Characterizations of Complete Embedded Minimal Surfaces: Finite Curvature, Finite Topology, and Foliations

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In classifying minimal surfaces, we start by requiring them to be complete and embedded. Having an embedding lets us work in real-space, and requiring completeness lets us focus on whole, nonextendable surfaces instead of all possible subsets of these surfaces. From here we can look at the class of surfaces of finite topological type, and the subclass of those surfaces with finite curvature. As conditions on curvature or topology can be strong enough to uniquely determine a surface, fully understanding the relationship between finite topology and finite curvature is useful in classifying complete, embedded minimal surfaces.

I begin by introducing the catenoid and mentioning how it can be characterized by various conditions on its topology and curvature. I then explore the relationship between finite curvature and finite topology, and outline the recent result that finite curvature and finite topology are equivalent when a complete, embedded, minimal surface has at least two ends. Finally, I state two beautiful theorems of Schiffman which give us some sufficient conditions for showing that a surface is foliated by convex curves, and prove one of these theorems.

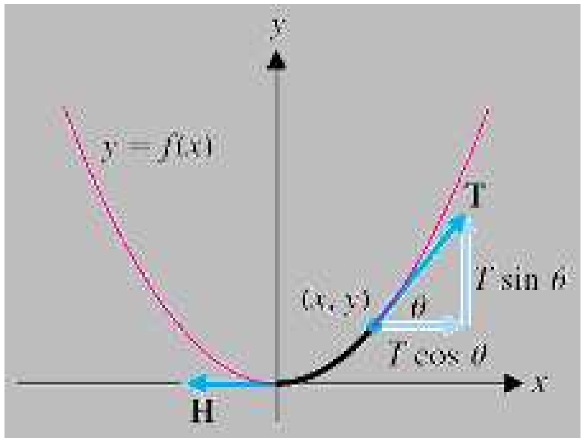

Figure 1: A hanging cable with the forces acting on it shown. From http://www.mhhe.com/math/calc/smithminton2e/cd/folder_structure/text/chap06/section09.htm

1 Minimal Surfaces and the Catenoid

1.1 The Catenoid

The **catenoid** is a surface obtained by rotating a catenary around the z-axis. It is the only minimal surface of revolution, and can also be characterized uniquely by other geometric or topological properties, like being the only minimal surface foliated by Jordan curves. I will use the catenoid as an example of showing how we can characterize a minimal surface through purely geometric and topological properties. First I derive the equation for the catenary from the physical description that the catenary is the curve formed by a cable hangle from two poles at equal height.

Let our hanging cable be be given by a function $f(t)$, and assume the cable has linear density ρ. The lowest point of the catenary is set to be the origin. Denote the horizontal tension pulling the cable, at the origin, to the left by H. Since the cable is not moving, all forces must be in equilibrium (see Figure 1.) The horizontal equilibrium yields $H = Tcos\theta$, and the vertical equilibrium yields $\rho \cdot \int \sqrt{1 + f'(t)^2}dt = Tsin\theta$ (noting that the left-hand side of this equation is the curve's weight, which is the density of the curve times its length.)

Multiplying the equation for horizontal equilibrium by $tan\theta$, we get $Htan\theta = \rho \cdot \int \sqrt{1 + f'(t)^2}dt$.

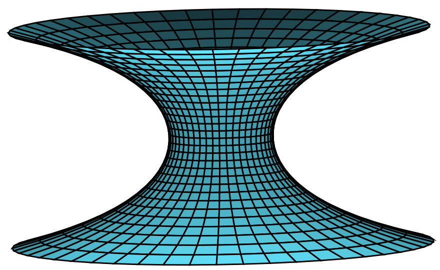

Figure 2: The catenoid. From http://www.gang.umass.edu/gallery/min/mingallery0101.html

The above force diagram shows us that $tan\theta$ is $f'(x)$, so $Hf'(x) = \rho \cdot \int \sqrt{1+f'(t)^2}dt$. Differentiating yields $Hf''(x) = \rho \cdot \sqrt{1+f'(x)^2}$, which can be rewritten as $\int(f''(x)/(1+f'(x)^2))dx = \int(\rho/H)dx$. Thus $sinh^{-}1(f'(x)) = (\rho/H)x + c$.

