

# On the Second Eigenvalue and Random Walks in Random $d$ -Regular Graphs

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## Abstract

The main goal of this paper is to estimate the magnitude of the second largest eigenvalue in absolute value,  $\lambda_2$ , of (the adjacency matrix of) a random  $d$ -regular graph,  $G$ . In order to do so, we study the probability that a random walk on a random graph returns to its originating vertex at the  $k$ -th step, for various values of  $k$ . Our main theorem about eigenvalues is that

$$E \{ |\lambda_2(G)|^m \} \leq \left( 2\sqrt{2d-1} \left( 1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right) \right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right) \right)^m$$

for any  $m \leq 2 \lfloor \log n \lfloor \sqrt{2d-1}/2 \rfloor / \log d \rfloor$ , where  $E \{ \}$  denotes the expected value over a certain probability space of  $2d$ -regular graphs. It follows, for example, that for fixed  $d$  the second eigenvalue's magnitude is no more than  $2\sqrt{2d-1} + 2 \log d + C'$  with probability  $1 - n^{-C}$  for constants  $C$  and  $C'$  for sufficiently large  $n$ .

## 1 Introduction

Let  $G$  be a  $d$ -regular (i.e. each vertex has degree  $d$ ) undirected graph, and let  $A$  be its adjacency matrix. We allow  $G$  to have multiple edges, in which case the corresponding entry of  $A$  is that multiplicity; we allow (possibly multiple) self-loops, in which case the corresponding entry of  $A$  is twice the multiplicity of the self-loop. Since  $A$  is symmetric it is diagonalizable with (an orthogonal set of eigenvectors and with) real eigenvalues. It is easy to see that  $d$  is the largest eigenvalue in absolute value (with the all 1's eigenvector). Let  $\lambda_2 = \lambda_2(G)$  be the next largest eigenvalue in absolute value.

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The magnitude of  $|\lambda_2|$  has received much attention in the literature: for example, it is useful to give some estimate of the expansion properties of  $G$  (see [Tan84],[Alo86]) or the rate of convergence of the Markov process on  $G$  (with probabilities  $1/d$  at each edge) to the stable distribution (see [BS87] for more on this and references). The smaller  $|\lambda|$  the better, usually; intuitively it measures the difference (in  $L_2$  operator norm) between  $A$  and the all  $d/n$ 's matrix, and more generally that between  $A^m$  and the all  $d^m/n$ 's matrix.

The main goal of this paper is to estimate the magnitude of  $|\lambda_2|$  of a random  $d$ -regular graph. We view  $d$  as fixed and  $n \rightarrow \infty$ ; if  $d$  is sufficiently large with respect to  $n$ , it is well known that  $|\lambda_2| = O(\sqrt{d})$  with high probability. In order to estimate  $|\lambda_2|$ , we study the probability that a random walk on a random graph returns to its originating vertex at the  $k$ -th step, for various values of  $k$  (for us,  $k \approx \log n$  or  $(\log n)^2$  are important). We will actually work with  $2d$ -regular graphs, drawn from a probability space  $\mathcal{G}_{n,2d}$  to be defined shortly. Our main theorem is

**Theorem A** *For  $G \in \mathcal{G}_{n,2d}$  we have*

$$E\{|\lambda_2(G)|\} \leq 2\sqrt{2d-1} \left( 1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right) \right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right)$$

(with an absolute constant in the  $O(\cdot)$  notation), where  $E\{\cdot\}$  denotes the expected value over  $\mathcal{G}_{n,2d}$  and where the logarithm is taken base  $e$ ; more generally we have

$$E\{|\lambda_2(G)|^m\} \leq \left( 2\sqrt{2d-1} \left( 1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right) \right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right) \right)^m$$

for any  $m \leq 2\lfloor \log n \lfloor \sqrt{2d-1}/2 \rfloor / \log d \rfloor$ .

All logarithms in this paper are taken base  $e$ . A corollary of this theorem is

**Theorem B** *For any  $\beta > 1$  we have*

$$|\lambda_2(G)| \geq \left( 2\sqrt{2d-1} \left( 1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right) \right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right) \right)$$

with probability

$$\leq \frac{2}{n^{2\lfloor \sqrt{2d-1}/2 \rfloor \log \beta / \log d}}.$$

In particular, we have that for any  $C$  there is a  $C'$  such that for any  $d$  there is an  $n_0$  such that if  $n \geq n_0$  then a  $G \in \mathcal{G}_{n,2d}$  has

$$|\lambda_2(G)| \leq 2\sqrt{2d-1} + 2 \log d + C'$$

with probability at least  $1 - n^{-C}$ .

