Modern Examples of Complete Embedded Minimal Surfaces of Finite Total Curvature

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1. Introduction. A natural class of minimal surfaces to study are the embedded minimal surfaces — those which have no self-intersections as a subset of $\mathbb{R}^3$. Of course, every point of a surface immersed in $\mathbb{R}^3$ has a neighborhood which is embedded, so by itself this isn't a very interesting restriction. However, an embedded minimal surface produced in this way is clearly extendable and thus not complete. Because of this, when characterizing embedded minimal surfaces, it is most interesting to restrict ourselves to those that are complete. There are many known examples of complete embedded minimal surfaces. Most of these surfaces, such as Scherk's First and Second Surfaces and the Schwarz P surface, have some sort of "hole" that appears periodically, leading to complicated topology; until the eighties the only known examples of finite topology were the plane, the catenoid, and the helicoid. When looking for further examples of complete minimal surfaces of finite topology, it is convenient to make the slightly stronger assumption of finite total curvature; only the plane and the catenoid were known in this category. With these conditions, we can apply a theorem of Osserman [**10**] to show that any such surface must be homeomorphic to a compact surface with a finite number of points removed; we refer to the removed points as *ends*.

Thus every complete embedded minimal surfaces of finite total curvature can be characterized by two integers: its genus k and its number of ends r. Viewed this way, both the plane and the catenoid have genus 0 (the sphere); the plane has one end and the catenoid has two. Jorge and Meeks [**8**] made significant progress in proving nonexistence of such surfaces for a variety of k and r, but at that point the only examples known were still the plane and the catenoid. In 1982, Costa [**2**] found a complete minimal surface of genus 1 and three ends which seemed likely to be embedded, and Hoffman and Meeks proved that it was in fact embedded [**6**]. This quickly led to the further discovery of a wide assortment of related surfaces, mostly also with three ends. In this paper, we describe a family of complete embedded minimal surfaces with three ends and arbitrary positive genus which includes Costa's surface.

In Section 2 we present general background on complete minimal surfaces of total finite curvature, and in Section 3 describe the Weierstrass-Enneper Representation. In Section 4 we describe the state of knowledge prior to Costa's discovery, the story of which is told in Section 5. In Section 6, we define the generalized Costa surface of genus k and sketch proofs of its properties; this section is based heavily on [**5**]. In Section 7, we describe newer examples of complete minimal surfaces of finite topology. Finally, in Section 8 we describe the techniques and benefits of creating physical models of minimal surfaces.

2. Background. Given a surface S, we define H to be the mean of the principal (maximum and minimum) curvatures at a point, and the Gaussian curvature K to be their product. We call a surface on which H is identically zero a *minimal surface*. For the purposes of this paper, we will assume that all surfaces are regular, connected, and orientable. A *complete* surface is one where every curve $s\colon [0,\epsilon) \to S$ which is geodesic can be extended to a geodesic defined on all of $\mathbb{R}$. Alternatively, a surface is complete if every divergent curve has unbounded length, where a divergent curve on S is a differentiable map $\alpha\colon [0,\infty) \to S$ which eventually leaves every compact subset of S. A surface S is said to be of *finite topological type*, or more simply to be of *finite topology*, if it is homeomorphic to a compact surface $\overline{M}$ of genus k with r points $\{p_1,\ldots,p_r\}$ removed; let $M = \overline{M} - \{p_1,\ldots,p_r\}$. We will continue to use S, k, r, $\overline{M}$, and M in this way for the remainder of this essay. We define *total curvature* of a surface to be $C(S) = \iint K\,dA$; for a minimal surface, K is nonpositive so $C(S)$ is either a finite nonpositive number or $-\infty$. The total curvature of a minimal surface is equal to the negative of the spherical area (counting multiple covers) of the surface under the Gauss map.

