18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Admissible Monomials (Lecture 6)

Recall that we have define the *big Steenrod algebra* $\mathcal{A}^{\mathrm{Big}}$ to be the quotient of the free associated $\mathbf{F}_2$-algebra

$$\mathbf{F}_2\{\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \mathrm{Sq}^1, \ldots\}$$

obtained by imposing the Adem relations:

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_k (2k-a, b-k-1) \mathrm{Sq}^{b+k} \mathrm{Sq}^{a-k}$$

for $a < 2b$, and the *Steenrod algebra* $\mathcal{A}$ to be the quotient of of $\mathcal{A}^{\mathrm{Big}}$ by imposing the further relation $\mathrm{Sq}^0 = 1$. Our goal in this lecture is to explain some consequences of the Adem relations for the structure of the algebras $\mathcal{A}^{\mathrm{Big}}$ and $\mathcal{A}$.

We say that a monomial $\mathrm{Sq}^a \mathrm{Sq}^b$ is *admissible* if $a \geq 2b$. If $\mathrm{Sq}^a \mathrm{Sq}^b$ is not admissible, then the Adem relations allow us rewrite the monomial $\mathrm{Sq}^a \mathrm{Sq}^b$ as a linear combination of other monomials. We observe that the coefficient $(2k-a, b-k-1)$ appearing in the Adem relations vanishes unless $\frac{a}{2} \leq k < b$. Using the inequality $k \geq \frac{a}{2}$, we deduce

$$b + k \geq b + \frac{a}{2} > \frac{a}{2} + \frac{a}{2} = 2(a - \frac{a}{2}) \geq 2(a-k).$$

In other word, the Adem relations allow us rewrite each inadmissible expression $\mathrm{Sq}^a \mathrm{Sq}^b$ as a sum of admissible monomials.

We would like to generalize the preceding observation. For every sequence of integers $I = (i_n, i_{n-1}, \ldots, i_0)$, we let Sq^I denote the product $\mathrm{Sq}^{i_n} \mathrm{Sq}^{i_{n-1}} \ldots \mathrm{Sq}^{i_0}$. We will say that the sequence I is *admissible* if

$$i_j \geq 2 i_{j-1}$$

for $1 \leq j \leq n$. In this case, we will also say that Sq^I is an *admissible monomial*.

Proposition 1. *The big Steenrod algebra $\mathcal{A}^{Big}$ is spanned (as an $\mathbf{F}_2$-vector space) by the admissible monomials Sq^I. The usual Steenrod algebra $\mathcal{A}$ is spanned by the admissible monomials Sq^I where I is a sequence of* positive *integers.*

Proof. Recall that Sq^i is equal to zero in $\mathcal{A}$ if $i < 0$. It follows that Sq^I vanishes in $\mathcal{A}$ unless I is a sequence of nonnegative integers. Moreover, if I' is the sequence of integers obtained from I by deleting all occurences of 0, then $\mathrm{Sq}^I = \mathrm{Sq}^{I'}$ in $\mathcal{A}$ (since $\mathrm{Sq}^0 = 1$); moreover, if Sq^I is admissible then $\mathrm{Sq}^{I'}$ is also admissible. Thus, the second assertion follows from the first.

The idea of the proof is now simple: let I be an arbitrary sequence of integers. We wish to show that we can use the Adem relations to rewrite Sq^I as a linear combination of admissible monomials. The proof will use inducation. In order to make the induction work, we will need the following slightly stronger inductive hypothesis:

(∗) Let $I = (i_n, \ldots, i_0)$ be a sequence of integers, and let x be an integer such that $i_j \leq 2^j x$ for $0 \leq j \leq n$. Then in $\mathcal{A}^{\mathrm{Big}}$ there is a relation of the form

$$\mathrm{Sq}^I = \sum_\alpha \mathrm{Sq}^{I(\alpha)},$$

where each $I(\alpha) = (i_n(\alpha), \ldots, i_0(\alpha))$ is an *admissible* sequence satisfying $i_j(\alpha) < 2^j x$ for $0 \leq j \leq n$.

We will prove this result by induction on n. For fixed n and x, we will use descending induction on i_n (this is justified since i_n is bounded above by $2^n x$, by assumption).

If $n = 0$, then assertion (∗) is vacuous, since the expression Sq^I is automatically admissible. Let us therefore assume that $n > 0$. Let $I = (i_n, \ldots, i_0)$, and let $I' = (i_{n-1}, \ldots, i_0)$. By the inductive hypothesis, we get an equation of the form

$$\mathrm{Sq}^{I'} = \sum_\beta \mathrm{Sq}^{I'(\beta)},$$

so that

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_n} \mathrm{Sq}^{I'} = \sum_\beta \mathrm{Sq}^{i_n} \mathrm{Sq}^{I'(\beta)}.$$

It therefore suffices to prove (∗) for the sequences $(i_n, i_{n-1}(\beta), \ldots, i_0(\beta))$. In other words, we may assume without loss of generality that the sequence $I' = (i_{n-1}, \ldots, i_0)$ is already admissible.