One can compare this to the well-known lower bound

$$|\lambda_2(G)| \geq 2\sqrt{2d-1} + O\left(\frac{1}{\log_d n}\right)$$

for any  $2d$ -regular graph,  $G$ , with, say,  $d \leq \sqrt{n}$  (the bound as stated is easy, and has been discovered by many people; for a proof see section 3; a similar result also holds for the second largest *positive* eigenvector, see [McK81] and Alon and Boppana in [Alo86]). Previously Broder and Shamir, in [BS87] had shown that for fixed  $d$ ,

$$E\{|\lambda_2(G)|\} \leq 2^{5/4}d^{3/4}(1 + \epsilon + o(1))$$

as  $n \rightarrow \infty$ , and the analog to the second equation in theorem A with  $m \leq (2 - \epsilon') \log_{d/2} n$ . Our method refines Broder and Shamir's estimate of the probability that a random walk in a random graph returns to its original vertex after some specified number of steps; this therefore improves the estimates on  $|\lambda_2(G)|$ . Also, Kahn and Szemerédi, in [KS], have independently given a much different approach, which modifies the standard counting argument (the standard counting argument gives no interesting bound), to get  $|\lambda_2(G)| = O(\sqrt{d})$  with probability  $\rightarrow 1$  as  $n \rightarrow \infty$ . The following conjecture of Alon is still unresolved, which states that for *fixed*  $d$ , as  $n \rightarrow \infty$ ,  $|\lambda_2| \leq 2\sqrt{2d-1} + \epsilon$  with high probability for any *epsilon*  $> 0$ .

The standard approach to estimating eigenvalues is to estimate the trace of a high power of the adjacency matrix, i.e. estimate the probability that a random walk of a given length will return to the starting vertex on a random graph. Since

$$\text{Trace}(A^k) = \sum_{i=1}^n \lambda_i^k$$

for a matrix,  $A$ , with eigenvalues  $\lambda_1, \dots, \lambda_n$ , this gives bounds on the magnitude of the largest unknown eigenvalue. This method was used by Wigner in [Wig55] to find the limiting distribution (the "semi-circle law") of the eigenvalues in a random  $n \times n$  symmetric matrix with independently chosen entries drawn from a fixed distribution, as  $n \rightarrow \infty$ ; implicit in his paper is that when the expected value of the entry is 0 the largest (second largest otherwise) eigenvalue is of magnitude roughly proportional to  $\sqrt{n}$ . His approach to estimating the trace of high powers of the matrix, and subsequent refinements (see [Gem80], [FK81] for example) do not work when the distribution of the entries varies too much with  $n$ , in particular with matrices of graphs whose edge density decreases too fast with  $n$ . In [McK81], McKay derived a semi-circle law for  $d$  regular graphs, i.e. the limiting distribution of the eigenvalues, but the method did not give bounds on  $|\lambda_2|$  (the limiting distribution only gives information on  $n - o(n)$  of the eigenvalues). In [Mar87] and [LPS86], Margulis and (independently) Lubotzky, Phillips, and Sarnak constructed a class of graphs with  $|\lambda_2| \leq 2\sqrt{d-1}$ . In [BS87], Broder and Shamir gave a method of estimating traces for random graphs which gave  $|\lambda_2| = O(d^{3/4})$  with high probability. In this paper we use Broder and Shamir's framework for trace estimates; we also make use of the cosine substitution

and resulting formulas as in [LPS86], and Alon's theorem that says expansion implies "small" second eigenvalue, in [Alo86]. We begin by reviewing the terminology and results in [BS87].

The model we take is for a random  $2d$ -regular graph on  $n$  vertices. Choose independently  $d$  permutations of the numbers from 1 to  $n$ , each permutation equally likely. We construct a directed graph,  $G = (V, E)$  with vertex set  $V = \{1, \dots, n\}$  and edges

$$E = \left\{ (i, \pi_j(i)), (i, \pi_j^{-1}(i)) \mid j = 1, \dots, d \quad i = 1, \dots, n \right\}.$$

$G$  is therefore a graph with possibly multiple edges and self-loops, and which is symmetric in the sense that for every edge in the graph the reverse edge is also in the graph. Although  $G$  is directed, we can view it as an undirected graph by replacing each pair of edges  $(i, \pi(i)), (\pi(i), i)$  with one undirected edge. We denote this probability space of random graphs  $\mathcal{G}_{n,2d}$ .