Having set $x = 0$ as a minimum for the catenary, $f'(0) = 0$ and thus $c = 0$. Setting $H/\rho = a$, we conclude that $f(x) = acosh(x/a)$, which is the general equation for a catenary.

Rotating this curve (with $a = 1$) around the z-axis provides a parameterization of the catenoid $\mathbf{x}$ from a domain $U \subset \mathbb{R}^2$ to $\mathbb{R}^3$: $\mathbf{x}(u, v) = (bcoshvcosu, bcoshvsinu, bv)$

1.2 Minimal Surfaces of Revolution

A surface is minimal if the mean curvature $H = (k_1 + k_2)/2$ is zero everywhere, where k_1 and k_2 are the principal curvatures. A surface with an isothermal parameterization can be shown to be a minimal surface if and only if its coordinate functions are harmonic. Using the given parameterization, this criterionshows that the catenoid is minimal. Furthermore, the catenoid can be shown to be the only minimal surface of revolution. I show this here by using Stoke's flux equation.

Let S be a minimal surface given by an isometric immersion $X : S \rightarrow \mathbb{R}^3$. Stoke's theorem states that for any C^2 function $f : S \rightarrow \mathbb{R}^n$,

$$\int_S \Delta f dA = \int_{\delta S} df(\vec{n}) ds \tag{1}$$

where $\vec{n}$ is the outward pointing conormal to S

Setting $n^* = dX(\vec{n})$, the image of the outward-pointing conormal, we get

$$\int_S \Delta f dA = \int_{\delta S} n^* ds. \tag{2}$$

As S is minimal, $\Delta X = 0$ and thus $\int_{\delta S} n^* = 0$. Equivalently, $\int_{\delta S} n^* \cdot \vec{v} ds = 0$.

This last equation can be interpreted as looking at the flux through the boundary δS with constant velocity vector $\vec{v}$.

Now take a minimal surface of revolution, obtained by rotating a curve $r(t)$ around the z-axis. Set S to be the subset bounded by the z-planes $z = t_1$ and $z = t_2$ (for $t_1 > t_2$) and define $\alpha(t)$ to be the angle between the tangent plane to S at $z = t$ and the z-plane $z = t$.

The boundary of S is two circles in the $z = t_1$ and $z = t_2$ planes. By using the flux equation for a minimal surface with $\vec{v} = (0, 0, 1)$ to compute the flux in the z-direction through S, we find

$$\int_{S \cap \{z=t_1\}} sin\alpha(t_1) ds = \int_{S \cap z = t_2} sin\alpha(t_2) ds. \tag{3}$$

and so $2\pi r(t_1) sin\alpha(t_1) = 2\pi r(t_2) sin\alpha(t_2)$. Since t_1 and t_2 are arbitrary, we can write $r(t) sin\alpha(t) = c$. This gives us the ODE:

$$r(t)\frac{1}{\sqrt{1+r'(t)^2}} = c. \tag{4}$$

This is solved by

$$r(t) = \frac{cosh(At+B)}{A}, \tag{5}$$

for real numbers $A \neq 0$ and B, which is the general equation for a catenoid.

1.3 Basic Properties about the Catenoid

We have just seen that requiring a minimal surface to be a surface of revolution forces it to be the catenoid. Similarly, specifying a minimal surface's total curvature and/or topological type appropriately will again ensure that the surface is the catenoid. For example:

* The only properly embedded, non-planar, minimal surface with genus zero is the catenoid. [**?**]

* The only properly embedded minimal surface with total curvature -4π is the catenoid. [**?**]

* The only properly embedded minimal surface with two ends and finite curvature is the catenoid (loosely, an **end** is an unbounded portion of the catenoid that lies outside of a compact subset.) [**?**]

* The only properly embedded minimal surface which forms a Jordan curve when intersected with any z-plane is the catenoid (this is the **generalized Collin-Nitsche theorem**, which will be discussed shortly.)

These facts illustrate the power of specifying topological or geometric conditions of a surface, as a few given parameters on a minimal surface can uniquely determine the entire surface. This makes notions of topology and curvature very useful when attempting to classify all complete, embedded, minimal surfaces.

2 Finite Topology and Finite Curvature

In this section, I will discuss the notions of finite topology and finite curvature, and discuss both how they are related to one another as well as how they connect to the classical problem of the Nitsche Conjecture. I begin by defining these notions and then describe a few surfaces with and without finite curvature.