3. The Weierstrass-Enneper Representation on Manifolds. Given any surface in $\mathbb{R}^3$, we can always locally introduce isothermal coordinates (u_1,u_2). We can then consider it to be a Riemann surface of one complex parameter $z = u_1 + u_2 i$. We then consider the function $g = \sigma \circ G \circ X$, where $X\colon D \to S$ is the coordinate map on a simply connected domain in $\mathbb{C}$, $G\colon S \to S^2$ is the Gauss map, and $\sigma\colon S^2 \to \mathbb{C}\cup\{\infty\}$ is stereographic projection; if S is minimal then g is meromorphic. (For the rest of this paper, we will refer to both g and G as "the Gauss map", but will continue to distinguish between the two by capitalization.) Incredibly, it turns out out that *any* meromorphic function $D \to \mathbb{C}$ is the Gauss map of some minimal surface immersed in $\mathbb{R}^3$. In fact, there are many minimal surfaces with g as its Gauss map: given any analytic function $f\colon D \to \mathbb{C}$, the map $X\colon D \to \mathbb{R}^3$ defined by

$$X(z) = \Re \int_{z_0}^{z} \Phi\, dz$$

where

$$\Phi = (\phi_1,\phi_2,\phi_3) = ((1-g^2)f, i(1+g^2)f, 2fg)$$

is a conformal minimal immersion of D into $\mathbb{R}^3$ with Gauss map g. This immersion is regular if the poles of g coincide with the zeros of f, and at each such point the order of the pole is exactly half the order of the zero.

This representation is called the Weierstrass-Enneper representation; its discovery made it very easy to find new minimal surfaces, because all one has to do is choose two simple functions and integrate. However, an complete embedded minimal surface created in this way will be topologically equivalent to the domain it is mapped from, and thus will have genus zero. So in order to use the Weierstrass-Enneper Representation to find a minimal surface of higher genus, we have to extend it to be defined on manifolds other than simply connected domains in $\mathbb{C}$. This can be done naturally by changing $f\,dz$ to a holomorphic one-form η; however, this does give us one extra condition that we need to check to ensure that the surface is well-defined.

Theorem 3-1 (Weierstrass-Enneper Representation Theorem, [**5**] Theorem 1.0). *Let M be a Riemann surface, η a holomorphic one-form on M, and $g\colon M \to \mathbb{C}\cup\{\infty\}$ a* *meromorphic function. Consider the vector-valued one-form*

$$\Phi = (\phi_1, \phi_2, \phi_3) = ((1 - g^2)\eta, i(1 + g^2)\eta, 2g\eta)$$

Then

$$X(p) = \Re \int_{p_0}^{p} \Phi$$

is a conformal minimal immersion which is well-defined on M and regular, provided that no component of Φ has a real period on M and that the poles of g coincide with the zeros of η and the order of each pole of g is half the order of the corresponding zero of η. g is the Gauss map of the surface.

4. Results Prior to Costa. The most important result for the classification of complete minimal surfaces of finite total curvature is the theorem of Osserman [**10**] which allows us to completely characterize the topology of such a surface by a nonnegative integer k and a positive integer r.

Theorem 4-1 (Osserman, [**10**] Lemma 9.5). *If $X\colon M \to S \subset \mathbb{R}^3$ defines a complete minimal surface, then there exists a compact 2-manifold $\overline{M}$ and a finite number of points p_1, p_2, …, p_r on $\overline{M}$ such that M is isometric to $\overline{M} - \{p_1, p_2, \ldots, p_r\}$. Moreover, the Gauss map $g\colon M \to \mathbb{C}$ (that is, the composition of X, the Gauss map N on the surface, and stereographic projection) extends to a meromorphic function on all of $\overline{M}$.*

Note that r must be positive because there are no compact minimal surfaces. For the rest of this paper, we will refer to the underlying compact manifold of a surface as $\overline{M}$, the punctured manifold as M, its coordinate map as X, and the surface itself as S. We refer to the image under X of a punctured neighborhood of one of the points p_i in $\overline{M}$ as an as an *end*. There are three key results in this theorem. First, in the context of complete minimal surfaces, finite total curvature implies finite topological type. (The converse is now known to be almost true: it was shown in 1997 by Collin [**1**] based on work by Meeks and Rosenberg [**9**] that every nonplanar complete embedded minimal surface of finite topological type either has finite total curvature or has exactly one end, but not both.) Because S is complete, as $q \in M$ approaches one of the deleted points p_i, $|X(q)|$ must grow without bound. Because the Gauss map g of the surface can be extended to $\overline{M}$, $\lim_{q \to M} g(X(q))$ must exist, so each end has a well-defined normal vector. (Note that it does not necessarily have an asymptotic plane — it can also continue to grow logarithmically in the direction of its normal vector.) Jorge and Meeks [**8**] proved the somewhat intuitive result that for any complete embedded minimal surface of finite total curvature in $\mathbb{R}^3$, the normal vectors on the ends of M are parallel; this makes sense, because otherwise the planes which the ends approach would intersect. By rotation, we can assume that the normal vectors on the ends all are the north and south poles of the sphere. That is, we can assume that the Gauss map g has a zero or a pole at each end of M.

