigma_2 \rtimes \Sigma_2$, which we can identify with a 2-Sylow subgroup of Σ_4. Let us use our usual basis $\{x_i\}_{i \leq 0}$ for the homology $\mathrm{H}_*(B\Sigma_2)$. As we saw in the second lecture, this determines a basis for $\mathrm{H}_*(BG) \simeq \mathrm{H}^{-*}\,D_2(C_*(B\Sigma_2)$, consisting of pairwise products $\{x_i x_j\}_{i<j}$ and Steenrod operations $\{\overline{\mathrm{Sq}}^k\,x_i\}_{k \leq -i}$. We have an isomorphism

$$\mathrm{H}_*(BG) \simeq \mathrm{H}^{4n-*}(D_2(D_2(V))),$$

which carries $\overline{\mathrm{Sq}}^k\,x_i$ to $\overline{\mathrm{Sq}}^{2n+k}\,\overline{\mathrm{Sq}}^{n-i}(v)$. Consequently, Lemma 14 is an immediate consequence of the following assertion:

Lemma 15. *Let p and q be positive integers. Then the expressions*

$$\sum_l (p-2l, l)\,\overline{\mathrm{Sq}}^{-q-l}\,x_{p-l} \in \mathrm{H}_{p+q}(BG)$$

$$\sum_l (q-2l', l')\,\overline{\mathrm{Sq}}^{-p-l'}\,x_{q-l'} \in \mathrm{H}_{p+q}(BG)$$

have the same image in $\mathrm{H}_{p+q}(B\Sigma_4)$.

We will prove Lemma 15 in the next lecture.