A surface has **finite topology** if it is homeomorphic to a compact surface with a finite number of points removed. An **end** of a surface is the image of a small disk neighborhood around a removed point of the compact surface under this homeomorphism. A surface has **finite curvature** if $\int_S KdA$ is finite (equivalently, if the surface area of the image of the Gauss map is finite.)

As a basic example, let's look at the catenoid, which has both finite topology and finite curvature. The catenoid's image under the Gauss map is all of the sphere except for its north and south poles, and because the Gauss map is 1-to-1 for the catenoid, this shows that it has finite topology. The image of a small disk around the north or south pole of the sphere under the inverse of the Gauss map corresponds to the top and bottom of the catenoid, so the catenoid has two ends, each going off to infinity. The catenoid has finite curvature since the Gauss map covers the sphere, with the exception of its poles, once and so the total curvature is -4π.

While the use of the Gauss map in the example of the catenoid shows a strong relationship between notions of finite curvature and finite topology, it's also easy to see that the two properties are not equivalent. Consider the example of the helicoid. The helicoid is obtained by taking a helix and ruling it along the xy-plane. We can parameterize it by $\mathbf{x}(u, v) = (vsinu, vcosu, u)$.

The helicoid is simply-connected, as we can deformation retract the helicoid onto a helix, and then deformation retract that to a point. That means that the helicoid has the same topological type as a sphere minus a point, and thus has finite topology. However, as the helicoid is periodic and winds around the z-axis infinitely many times, it doesn't have finite curvature, as the Gauss map covers the sphere (minus its poles) infinitely many times.

A result of Osserman states that a complete minimal surface of finite curvature is conformally equivalent to a compact Riemann surface with a finite number of points missing (conformally equivalent means that

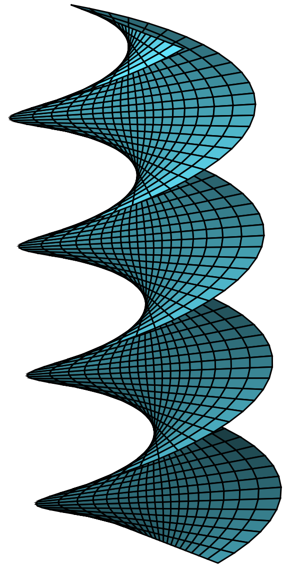

Figure 3: The helicoid. From http://www.gang.umass.edu/gallery/min/mingallery0101.html

the homeomorphism between the two spaces is conformal.)

Thus finite curvature implies finite topology, but without additional conditions, as the example of the helicoid shows, the converse is not true. Many additional requirements for a surface have been suggested, based on the approach of finding a topological or geometric condition that ruled out the helicoid (pun intended) – the only known counterexample to the equivalence of finite curvature and finite topology. In [1], published in 1985, Hoffman and Meeks explored the conjecture that all complete, embedded, minimal surfaces that are not simply-connected have finite topology if and only if they have finite curvature.

A counter-example to this conjecture was produced in 1992 by Hoffman, Karchner, and Wei [**?**]. This was the genus-one helicoid. The genus-one helicoid looks like the original helicoid on all turns but one, and this turn contains a torus-like hole.

The genus-one helicoid is a counterexample to the conjecture that finite topology and finite curvature are equivalent even when restricting to the class of non-simply connected minimal embedded surfaces, as the genus-one helicoid is not simply connected and has finite topology and infinite curvature.

Both the ordinary helicoid and the genus-one helicoid have just one end. Thus, looking at surfaces with

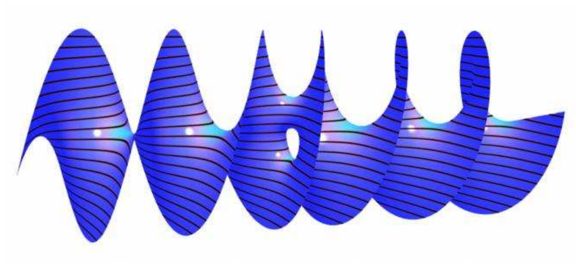

Figure 4: The genus one helicoid. From http://www.math.uni-bonn.de/peoplse/weber/research/minimal/notes/helicoid/

at least two ends would eliminate both counterexamples, and in fact it turns out that finite topology is indeed equivalent to finite curvature for surfaces with at least two ends. I will now go through the major facts used in this theorem, but will only state them in order to show how they connect to the equivalence of finite topology and finite curvature for minimal surfaces with at least two ends. The main idea is to show that this equivalence can be reduced to the recently proven **generalized Nitsche Conjecture.**