It was known that for a complete minimal surface M in $\mathbb{R}^n$ of finite total curvature $C(M)$, $C(M) \leq 2\pi(\chi(M) - r)$. For the surface of genus k with r ends we have $\chi(M) = 2 - 2k - r$, so $C(M) \leq -4\pi(k + r - 1)$. Jorge and Meeks [**8**] showed that for a surface immersed in $\mathbb{R}^3$, equality holds here if and only if all of the ends are embedded. Thus, any minimal surface with total curvature $-4\pi(k + r - 1)$ is embedded outside of a compact set (the image of M minus the neighborhood of the punctures). They then used their results to show that the plane is the only embedded complete minimal surface of finite total curvature with $r = 1$, and that the only such surfaces with $k = 0$ and $r \leq 5$ are the plane and the sphere.

5. Discovery of the Costa Surface. In his 1982 thesis [**2**], Costa identified a complete minimal surface homeomorphic to the torus with three points removed. The Costa surface is an conformal embedding of the square torus — the torus defined by identifying opposite sides of the unit square in the complex plane. Any map f defined on this torus can be considered as a map defined on the entire complex plane, which is the universal covering space of the torus; the lifted map will satisfy the condition that $f(z+1) = f(z+i) = f(z)$ for all z. That is, maps defined on the square torus are equivalent to maps defined on $\mathbb{C}$ with periods 1 and i; such doubly periodic functions are called *elliptic*. Every non-constant elliptic function has at least two poles (in each unit square); it turns out that every elliptic function is a rational combination of a function with two poles called the Weierstrass $\wp$-function and its first derivative. Thus $\wp$ and $\wp'$ are natural candidates for use in constructing a minimal surface homeomorphic to a punctured torus.

Costa defined his surface by using a Weierstrass-Enneper representation of $g = \frac{2\sqrt{2\pi}\wp(\frac{1}{2})}{\wp'(\zeta)}$ and $\eta = \wp(\zeta)d\zeta$; the domain used was the unit square torus with the points at 0, $\frac{1}{2}$, and $\frac{i}{2}$ removed. Costa showed that this surface had total curvature -12π. Since $-12\pi = -4\pi(k+r-1)$, by the result of Jorge and Meeks above this implied that the ends of the Costa surface are embedded. In order to prove that this was the first discovered complete embedded minimal surface of finite total curvature and positive genus, it remained only to show that the complement of the ends in the surface (a compact set) is embedded.

However, while the Weierstrass-Enneper representation provides a simple way of demonstrating the existence of a minimal surface, it does not make it particularly easy to visualize it or discover many of its geometric properties. Specifically, the Weierstrass-Enneper representation does not provide any information about whether or not the surface is embedded. D. Hoffman and Meeks, as described in [**3**], sought out to understand the Costa surface; with help from J. Hoffman, they created computer images of the Costa surface that revealed its structure. As this was 1984, the computer work was not a simple matter of plugging some equations into a commercial software package; rather, it was a significant programming project which evolved over the course of their research. They saw that the Costa surface appears similar to the union of a catenoid with a plane through its waist circle, with two pairs of "tunnels" reminiscent of Scherk's Second Surface passing between the plane and the catenoid ends in order to make it into one coherent surface.

