18.969 Topics in Geometry: Mirror Symmetry
Spring 2009

MIRROR SYMMETRY: LECTURE 17

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1. Coherent Sheaves on a Complex Manifold (contd.)

We now recall the following definitions from category theory.

Definition 1. *An* additive category *is one in which* Hom(A, B) *are abelian groups, composition is distributive, and there is a direct sum* $\oplus$ *and a zero object* 0*. An* abelian category *is an additive category s.t. every morphism has a kernel and cokernel, e.g. a kernel of* $f : A \to B$ *is a morphism* $K \to A$ *s.t.* $g : C \to A$ *factors through* K *uniquely iff* $f \circ g = 0$*.*

One can define complexes in an additive category, but one needs to be in an abelian category to have notions of exact sequences and cohomology. Recall that, given chain complexes C_*, D_*, a chain map $f : C_* \to D_*$ is a collection of maps $f_i C_i \to D_i$ commuting with δ. Given two such maps $f = \{f_i\}, g = \{g_i\}$, we call them *homotopic* if there is a map $h : A \to B[-1]$ (B shifted down by 1) s.t. $f - b = d_B h + h d_A$, i.e.

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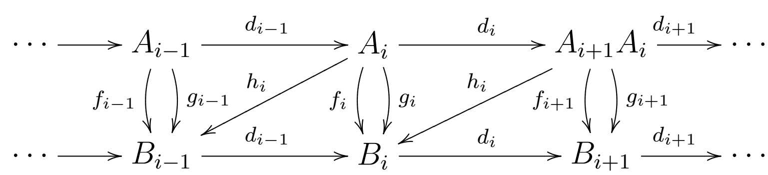

A chain map is a *quasi-isomorphism* if the induced maps on cohomology are isomorphisms. This is stronger than $H^*(C_*) \cong H^*(D_*)$. For $\mathcal{A}$ an abelian category, the category of bounded chain complexes is the differential graded category whose objects are bounded chain complexes in $\mathcal{A}$ and whose morphisms are "pre-homomorphisms" of complexes $\mathrm{Hom}^k(A_*, B_*) = \bigoplus_i \mathrm{Hom}_{\mathcal{A}}(A_i, B_{i+k})$: it is equipped with a differential δ where

$$f \in Hom^k(A_*, B_*) \implies \delta(f) = d_B f + (-1)^{k+1} f d_A \in \mathrm{Hom}^{k+1}(A_*, B_*) \tag{2}$$

Chain maps are precisely the elements of Ker ($\delta : \mathrm{Hom}^0 \to \mathrm{Hom}^1$), and the nullhomotopic maps are elements of im ($\delta : \mathrm{Hom}^{-1} \to \mathrm{Hom}^0$), so $H^0\mathrm{Hom}(A, B)$ gives the space of chain maps up to homotopy.

Definition 2. *For* $\mathcal{A}$ *an abelian category, the* bounded derived category $D^b(\mathcal{A})$ *is the triangulated category whose objects are bounded chain complexes in* $\mathcal{A}$ *and* *whose morphisms are given by chain maps up to homotopy localizing w.r.t. quasi-isomorphisms. That is, quasi-isomorphisms are formally inverted; for any quasi-isomorphism s, we add a morphism s^{-1}. More precisely,* $\mathrm{Hom}_{D^b(\mathcal{A})}(A_*, B_*) = \{A \xleftarrow{s} A' \xrightarrow{f} B\}/\sim$ *where s is a quasi-isomorphism, f is a chain map, and $\sim$ is homotopy equivalence. We similarly define the categories $D^+(\mathcal{A}), D^-(\mathcal{A})$ of chain complexes bounded above/below.*

To explain the notion of triangulated category, recall the following:

- In the category of topological spaces (or simplicial complexes), there are no kernels and cokernels. Given a map f, however, the mapping cone $C_f = (X \times [0,1]) \sqcup Y/(x,0) \sim (x',0), (x,1) \sim f(x)$ acts as both simultaneously. There are natural maps $i : Y \to C_f$ (inclusion) and $q : C_f \to \Sigma X$ (collapsing Y), and we obtain a sequence of topological spaces

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{q} \Sigma X \to \cdots \tag{3}$$

  with compositions null-homotopic. This gives a long exact sequence of

$$H_i(X) \to H_i(Y) \to H_i(C_f) \to H_i(\Sigma X) = H_{i-1}(X) \to H_i(\Sigma Y) = H_{i-1}(Y) \tag{4}$$

- If X, Y are simplicial complexes, f a simplicial map, C_f defined analogously is a simplicial complex, with i-cells given by cones on $(i-1)$-cells of X and i-cells of Y. The boundary map is given by the matrix $\begin{pmatrix} \partial_X & 0 \\ f & \partial_Y \end{pmatrix}$.
- If A^* and B^* are complexes, f a chain map, we define $C_f = A[1] \oplus B$, i.e. $C_f^i = A^{i+1} \oplus B^i$. The boundary map is $\delta = \begin{pmatrix} \delta_A[1] & 0 \\ f & \delta_B \end{pmatrix}$. Note that, if A, B are single objects, $\mathrm{Cone}(f : A \to B)$ is just $\{0 \to A \xrightarrow{f} B \to 0\}$. We have natural chain maps $B^* \xrightarrow{i} C_f^*$ (subcomplex) and $C_f^* \xrightarrow{q} A^*[1]$ (quotient complex). As before, $A^*[1]$ is quasi-isomorphic to $\mathrm{Cone}(i : B^* \to C_f^*)$.
- Finally, in the derived category, the inversion of quasi-isomorphisms gives us *exact triangles*