2.1 The Nitsche Conjecture

In 1962, Nitsche showed that the only minimal annulus that meets every z-plane in a star-shaped curve is the catenoid. (A *star-shaped curve* is one that bounds a star-shaped set. A *star-shaped set* is a set S that has a point $a \in S$ such that for all other points $x \in S$, the line from a to x is contained in S.) The Nitsche Conjecture asked if this was also true for Jordan curves instead of star-shaped curves.

In light of a theorem of Schoen's, which says that the only complete embedded minimal surface with two ends and finite total curvature is the catenoid, we can generalize the Nitsche Conjecture.

Theorem (Collin-Nitsche Theorem): If A is a minimal annulus whose intersection with every z-plane is a Jordan curve, then A must have *finite curvature.*

Since a minimal annulus has two ends, Schoen's theorem reduces this to the original Nitsche conjecture. This had been a conjecture until proven by Collin in 1997 [**?**]. The explicit mention of finite curvature in this generalizations establishes a link to showing under what conditions finite curvature and finite topology are equivalent.

Using a theorem of Meeks and Rosenberg, [?] we can reduce the statement that for surfaces with at least two ends, finite topology implies finite curvature.

Theorem (Meeks/Rosenberg): Let $S \subset \mathbb{R}^3$ be a properly embedded minimal surface with more than one end. If A is an annular end of M, then it is conformally diffeomorphic to the punctured disk.

From looking at the images of rings on the punctured disk, we get

Corollary: If $S \subset \mathbb{R}^3$ is a properly embedded minimal annulus, then after a rotation of $\mathbb{R}^3, M$ intersects every horizontal plane in a single closed curve.

Thus if S is a properly embedded minimal surface with at least two ends, each of its ends are annular (being homeomorphic to a punctured disk by definition) and so satisfy the hypothesis of the Collin-Nitsche theorem, meaning they each have finite curvature. Assuming finite topology, we have finitely many ends each with finite curvature, so the total curvature of surface must be finite.

3 Foliated Surfaces and Schiffman's Theorems

The prior discussion shows that looking at if and how a surface is foliated – what each intersection of the surface with the z-plane is, for instance – can be a useful tool in verifying various geometric properties about surfaces. We first take a look at the simplest example of a minimal foliated surface, the catenoid, and then state Schiffman's theorems and prove one of them. These theorems tell us that a minimally immersed annulus is foliated by convex curves or circles when its boundary consists of convex curves or circles, respectively.

3.1 Foliation by Circles and the Catenoid

The catenoid is clearly foliated by circles: as it is a surface of revolution, every cross-section we look at along the z-axis will be a circle. We can also show, through the Gauss-Bonnet theorem and a characterization theorem for the catenoid, that any complete minimal annulus foliated by circles must be the catenoid.

The Gauss-Bonnet theorem tells us that for a region R of an oriented surface S,

$$\int_R K dA = 2\pi\chi(R) + \sum_{i=0} \beta_i + \int_{\delta R} \kappa_g ds$$

where $\chi(R)$ is the Euler characteristic of the region R, β_i is the i-th exterior angle of the corners made by the boundary δR and κ_g is the geodesic curvature of δR. Often, the Gauss-Bonnet Theorem is presented for surfaces without boundary, which gives us the (even more) elegant $\int_S K dA = 2\pi\chi(S)$. This theorem's power lies in relating topological information of a surface to geometric information; the simple, immediate implication from the Gauss-Bonnet Theorem that the total curvature of a surface is invariant under homotopic deformations is a pretty startling result. (For more on the Gauss-Bonnet Theorem, see [doCarmo], pages 264 - 283.)