It appeared to Hoffman and Meeks that the Costa surface had a D_4 symmetry group, and that the section of the surface in each octant of $\mathbb{R}^3$ was a graph over a plane. Using these insights, they were able to prove that it was in fact embedded. Even more impressively, they were quickly able to generalize their proofs to demonstrate the existence of a complete embedded minimal surface of finite total curvature with three ends of any positive genus. The Weierstrass $\wp$-function and its derivative $\wp'$ used in Costa's definition satisfy the differential equation $\wp'^2 = 4\wp(\wp^2 - t)$, where t is a constant. Let us set $z = \wp(\zeta)$ and $w = \wp'(\zeta)$. Then we have $g = \frac{c}{w}$ and $\eta = zd\zeta$. We see that $dz = d\wp(\zeta) = \wp'(\zeta)d\zeta = wd\zeta$, so $\eta = \frac{z}{w}dz$. Thus, instead of using the Weierstrass-Enneper representation $g = \frac{c}{\wp'(\zeta)}$ and $\eta = \wp(\zeta)d\zeta$ defined on $\mathbb{C}$ modulo the lattice of integers, we can equivalently use $g = \frac{c}{w}$ and $\eta = \frac{z}{w}dz$ defined on the manifold $\{(z, w) \in (\mathbb{C} \cup \{\infty\}^2 | w^2 = z(z^2 - 1)\}$. Because this representation avoids the explicit use of $\wp$, it is no longer tied intrinsically to the torus and has a simple generalization to surfaces of higher genus, as we will show in the following section.

6. Definition of the Costa Surfaces of Arbitrary Genus. We will demonstrate the existence of a complete minimal surface of finite total curvature of genus k and 3 embedded ends, following the method of [**5**]. We first consider the 2-manifold defined by

$$\overline{M} = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 | w^{k+1} = z^k(z^2 - 1)\}$$

This equation is inspired by the differential equation for $\wp$ demonstrated in Section 5. We can imagine this as a 2-manifold embedded in $\mathbb{C}^2$ or $\mathbb{R}^4$ (with the caveat that z and w can both be the point ∞ on the Riemann sphere). Considered as a covering surface of the z-plane, $\overline{M}$ has a branch point at $z = 0$, 1, and -1, where the defining equation becomes $w^{k+1} = 0$; each of these branch points are thus of order k. In addition, because there are an odd number of branch points evident from the equation, it also has a branch point of order k at $z = \infty$. By the Riemann-Hurwitz formula ([**7**], Theorem 4.16.3), we know that the genus g of the surface is given by $g = 1 - n + \frac{1}{2}\sum n_i$, where n is the order w in the defining equation $0 = w^{k-1} - z^k(z^2 - 1)$ and n_i is the order of the ith branch point. So $n = k + 1$ and $\sum n_i = 4k$, so $g = 1 - (k + 1) + \frac{1}{2}4k = k$; thus $\overline{M}$ is in fact a compact surface of genus k.

We let $p_0 = (0, 0)$, $p_1 = (1, 0)$, $p_{-1} = (-1, 0)$, and $p_\infty = (\infty, \infty)$; these are the branch points of the surface and are each the unique point of $\overline{M}$ with its z-coordinate. We remove the points p_1, p_{-1}, and p_∞ from $\overline{M}$ to create M, a genus-k surface with three ends. Note that the branch point p_0 is still in M. We let $g = \frac{c}{w}$ (where c is a positive real constant that will be determined later) and $\eta = \left(\frac{z}{w}\right)^k dz = \frac{w}{z^2-1}dz$. (While the first form of η follows more naturally from Costa's original definition, the second form is easier to work with because it has no poles on M.) We see here that the pole of g does coincide with the zero of η. The minimal surface is thus defined by $X(p) = \Re \int_{p_0}^{p} \Phi$ for $p \in M$, where

$$\begin{aligned}\Phi &= (\phi_1, \phi_2, \phi_3) = ((1 - g^2)\eta, i(1 + g^2)\eta, 2g\eta) \\ &= \left(\left(\frac{z^k}{w^k} - \frac{c^2 z^k}{w^{k+2}}\right) dz, i\left(\frac{z^k}{w^k} + \frac{c^2 z^k}{w^{k+2}}\right) dz, \frac{2c}{z^2 - 1}dz\right)\end{aligned}$$