If $i_n \geq 2i_{n-1}$, then the sequence I is admissible and there is nothing to prove. Otherwise, we can invoke the Adem relations to deduce

$$\mathrm{Sq}^{i_n} \mathrm{Sq}^{i_{n-1}} = \sum_k (2k - i_n, i_{n-1} - k - 1) \mathrm{Sq}^{i_{n-1}+k} \mathrm{Sq}^{i_n - k}.$$

The terms on the right side vanish unless $\frac{i_n}{2} \leq k < i_{n-1}$. In particular, we get

$$i_{n-1} + k < 2i_{n-1} \leq 2^n x$$

$$i_n - k \leq i_n - \frac{i_n}{2} \leq 2^{n-1} x$$

so that the new sequence $J = (i_{n-1} + k, i_n - k, i_{n-2}, \ldots, i_0)$ satisfies the hypotheses of (∗). Moreover,

$$i_{n-1} + k > \frac{i_n}{2} + \frac{i_n}{2} = i_n,$$

so the inductive hypothesis implies that Sq^J can be rewritten in the desired form. □

Scholium 2. Let $\mathcal{B}$ be the subspace of $\mathcal{A}^{\mathrm{Big}}$ generated by Sq^I, where $I = (i_n, \ldots, i_0)$ is an admissible sequence of nonpositive integers. Then $\mathcal{B}$ is a subalgebra of $\mathcal{A}^{\mathrm{Big}}$.

Proof. Apply (∗) in the case $x = 0$. □

The subalgebra $\mathcal{B} \subseteq \mathcal{A}^{\mathrm{Big}}$ is usually called the *Dyer-Lashof algebra*.

Proposition 1 is subsumed by the following stronger result:

Proposition 3. *The admissible monomials Sq^I form a basis for the big Steenrod algebra $\mathcal{A}^{Big}$. The admissible monomials of the form Sq^I, where I is a sequence of* positive *integers, form a basis for the usual Steenrod algebra $\mathcal{A}$.*

Proposition 1 already implies that $\mathcal{A}^{\mathrm{Big}}$ is generated (as a vector space) by the admissible monomials. Hence, the only thing we need to check is that the admissible monomials are linearly independent. This is a consequence of a more precise result, which we now formulate. First, we recall a bit of terminology. Let M be a module over $\mathcal{A}^{\mathrm{Big}}$ (always assumed to be graded). We say that M is *unstable* if $\mathrm{Sq}^k(m) = 0$ whenever $k > \deg(m)$.

Let $I = (i_n, i_{n-1}, \ldots, i_0)$ be an admissible sequence of integers, so we can write $i_j = 2i_{j-1} + \epsilon_j$ where $\epsilon_j \geq 0$. The sum $\epsilon_n + \ldots + \epsilon_1 + i_0$ is called the *excess* of I. Our reason for introducing this notion is the following:

Lemma 4. *Let M be an unstable $\mathcal{A}^{Big}$-module, and let $I = (i_n, \ldots, i_0)$ be an admissible sequence of integers. Then $\mathrm{Sq}^I(m)$ vanishes whenever the excess of I is larger than the degree of m.*

Proof. Let $I' = (i_{n-1}, \ldots, i_0)$. To show that $\mathrm{Sq}^I(m)$ vanishes, it will suffice to show that $i_n > \deg(\mathrm{Sq}^{I'}(m))$. We now observe that

$$i_n - \deg(\mathrm{Sq}^{I'}(m)) = i_n - (i_{n-1} + \ldots + i_0 + \deg(m)) = (i_n - 2i_{n-1}) + (i_{n-1} - 2i_{n-2}) + \ldots + i_0 - \deg(m)$$

is positive if the excess of I is larger than the degree of m. □

Given any graded $\mathcal{A}^{\mathrm{Big}}$-module M, we can construct an unstable $\mathcal{A}^{\mathrm{Big}}$-module by taking the quotient of M by the submodule generated by elements of the form $\mathrm{Sq}^i(m)$, $i > \deg(m)$. In particular, if we take M to be the free $\mathcal{A}^{\mathrm{Big}}$-module generated by a single class in degree n, then we obtain an unstable $\mathcal{A}^{\mathrm{Big}}$-module which we will denote by $\mathrm{F}^{\mathrm{Big}}(n)$: we call $\mathrm{F}^{\mathrm{Big}}(n)$ the *free unstable $\mathcal{A}^{Big}$-module on one generator in degree n*. There is a canonical element $\overline{\nu}_n \in \mathrm{F}^{\mathrm{Big}}(n)^n$. By construction, this element has the following universal property: if N is any unstable $\mathcal{A}^{\mathrm{Big}}$-module, then evaluation at $\overline{\nu}_n$ induces an isomorphism of $\mathbf{F}_2$-vector spaces $\mathrm{Hom}_{\mathcal{A}^{\mathrm{Big}}}(\mathrm{F}^{\mathrm{Big}}(n), N) \to N^n$.