Now take A to be a complete minimal annulus, fibred by circles, and R to be some compact subset of A. As R is compact and bounded by parallel circles, it is an annulus. An annulus has Euler charateristic zero. This can be seen by noting that an annulus is topologically a sphere minus two points. A standard triangulation of the sphere is a tetrahedron, so applying the Euler characteristic formula to the tetrahedron with two vertices removed, we get $\chi(R) = vertices - edges + faces = (4-2) + 6 + 4 = 0$. The $\sum_i \beta_i$ terms amount to zero because the boundary of R is a pair of circles, neither of which have any corners. Finally, the geodesic curvature of each circle contributes $+/-2\pi$, so we get that $|\int_R K dA| \leq 4\pi$

Now, it's been shown that the total curvature of a complete embedded minimal surface (and while we did not explicitly state that A is embedded, it is because it is fibred by circles) must be a multiple of 4 π (for a surface S with genus k and r ends, $\int K dA = -4\pi(r+k-1)$.) This means our surface could have total curvature of $0, 4\pi$, or -4π. Our surface is minimal, so it has non-positive Gaussian curvature everywhere, meaning its total curvature is either 0 or 4π. Having total curvature zero would imply A was the plane, but it can't be because we are looking at a surface fibred by circles. Thus, the total curvature of A is -4π.

Using the characterization, also proved by Schoen [**?**], that the only complete and not simply connected surface with total curvature -4π is the catenoid, we have shown that a minimal annulus fibred by circles must be the catenoid.

This is also another proof that the only minimal surface of rotation must be the catenoid.

3.2 Schiffman's Theorems.

Schiffman's theorems, from [**?**] tell us about the fibres of a minimally immersed annulus, given some basic information about the boundary of the annulus. They are best described by their statements.

Let A be a minimally immersed annulus, and γ be the boundary of the annulus.

Theorem [Schiffman]: If γ is a pair of smooth convex Jordan curves, sitting in parallel planes, then the intersection of A with any plane parallel to and between those planes is a convex curve.

Additionally, Schiffman proved that for the same situation as above,

Theorem [Schiffman]: If γ is a pair of circles, then every intersection of A with a plane is a circle.

Schiffman's first theorem also implies that the surface A is actually embedded, as we have a foliation for it.

We now prove Schiffman's first theorem. First, we recall the maximum principle: a non-constant harmonic function u defined on an open region Ω attains its maximum and minimum on the boundary of Ω. Thus, a harmonic function is uniquely determined by its values on the boundary (for two harmonic functions f and g that agree on the boundary of Ω, their difference $f - g$ must be zero everywhere as the maximum and minimum of $f - g$ are zero.)

Also, we will use the notation that P_t denotes the z-plane where $z = t$.

We start our proof by noting that, through uniformization, any A is conformally equivalent to an annulus $D(r) = \{z \in \mathbb{C} | 1 \leq |z| \leq r\}$ for a unique $r > 1$. We can assume that our annulus A is the image of some conformal minimal immersion $X : D(r) \to \mathbb{R}^3$ where, using polar coordinates for $D(r)$, $X(re^{i\theta}) \subset P_{ln(r)}$ (and so $X(e^{i\theta}) \subset P_0$, the xy-plane.) Since we are looking at a minimal immersion, we know that the third coordinate function X_3 is harmonic, and by the construction of X, X_3 agrees with $ln|z|$ on the boundary of A. As $ln|z|$ is also harmonic, we can apply the maximum principle to conclude that $X_3 = ln|z|$.

Now, having written down X_3, we know that the gradient of X_3 is never zero on $D(r)$, and so the tangent plane to the minimal annulus A is never horizontal. That means that the function $g(z)$, the composition of the Gauss map with stereographic projection, is never zero or ∞, and so $\phi = arg\ g(z)(mod 2\pi)$ is well-defined. Observe that the horizontal projection of the normal to A at a point p in the intersection $P_t \cap A$ is orthogonal to the plane curve $P_t \cap A$ at p and never zero. Thus, ϕ can also be considered as the angle of the normal to the intersection $P_t \cap A$.

The convexity of a curve $P_{ln(c)} \cap A$ (equivalently, the image of X on $ce^{i\theta}$) is equivalent to $\frac{\delta}{\delta\theta}\phi(ce^{i\theta})$ never changing sign. The function g is analytic, and so $\frac{\delta}{\delta\theta}(argg)$ is harmonic, since this is true for analytic functions in general. As both boundary curves of A have the same orientation, $\frac{\delta}{\delta\theta}\phi$ has fixed sign on the boundary δA. Thus, the max and min of ϕ both have the same sign, and ϕ can never be zero in the interior. This means that our level curves of A are all convex, and so we have shown our result.

For a proof of Schiffman's second theorem, see his paper [**?**]

References