In order to show that this in fact a conformal minimal immersion, we need to show that this is independent of path chosen in the integral; that is, we must show that $\Re \int_C \Phi$ is zero for any closed path C. Studying the symmetry of M and S is very useful for this. The mappings $\kappa(z, w) = (\bar{z}, \bar{w})$ and $\lambda(z, w) = (-z, \rho^k w)$ where $\rho = e^{\pi i/(k+1)}$ are conformal mappings of order 2 and $2k + 2$ respectively which generate the dihedral group D_{2k+2}. This group leaves M invariant, and fixes both p_0 and p_∞; κ fixes p_1 and p_{-1} while λ interchanges them. These roughly correspond to the transformations of $\mathbb{R}^3$

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; L = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \; \theta = \frac{\pi}{k+1}$$

That is, we can show that $\kappa * \Phi = K\bar{\Phi}$ and $\lambda * \Phi = L\Phi$, where $\alpha * \Psi$ means the operator α should be applied to M before it is given to the one-form Ψ. This implies that S also has a D_{2k+2} symmetry group, generated by K and L.

To demonstrate that $\int \Phi$ has no real period, we need to show that $\int_C \Phi$ is purely imaginary, for C a closed curve around each of the ends and for C each element of a homology basis of $\overline{M}$. First let $\tilde{\beta}$ be the loop in the z-plane defined by $\tilde{\beta}(t) = 1 + \frac{1}{2}e^{it}$ where $t \in [0, 2\pi]$, and let β be the lift of $\tilde{\beta}$ to M, which winds once around the end p_1. Let $-\beta$ be β followed in reverse; we note that this is the same as $\bar{\beta} = \kappa \circ \beta$. Thus

$$\int_\beta \Phi = -\int_{-\beta} \Phi = -\int_{\kappa\circ\beta} \Phi = -\int_\beta \kappa * \Phi = -K\int_\beta \Phi$$

Because K preserves the first and third coordinates, this implies that $\Re\int_\beta \phi_1 = \Re\int_\beta \phi_3 = 0$; that is, $\Re\int_\beta \Phi$ is orthogonal to the (x_1, x_3)-plane. The same argument can be repeated with $\lambda^{-1}\kappa\lambda$ and $L^{-1}KL$ in place of κ and L, which will show that $\Re\int_\beta \Phi$ is orthogonal to a vertical plane making an angle of $\frac{\pi}{k+1}$ with the x_1-axis, so together this shows that $\Re\int_\beta \Phi$ is zero. So Φ has no real periods at the end p_1. A similar argument shows that Φ has no real periods at its other two ends.

It is not too hard to analyze $\phi_3 = \frac{2c\,dz}{z^2-1}$; in fact, $\phi_3 = c\,d\left(\ln\left(\frac{z-1}{z+1}\right)\right)$. Thus we can explicitly show that

$$X_3(p) = \Re\int_{p_0}^{p} \phi_3 = c\ln\left|\frac{z-1}{z+1}\right| \tag{6-1}$$

(where z is the z-coordinate of p); this shows that X_3 is independent of path. It remains to show that ϕ_1 and ϕ_2 are independent of path. It is a slightly lengthy process to show that their integrals are zero around a homology basis of $\overline{M}$ as long as c is chosen to be a specific positive real constant; see [**5**] for details.

We show that the surface is regular by showing that the functions ϕ_i never all vanish. In fact, ϕ_3 is only zero at p_0, and ϕ_1 is nonzero there. It is now simple to prove that the total curvature of the surface is $-4\pi(k+r-1) = -4\pi(k+2)$, which is to say that the Gauss map covers the Riemann sphere $k+2$ times. Ignoring the finite number of punctures, the Gauss map $g = \frac{c}{w}$ hits each point of the Riemann sphere for exactly one value of w. Therefore, the multiplicity of the covering of the sphere is equal to the number of points of M having the same value of w. For fixed w, the equation $w^{k+1} = z^k(z^2-1)$ is a polynomial of degree $k+2$ in z, so there are $k+2$ points (z, w) with any given value of w. Thus the Gauss map covers the Riemann sphere $k+2$ times, so that the total curvature of the generalized Costa surface of genus k is $-4\pi(k+2) = -4\pi(k+r-1)$. This shows both that the surface has finite total curvature and, by the result of Jorge and Meeks mentioned in Section 4, that the ends of the surface are embedded, parallel, and mutually disjoint.

