# 25 Differential forms and de Rham's Theorem

## 25.1 The exterior algebra

Let $V$ be a finite dimensional vector space over the reals. The tensor algebra of $V$ is direct sum

$$\mathbf{Ten}(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \ldots \oplus V^{\otimes k} \ldots$$

It is made into an algebra by declaring that the product of $a \in V^{\otimes k}$ and $b \in V^{\otimes l}$ is $a \otimes b \in V^{\otimes (k+l)}$. It is characterized by the universal mapping property that any linear map $V \to A$ where $A$ is an algebra over $\mathbb{R}$ extends to a unique map of algebras $\mathbf{Ten}(V) \to A$.

The exterior algebra algebra is the quotient of exterior algebra by the relation

$$v \otimes v = 0.$$

The exterior algebra is denoted $\Lambda^*(V)$ or $\Lambda(V)$. It is customary to denote the multiplication in the exterior algebra by $(a, ) \mapsto a \wedge b$ If $v_1 \ldots v_k$ is a basis for $V$ then this relation is equivalent to the relations

$$\begin{aligned} v_i \wedge v_j &= -v_j \wedge v_i \quad \text{for} \quad i \neq j, \\ v_i \wedge v_i &= 0 \end{aligned}$$

Thus $\Lambda^*(V)$ has basis the products

$$v_{i_1} \wedge v_{i_2} \ldots v_{i_k}$$

where the indices run over all strictly increasing sequences of numbers between 1 and $n$.

$$1 \leq i_i < i_2 < \ldots < i_k \leq n.$$

Since for each $k$ there are $\binom{n}{k}$ such sequences of length $k$ we have

$$\dim(\Lambda^*(V)) = 2^n.$$

$\Lambda^*(V)$ since the relation is homogenous the grading of the tensor algebra descends to a grading on the exterior algebra (hence the *).

We can apply this construction fiberwise to a vector bundle. The most important example is the cotangent bundle of a manifold $T^*X$ in which case we get the bundle of differential forms

$$\Lambda^*(T^*X) \quad \text{or} \quad \Lambda^*(X).$$

We will denote the space of smooth sections of $\Lambda^*(X)$ by $\Omega^*(X)$. In local coordinates a typical element of $\Omega^*(X)$ looks like

$$\omega = \sum_{1\le i_i<i_2<\ldots<i_k\le n} \omega_{i_i i_2\ldots<i_k} dx^{i_1}\wedge dx^{i_2}\wedge\ldots dx^{i_k}.$$

Since the construction of $\Lambda^*(X)$ was functorial in the cotangent bundle these bundles naturally pull back under diffeomorphism and if $f : X \to Y$ is any smooth map there is natural map

$$f^* : \Omega^*(Y) \to \Omega^*(X).$$

The most important thing about differential forms is the existence of a natural differential operator the exterior differential defined locally by the following rules

$$\begin{aligned} df &= \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \\ d\omega &= \sum_{1\le i_i<i_2<\ldots<i_k\le n} d\omega_{i_i i_2\ldots<i_k} \wedge dx^{i_1}\wedge dx^{i_2}\wedge\ldots dx^{i_k}. \end{aligned}$$

Notice that we can't invariantly define a similar operator on the tensor algebra. If we have a one form

$$\theta = \sum_{i=1} f_i dx^i$$

and try to define

$$D\theta = \sum_{i=1} \frac{\partial f_i}{\partial x^j} dx^j \otimes dx^i$$

then when if we have new coordinates $y^1 \ldots y^n$ we have

$$dx^i = \sum_{j=1}^n \frac{\partial x^i}{\partial y^j} dy^j$$

and

$$\theta = \sum_{m=1}^n g_m dy^m$$

where

$$g_m = f_i \frac{\partial x^i}{\partial y^m}$$

$$
\begin{aligned}
D\theta &= \sum_{i=1} \frac{\partial f_i}{\partial x^j} dx^j \otimes dx^i \\
&= \frac{\partial f_i}{\partial x^j} \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^m} dy^m \otimes dy^l \\
&= \frac{\partial f_i}{\partial y^k} \frac{\partial y^k}{\partial x^j} \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^m} dy^m \otimes dy^l \\
&= \frac{\partial f_i}{\partial y^m} \frac{\partial x^i}{\partial y^l} dy^m \otimes dy^l \\
&= \frac{\partial f_i}{\partial y^m} \frac{\partial x^i}{\partial y^l} dy^m \otimes dy^l \\
&= \Big(\frac{\partial}{\partial y^m}( f_i \frac{\partial x^i}{\partial y^l}) - f_i \frac{\partial^2 x^i}{\partial y^m \partial y^l}\Big) dy^m \otimes dy^l \\
&= \sum_{m=1}^{n} \frac{\partial g_l}{\partial y^m} dy^m \otimes dy^l - f_i \frac{\partial^2 x^i}{\partial y^m \partial y^l} dy^m \otimes dy^l .
\end{aligned}
$$

Thus our definition depends on the choice of coordinates. Notice that when we pass to the exterior algebra this last expression vanishes that exterior derivative is well defined.

**Theorem 25.1.** $d^2 = 0$.

*Proof.* From the definition in local coordinates it suffices to check that $d^2 = 0$ on functions.

$$d^2(f) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$$

since the $f$ smooth so the matrix of second derivatives is symmetric. $\square$

**Proposition 25.2.**

$$d(a \wedge b) = da \wedge b + (-1)^{\deg(a)} \wedge db.$$

*Proof.* The bilinearity of the wedge product implies that it suffices to check the result when

$$a = f\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

$\square$

**Definition 25.3.** A cochain complex is a graded vector space $C = \sum_{i=0}^{\infty} C_i$ together with a map $d : C \to C$ so that $dC_i \subset C_{i+1}$ and $d^2 = 0$. The cohomology groups of a cochain complex are defined to be

$$H^i(C, d) = \ker(d : C^i \to C^{i+1})/\mathrm{Ran}(d : C^{i-1} \to C^i)$$