Let  $A$  be the adjacency matrix of a graph  $G \in \mathcal{G}_{n,2d}$ . Let  $\Pi$  be the alphabet of symbols

$$\Pi = \{\pi_1, \pi_1^{-1}, \pi_2, \dots, \pi_d^{-1}\}.$$

We think of  $\pi_1, \dots, \pi_d$  as being the  $d$  permutations from which  $G$  was constructed, and any word,  $w = \sigma_1 \dots \sigma_k$  of  $\Pi^*$  as the permutation which is the composition of the permutations  $\sigma_1, \dots, \sigma_k$ . Let

$$i \xrightarrow{w} j \equiv \begin{cases} 1 & \text{if } w(i) = j \\ 0 & \text{otherwise} \end{cases}.$$

We have that the  $i, j$ -th entry of  $A^k$  is

$$\sum_{w \in \Pi^k} i \xrightarrow{w} j.$$

We wish to estimate the expected value of the above sum for  $i = j$ . In evaluating  $i \xrightarrow{w} i$ , we can cancel in  $w$  any consecutive pair of letters  $\pi\pi^{-1}$  with  $\pi \in \Pi$ . We say that a word  $w \in \Pi^k$  is *irreducible* if  $w$  has no pair of consecutive letters of the form  $\pi\pi^{-1}$ . We denote the set of irreducible words of length  $k$  by  $\text{Irred}_k$ . Clearly  $\text{Irred}_k$  has size  $2d(2d-1)^{k-1}$ . It turns out that to estimate the second eigenvalue it suffices to get an estimate of the form

$$\sum_{w \in \text{Irred}_k} i \xrightarrow{w} i = 2d(2d-1)^{k-1} \frac{1}{n} + \text{error} \quad (1.1)$$

for all fixed  $i$  with a small error term; to estimate the expected second eigenvalue it suffices to estimate the expected value of the left-hand-side of equation 1.1. Intuitively, for  $k \geq 1$  we expect that words in  $\text{Irred}_k$  will send a fixed vertex to each vertex with more or less equal probability,  $1/n$ . The corresponding eigenvalue estimate is roughly  $O(n^{1/k}(\sqrt{d} + \text{error}^{1/k}))$ . In [LPS86], the error terms for a specific family of graphs

were each bounded by  $(cd)^{k/2}$  for all  $k$ , yielding  $|\lambda_2| \leq 2\sqrt{d-1}$ . In [BS87] the error term for the expected value over  $\mathcal{G}_{n,2d}$  was bounded as

$$O\left(\frac{k^4}{n^2} 2d(2d-1)^{k-1} + \frac{k}{n} (cd)^{k/2}\right),$$

which for  $k = 2\log_d n$  is  $\leq (cd)^{3k/4}/n$ .

In this paper we obtain sharper estimates on the error term for the expected value over  $\mathcal{G}_{n,2d}$ . One that is useful for information on the eigenvalues is (corollary 3.6)

**Theorem C** *There is a constant  $c$  such that the error term in equation 1.1 is no more than*

$$2d(2d-1)^{k-1} (ckd)^{\ell} \left( \frac{k^{2\sqrt{2d}}}{n^{1+\lfloor \sqrt{2d-1}/2 \rfloor}} + \frac{(2d-1)^{-k/2}}{n} \right). \quad (1.2)$$

for all  $k$  with  $k \leq n^{1/(c\sqrt{d})}/(cd^{3/2})$  and  $d \geq 4$ .

(See theorem 2.18 for other estimates on the error term.) From the above theorem we derive estimates on the second eigenvalue.