The equation 6-1 for $X_3(p)$ is especially useful for understanding the geometry of S. If we let $\zeta = \frac{z-1}{z+1}$, then 6-1 becomes $X_3(p) = c\ln|\zeta|$; this tells us that the intersection of S with the plane $x_3 = t$ is the image of the circle of radius $e^{\frac{t}{c}}$ around the origin in the ζ-plane. Thus the part of M over the unit circle in the ζ-plane is sent to the x_3-plane, the part of M over the unit disk in the ζ-plane is sent to the the half of the surface below the x_3-plane, and the part of M over the rest of the ζ-plane is sent to the half of the surface above the x_3-plane. We can see directly from equation 6-1 that $X_3(p) \to +\infty$ as $p \to p_{-1}$, that $X_3(p) \to -\infty$ as $p \to p_1$, and that $X_3(p) \to 0$

as $p \to p_\infty$. This implies that the ends at p_1 and p_{-1} are catenoid-style ends whose X_3 coordinates grow logarithmically; p_∞ is a planar end asymptotic to the (x_1, x_2)-plane. Because the map between ζ and z is a Möbius transformation, circles separating $0 = \zeta(p_1)$ from $\infty = \zeta(p_{-1})$ in the ζ-plane are equivalent to circles separating p_1 from p_{-1} in the z-plane, with the exception of the unit circle in the ζ-plane, which is mapped onto the union of the imaginary line and ∞ (a circle in the Riemann sphere). Because each of the circles in the z-plane (not counting the exceptional one) separate one branch point (p_1 or p_{-1}) from the other three, they each cross over a branch cut. So their lift to M (that is, the set of all points of M whose z-coordinate lies on the circle) is a single connected simple closed curve. So the cross-section of S at $x_3 = e^{\frac{t}{c}}$ (for $t \neq 0$) is a closed curve, though we have not yet shown that it is simple. On the other hand, the imaginary line contains the branch point p_0, so its lift is the union of $k+1$ straight lines with one point in common; thus the intersection of S with the x_3-plane is $k+1$ straight lines intersecting at the origin.

We have already shown that the symmetry group of S contains K and L and the D_{2k+2} group generated by them; in fact, it turns out that this is the entire symmetry group. To describe the symmetries geometrically we use the following notation: P_0 is the (x_1, x_2)-plane and P_j is the plane containing the x_3 axis and the line in the (x_1, x_2)-plane making an angle of $(j-1)\pi/(k+1)$ with the positive x_1-axis (for $1 \leq j \leq k+1$). A *sector* of $\mathbb{R}^3$ is a component of $\mathbb{R}^3 - \cup_{j=0}^{k+1} P_j$; there are $4k+4$ sectors and D_{2k+2} acts transitively on them. We let l_j be the line in the (x_1, x_2)-plane making an angle of $\pi\frac{2j+1}{2k+2}$ with the positive x_1-axis, for $0 \leq j \leq k$; then each line l_j bisects the projection of a sector onto the (x_1, x_2)-plane. Then the orientation-preserving symmetries of S are L^{2j}, the rotation by $\frac{2\pi j}{k+1}$ radians around the x_3-axis ($0 \leq j \leq k$) and $L^{2j+1}K$, the rotation by π radians around l_j ($0 \leq j \leq k$). (Note that the identity is the first type with $j = 0$.) The orientation-reversing symmetries are $L^{2(j-1)}K$, the reflection through P_j ($1 \leq j \leq k+1$), and L^{2j+1}, the rotation by $\frac{\pi(2j+1)}{k+1}$ radians around the x_3-axis followed by reflection in P_0. Each catenoidal end is made up of pieces of $2k+2$ of the sectors; both the middle plane and a neighborhood of the origin $0 = X(p_0)$ are made up of all $4k+4$ sectors. As mentioned above, the intersection of S with the plane $x_3 = 0$ is the image of $k+1$ lines with one point of each identified; in fact, it is equal to the union of the $k+1$ lines l_j. The intersection of S with any other plane $x_3 = t$ is a simple closed curve.