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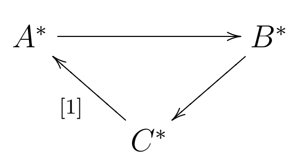

  with

$$H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots \tag{6}$$

Definition 3. *A* triangulated category *is an additive category with a shift functor* [1] *and a set of distinguished triangles satisfying various axioms:*

- $\forall X, X \xrightarrow{\text{id}} X \to 0 \to X[1]$ *is distinguished,*
- $\forall X \to Y$*, there is a distinguished triangle* $X \xrightarrow{u} Y \to Z \to X[1]$ *(*Z *is called the mapping cone of* f*).*
- *The rotation of any distinguished triangle is distinguished, i.e. for* $X \to Y \to Z \to X[1]$ *distinguised,* $Y \to Z \to X[1] \to Y[1]$ *and* $Z \to X[1] \to Y[1] \to Z[1]$ *are distinguished.*
- *Given a square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow[f']{} & Y \end{array} \tag{7}$$

there is a map between the mapping cones of f, f' *that makes everything commute in the induced map of distinguished triangles*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} \tag{8}$$

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- *Given a pair of maps* $X \xrightarrow{u} Y \xrightarrow{v} Z$*, there are maps between the mapping cones* $C_u, C_v, C_{v\circ u}$ *of* u, v*, and* $v \circ u$ *that make every commute in the induced maps of distinguished triangles.*

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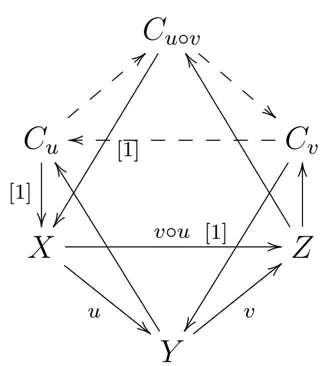

1.1. **Derived functors.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. $\mathcal{R} \subset \mathcal{A}$ is called an *adapted class of objects* for F if

- $\mathcal{R}$ is stable under direct sums,
- for C^* an acyclic complex of objects in $\mathcal{R}$, $F(C^*)$ is acyclic, and
- $\forall A \in \mathcal{A}, \exists R \in \mathcal{R}$ s.t. $0 \to A \xrightarrow{i} R$.

For instance, the set of injective objects is such an adapted class. Let $K^+(\mathcal{R})$ be the homotopy category of complexes bounded below of objects in $\mathcal{R}$. RF gives a composition $D^+(\mathcal{A}) \to K^+(\mathcal{R}) \xrightarrow{F} D^+(\mathcal{B})$, where the first map is induced by resolution by objects of R. The map $D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact, i.e. it maps exact triangles to exact triangles, and $R^iF = H^i(RF)$.

1.2. **Extensions.** Let $A, B \in \mathcal{A} \hookrightarrow D^b(\mathcal{A})$ be single object complexes concentrated in degree 0, so $B[k]$ is conentrated in degree $-k$.

Proposition 1. $\mathrm{Hom}_{D^b(\mathcal{A})}(A, B[k]) = \mathrm{Ext}^k_{\mathcal{A}}(A, B)$.

We can use this to define a product $\mathrm{Ext}^k_{\mathcal{A}}(A, B) \otimes \mathrm{Ext}^\ell_{\mathcal{A}}(B, C) \to \mathrm{Ext}^{k+\ell}_{\mathcal{A}}(A, C)$ as a composition $A \to B[k] \to C[k+\ell]$ in $D^b(\mathcal{A})$.

Example. For $k = 1$, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{10}$$

There are no chain maps, but we can invert quasi-isomorphisms. If we have an extension $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$, we have chain maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle g} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{11}$$

giving an element in $\mathrm{Hom}_{D^b(\mathcal{A})}(C, A[1]) = \mathrm{Ext}^1(C, A)$.

There are two ways to understand the above proposition. First, if $\mathcal{A}$ has enough injectives, take a resolution of B by a complex $I^0 \to I^1 \to \cdots$ quasi-isomorphic to B: the chain maps from A to I^* are, up to homotopy, isomorphic to $H^k(\mathrm{Hom}(A, I^*)) \cong \mathrm{Ext}^k(A, B)$. Second, we can check the definition of a derived functor. Given a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$, we get an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{w} A[1]$ quasi-isomorphic to a distinguished triangle with $\mathrm{Cone}(f)$.

Proposition 2. *For an exact triangle* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *and an object* E*, we have long exact sequences*

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$$\cdots \to \mathrm{Hom}(E, A[i]) \xrightarrow{f_*} \mathrm{Hom}(E, B[i]) \xrightarrow{g_*} \mathrm{Hom}(E, C[i]) \xrightarrow{h_*} \mathrm{Hom}(E, A[i+1]) \to \cdots$$

$$\cdots \to \mathrm{Hom}(A[i+1], E) \xrightarrow{h^*} \mathrm{Hom}(C[i], E) \xrightarrow{g^*} \mathrm{Hom}(B[i], E) \xrightarrow{f^*} \mathrm{Hom}(A[i], E) \to \cdots$$