In section 2 we do the most of the work, which is to prove that the sum in equation 1.1 has an  $r$ -th order asymptotic expansion in  $1/n$  for any  $r \leq \sqrt{2d-1}/2$  of the form

$$E \left\{ \sum_{w \in \text{Irred}_k} i \xrightarrow{w} i \right\} = 2d(2d-1)^{k-1} \left( \frac{1}{n} f_0(k) + \frac{1}{n^2} f_1(k) + \dots + \frac{1}{n^r} f_{r-1}(k) \right) + \text{error}, \quad (1.3)$$

where the error involves a  $1/n^{\ell+1}$  term and a term like the second term in equation 1.2, and where the  $f_i$ 's are polynomials of degree  $5i+2$  whose coefficients are bounded by  $(cdr)^{c\ell}$ . In section 3 we show why theorem C follows from equation 1.3, i.e. why  $f_0(k) = 1$  and all the other  $f_i$ 's must vanish, and derive information on the eigenvalues, such as theorem A.

## 2 An Asymptotic Expansion

The goal of this section is to develop the asymptotic expansion of equation 1.3. We first explain the reason that such an expansion should exist, reviewing the ideas in [BS87]. Throughout this section we will bound error terms by expressions involving various absolute constants. Rather than giving each one a distinct name, we shall denote them all by  $c$ ; in cases where confusion could arise we shall also use  $c'$ .

For  $w = \sigma_1 \dots \sigma_k \in \text{Irred}_k$  and  $i$  fixed, consider the random variable  $i \xrightarrow{w} i$ . We wish to estimate  $E\{i \xrightarrow{w} i\}$ . In [BS87] the analysis is as follows. It is helpful to consider the random variables  $t_1 = \sigma_1(i)$ ,  $t_2 = \sigma_2(t_1)$ , etc. which trace  $i$ 's path through  $G$  along  $w$ . We simplify the calculation of  $E\{i \xrightarrow{w} i\}$  over  $\mathcal{G}_{n,2d}$  by considering only the

values of  $\pi_1, \dots, \pi_d$  needed to determine the  $t_j$ 's and therefore the value of  $i \xrightarrow{w} i$ . So consider the process of selecting an element of  $\mathcal{G}_{n,2d}$  by first selecting  $t_1$ , then  $t_2$ , until  $t_k$ , and then determining the remaining values of the  $\pi_j$ 's.

To determine  $t_1$  it suffices to know the value of  $\sigma_1(i)$ , which clearly takes on the values  $\{1, \dots, n\}$  each with probability  $1/n$ . Next we determine the value of  $\sigma_2(t_1)$ ; there are a few cases to consider. If  $\sigma_2 \neq \sigma_1$ , then  $\sigma_2$  and  $\sigma_1$  represent different permutations (since  $w$  is irreducible,  $\sigma_2 \neq \sigma_1^{-1}$ ), and so  $t_2$  takes on the values  $\{1, \dots, n\}$  each with probability  $1/n$ . If  $\sigma_1 = \sigma_2$ , then either  $t_1 = i$  and so  $t_2$  is forced to be  $i$ , or  $t_1 \neq i$ , in which case  $t_2$  takes on the values  $\{1, \dots, n\} - \{t_1\}$  each with probability  $1/(n-1)$ . We can continue to determine  $t_3, \dots, t_k$  in this fashion, each  $t_j$ 's value conditional on the values of the previous ones. When  $t_j$ 's value is exactly determined by the previous values, i.e.  $\sigma_j(t_{j-1})$ 's value is known, we say that  $t_j$  is a *forced choice*; otherwise we say that  $t_j$  is a *free choice*. If  $t_j$  is a free choice, then clearly  $t_j$  takes on any one of  $n-m$  values with probability  $1/(n-m)$  for some  $m \leq j-1$ . If a free choice  $t_j$  happens to be a previously visited vertex (i.e.  $= i$  or  $= t_m$  for some  $m < j$ ) we say that  $t_j$  is a *coincidence*; given that  $t_j$  is a free choice,  $t_j$  will be a coincidence with probability  $\leq j/(n-j+1)$ . During this paper we will also call an edge  $(t_{j-1}, t_j)$  a free choice, forced choice, or coincidence if  $t_j$  is respectively a free choice, forced choice, or coincidence.

In [BS87], it is observed that two coincidences occur in any given  $w$  and  $i$  with probability no greater than

$$\binom{k}{2} \left( \frac{k-1}{n-k+1} \right)^2 = O\left(\frac{k^4}{n^2}\right).$$

This accounts for the first of their error terms. The rest of the work is to analyze the case of one coincidence, since if there are no coincidences then clearly  $i \xrightarrow{w} i$  is 0. We will analyze the case of  $\leq r$  coincidences for any  $r \leq \sqrt{2d-1}/2$ , and show that the error term in equation 1.1 actually has an  $r$  term asymptotic expansion in  $1/n$ , whose coefficients are polynomials in  $k$  for any fixed  $d$ . The key to the proof is to group together walks in the graph which have the same rough shape, which we call the *type* of the walk.