This symmetry structure can be used to show that each sector is in fact a graph over an appropriately chosen plane. Thus S is embedded. So S is a complete embedded minimal surface of finite total curvature, genus k, and 3 ends, and it has a symmetry group D_{2k+2} with $4(k+1)$ elements. In fact, Hoffman and Meeks showed that any such surface with at least $4(k+1)$ elements in its symmetry group is in fact similar to this generalized Costa surface.

Previously, the Costa surface was described as appear like the union of a catenoid and a plane with tunnels that appear like Scherk's Second Surface. These similarities are reflected by the limiting behavior of the surfaces, as described in [**4**]. If S_k is the generalized Costa surface of genus k, then the sequence S_k has a subsequence that converges as $k \to \infty$ to the union of the plane and the catenoid. In addition, if $\hat{S}_k$ is a normalized version of S_k such that the maximum value of K is equal to 1 and is at the origin, then the sequence $\hat{S}_k$ has a subsequence that converges to Scherk's Second Surface.

7. Other Complete Minimal Surfaces of Finite Topology. Hoffman and Meeks showed, as described in [**4**], that each surface S_k lies in a one-parameter family $S_{k,x}$ ($x \geq 1$) of embedded minimal surfaces of genus k and finite total curvature; $S_k = S_{k,1}$. The surfaces with $x > 1$ have all three ends of "catenoid" type (that is, growing logarithmically) and a symmetry group generated by k vertical reflection planes. Intuitively, this can be imagined as taking the "tunnels" that flow into the bottom catenoid half and pushing down on them until they are flowing *separately* into the bottom catenoidal plane.

Several other families of complete embedded minimal surfaces of finite total curvature are mentioned in [**4**], although for most of them the surface is not embedded for all values of x. These include four-ended examples with $c \geq 2$ vertical planes of symmetry, one horizontal plane of symmetry, genus $2(c-1)$, two flat ends and two catenoid ends; a one-parameter family deforming these surfaces through surfaces with four catenoid ends; a variant of the same surfaces with symmetric tunnels through their waist plane; and a series of five-ended examples. In addition, Hoffman, Karcher, and Wei constructed a new complete embedded minimal surface of finite topology with one end (and thus infinite total curvature); this is called the "helicoid with a handle".

8. Physical Modeling. In [**3**], Hoffman waxes eloquently about the value of computer graphics in his study of minimal surfaces. While I could never have even imagined the Costa surfaces without viewing computer graphics, and while the graphics I could find with a few seconds on Google in 2004 are vastly superior to what Hoffman could create with much labor in 1984, I still found it difficult to truly imagine the Costa surface just by looking at computer images. I decided to make physical models of the Costa surfaces. After experimenting with several different materials, I settled on Crayola Model Magic, an air-drying, flexible artificial clay-like product.

My first attempt at creating a Costa surface started by creating the two catenoid-like ends and trying to merge them together to form the central plane. After much trial and error, I ended up attaching two tubes to start to form the tunnels to each side; a little more massaging yielded a passable model of the Costa surface. For my second attempt, I wanted to highlight the symmetry properties of the surface, and specifically the eight octants that are mapped onto each other transitively by the symmetry group of the surface. I made four green and four white wedge-shaped pieces; the long edges formed the catenoid ends (two green and two white pieces each), while the shorter end split in half, with one tab going in to meet with the other seven pieces at the origin and the other tab spreading out to form the waist plane. This construction made it clear how the plane and the neighborhood of the origin were all composed of all eight pieces, and also showed (because each colored piece should lie entirely in one octant) where the directions of maximum and minimum height on the wait plane are. I plan to create similar models of the higher genus Costa surfaces as well.

While the computer graphics were a good starting place, the ability to physically hold a Costa surface in my hand allowed several of its properties to be immediately visible. For example, if one colors the two sides of the surface distinct colors, the fact that the Gauss map covers the Riemann sphere three times is clear: you can see exactly three places where either color faces in any given direction (for example, one color faces up at the origin and at the limits of the top of the upper catenoid and the bottom of the lower catenoid). The fact that the surface has genus one can be seen by trying to link one's fingers around the surface (without crossing any of the three infinite planes or putting the fingers into the same hole from two different directions) like holding a donut; this can be done exactly once, so the surface has genus 1. A similar operation can be performed on the higher genus surfaces.

References

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