Similarly, we can define the *free unstable $\mathcal{A}$-module* on a generator in degree ν_n, which we will denote by $F(n)$.

Proposition 3 is an immediate consequence of the following result:

Proposition 5. *Let n be an integer. Then:*

(1) *The free unstable $\mathcal{A}^{Big}$-module $\mathrm{F}^{\mathrm{Big}}(n)$ has a basis consisting of elements $\mathrm{Sq}^I\,\overline{\nu}_n$, where I is an admissible sequence of excess $\leq n$.*

(2) *The free unstable $\mathcal{A}$-module $F(n)$ has a basis consisting of elements $\mathrm{Sq}^I\,\nu_n$, where I is an admissible sequence of* positive *integers of excess $\leq n$.*

Once again, half of Proposition 5 is clear: since $\mathcal{A}^{\mathrm{Big}}$ is generated by admissible monomials, $\mathrm{F}^{\mathrm{Big}}(n)$ is generated by expressions of the form $\mathrm{Sq}^I\,\overline{\nu}$, where I is admissible. Lemma 4 implies that $\mathrm{Sq}^I\,\overline{\nu}$ vanishes if I has excess $> n$. Thus $\mathrm{F}^{\mathrm{Big}}(n)$ is generated by admissible monomials $\mathrm{Sq}^I\,\overline{\nu}_n$, where I is admissible and has excess $\leq n$. The same reasoning shows that $F(n)$ is generated by elements of the form $\mathrm{Sq}^I\,\nu_n$, where I is admissible, positive and has excess $\leq n$.

To complete the proof of Proposition 5, we need to show:

(1′) The elements $\{\mathrm{Sq}^I\,\overline{\nu}_n\}$ are linearly independent in $\mathrm{F}^{\mathrm{Big}}(n)$, where I ranges over admissible sequences of excess $\leq n$.

(2′) The elements $\{\mathrm{Sq}^I\,\nu_n\}$ are linearly independent in $F(n)$, where I ranges over positive admissible sequences of excess $\leq n$.

Our strategy is as follows. Let M be an unstable module over the Steenrod algebra $\mathcal{A}$, and let $v \in M^n$. Then, by construction, we get an induced map $F(n) \to M$ of modules over the Steenrod algebra. To show that the generators $\{\mathrm{Sq}^I\,\nu_n\}$ are linearly independent in $F(n)$, it will suffice to show that the elements $\{\mathrm{Sq}^I\,v\}$ are linearly independent in M. It will therefore suffice to find a particularly clever choice for the pair

(M,v). Fortunately, we have a host of examples of modules unstable $\mathcal{A}$-modules to choose from: namely, the cohomology $\mathrm{H}^*(X)$ of any space X is an unstable $\mathcal{A}$-module. We will therefore be able to deduce $(2')$ by finding a sufficiently nontrivial example of a cohomology class on a topological space. We will return to this point in the next lecture.

Let us assume $(2')$ for the moment, and show how to use $(2')$ can be used to deduce $(1')$. The proof is based on the following observation:

Lemma 6. *Let n and p be integers. Then there is a canonical isomorphism of vector spaces*

$$\phi : \mathrm{F}^{Big}(n) \to \mathrm{F}^{Big}(n+p)$$

described by the formula

$$\mathrm{Sq}^{i_m} \ldots \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_0} \overline{\nu}_n \mapsto \mathrm{Sq}^{i_m+2^k p} \ldots \mathrm{Sq}^{i_1+2p} \mathrm{Sq}^{i_0+p} \overline{\nu}_{n+p}.$$

Proof. The above formula defines a map

$$\widetilde{\phi} : \mathbf{F}_2\{\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \ldots\}\overline{\nu}_n \to \mathbf{F}_2\{\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \ldots\}\overline{\nu}_{n+p}$$

of free modules over the free algebra $R = \mathbf{F}_2\{\ldots, \mathrm{Sq}^{-1}, \mathrm{Sq}^0, \mathrm{Sq}^1, \ldots\}$. To show that ϕ is well-defined, we need to show that $\widetilde{\phi}$ descends to the quotient. This amounts to two observations:

(a) Let J denote the two-sided ideal of R generated by the Adem relations. Then $\widetilde{\phi}$ carries $J\overline{\nu}_n$ into $J\overline{\nu}_{n+p}$. This amounts to a "translation-invariance" feature of the Adem relations: if $a < 2b$, then we have an Adem relation

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_k (2k-a, b-k-1) \mathrm{Sq}^{b+k} \mathrm{Sq}^{a-k}.$$

But we also have $(a + 2^l p) < 2(b + 2^{l-1} p)$, and a corresponding Adem relation

$$\mathrm{Sq}^{a+2^l p} \mathrm{Sq}^{b+2^{l-1}p} = \sum_k (2k - a - 2^l p, b + 2^{l-1} - k - 1) \mathrm{Sq}^{b+2^{l-1}p+k} \mathrm{Sq}^{a+2^l p-k}.$$

Letting $k' = k + 2^{l-1}p$, we can rewrite this as

$$\mathrm{Sq}^{a+2^l p} \mathrm{Sq}^{b+2^{l-1}p} = \sum_{k'} (2k' - a, b - k' - 1) \mathrm{Sq}^{b+2^l p+k'} \mathrm{Sq}^{a+2^{l-1}-k'}$$

which is precisely the sort of term that appears in the image of $\widetilde{\phi}$.

(b) Let $x \in R\overline{\nu}_n$ have degree q, so that $\mathrm{Sq}^a(x)$ vanishes in $\mathrm{F}^{\mathrm{Big}}(n)$ for $a > q$. We wish to show that $\widetilde{\phi}(\mathrm{Sq}^a(x))$ vanishes in $\mathrm{F}^{\mathrm{Big}}(n+p)$. Without loss of generality, we may suppose that

$$x = \mathrm{Sq}^{i_m} \ldots \mathrm{Sq}^{i_0} \overline{\nu}_n,$$

where $q = i_m + \ldots + i_0 + n$. Then

$$\widetilde{\phi}(\mathrm{Sq}^a(x)) = \mathrm{Sq}^{a+2^{m+1}p} \mathrm{Sq}^{i_m+2^m p} \ldots \mathrm{Sq}^{i_0+p} \overline{\nu}_p = \mathrm{Sq}^{a+2^{m+1}p} \widetilde{\phi}(x)$$

vanishes in $\mathrm{F}^{\mathrm{Big}}(n+p)$ since

$$a + 2^{m+1}p > (i_m + \ldots + i_0 + n) + 2^{m+1}p = (i_m + 2^m p) + \ldots + (i_0 + p) + (n + p) = \deg(\widetilde{\phi}(x)).$$

This completes the proof that ϕ is well-defined. To show that ϕ induces an isomorphism $\mathrm{F}^{\mathrm{Big}}(n) \to \mathrm{F}^{\mathrm{Big}}(n+p)$, we observe that the same construction (applied to $n+p$ and $-p$) gives a map $\mathrm{F}^{\mathrm{Big}}(n+p) \to \mathrm{F}^{\mathrm{Big}}(n)$ which is inverse to ϕ. ◻

Proof of $(2') \Rightarrow (1')$. Fix an integer n. We wish to show the elements $\mathrm{Sq}^I \, \overline{\nu}_n$ are linearly independent in $\mathrm{F}^{\mathrm{Big}}(n)$, where I ranges over admissible sequences of integers of excess $\leq n$. Assume otherwise; then there exists a nontrivial relation of the form

$$\sum_\alpha \mathrm{Sq}^{I(\alpha)} \, \overline{\nu}_n = 0.$$

Choose $p \gg 0$, and let $\phi : \mathrm{F}^{\mathrm{Big}}(n) \to \mathrm{F}^{\mathrm{Big}}(n+p)$ be as in Lemma 6. We then get a nontrivial relation

$$\sum_\alpha \phi(\mathrm{Sq}^{I(\alpha)} \, \overline{\nu}_n) = \sum_\alpha \mathrm{Sq}^{J(\alpha)} \, \overline{\nu}_{n+p} = 0$$

in $\mathrm{F}^{\mathrm{Big}}(n+p)$. It follows that

$$\sum_\alpha \mathrm{Sq}^{J(\alpha)} \, \nu_{n+p} = 0$$

in $F(n+p)$. The sequences $J(\alpha)$ are distinct, admissible, and positive if p is chosen sufficiently large. Thus $(1')$ implies that the elements $\{\mathrm{Sq}^{J(\alpha)} \, \nu_{n+p}\}$ are linearly independent in $F(n+p)$, and we obtain a contradiction. $\square$