Let

$$i_0 \xrightarrow{w_1} i_1 \xrightarrow{w_2} \dots \xrightarrow{w_j} i_j \equiv \begin{cases} 1 & \text{if } w_k(i_{k-1}) = i_k \text{ for } k = 1, \dots, j \\ 0 & \text{otherwise} \end{cases}.$$

Consider

$$\mathbb{E} \left\{ \sum_{w \in \text{Irred}_k} i \xrightarrow{w} i \right\} = \sum_{\substack{w = \sigma_1 \dots \sigma_d \in \text{Irred}_k \\ t_j \in \{1, \dots, n\}, j=1, \dots, k-1}} \mathbb{E} \left\{ i \xrightarrow{\sigma_1} t_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_{k-1}} t_{k-1} \xrightarrow{\sigma_k} i \right\}.$$

We will group the terms of the right-hand-side summation by their geometric configuration. Fix a word  $w = \sigma_1 \dots \sigma_d$ , integer  $i$ , and sequence of integers  $\mathfrak{t} = (t_1, \dots, t_{k-1})$

and let  $t_k = i$ . With these values we associate a directed graph,  $\Gamma_{w,i,t}$ , with edges labeled by  $\Pi$ . The graph's vertex set is  $V = \{i_1, \dots, i_m\}$ , where  $m$  is the number of distinct integers among  $i$  and  $t$ ; we think of  $i_1$  as representing  $i$  and, more generally,  $i_l$  as representing the  $l$ -th distinct member of the sequence  $i, t_1, \dots, t_{k-1}$ . The edge set,  $E$ , has one edge  $e_j$  for each free choice  $t_j$ ;  $e_j$  is an edge  $(i_a, i_b)$  where  $i_a, i_b$  respectively represent the values of  $t_{j-1}$  and  $t_j$ ; in addition  $e_j$  is labeled with  $\sigma_j$ .  $\Gamma_{w,i,t}$  represents the geometric structure of the intended path from  $i$  through  $t$  and back to  $i$  along  $w$ ; it contains the amount of conditioning on  $\pi_1, \dots, \pi_d$  given that the intended path is actually taken. We say call  $\Gamma_{w,i,t}$  the *generalized form* of  $w, i, t$ .

Consider an *abstract* generalized form,  $\Gamma = (V_\Gamma, E_\Gamma)$ , i.e. a directed graph with vertices named  $V_\Gamma = \{i_1, \dots, i_m\}$  and with edges labelled by  $\Pi$ , which is  $= \Gamma_{w,i,t}$  for some  $(w, i, t)$  triple, but where we ignore the particular  $(w, i, t)$  from which it arises (at least for the time being). Let  $a_j(\Gamma)$  denote the total number of occurrences of  $\pi_j$  and  $\pi_j^{-1}$  as labels of edges.  $a_j$  gives the number of values of  $\pi_j$  determined by  $\Gamma$ . For a fixed  $w$  we have

$$\sum_{\{i,t\}|\Gamma_{w,i,t}=\Gamma} \mathbb{E} \left\{ i \xrightarrow{\sigma_1} t_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_{k-1}} t_{k-1} \xrightarrow{\sigma_k} i \right\} = \mathbb{E} (n-1) \dots (n - |V_\Gamma| + 1) \prod_{j=1}^D \frac{1}{n(n-1) \dots (n - a_j(\Gamma) + 1)},$$

where the  $j$ -th term in the above product is omitted if  $a_j(\Gamma) = 0$ . We denote the above by  $\text{Pr}(\Gamma)$ . We can expand

$$\text{Pr}(\Gamma) = \frac{1}{n^{\text{coin}(\Gamma)-1}} \left( 1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots \right) \quad (2.1)$$

where the  $c_j$ 's are constants depending on  $\Gamma$  and  $\text{coin}(\Gamma)$  denotes the number of coincidences in  $\Gamma$  (more precisely in any triple  $(w, i, t)$  with  $\Gamma_{w,i,t} = \Gamma$ ), which does not depend on  $(w, i, t)$ . Clearly

$$\sum_{\substack{w \in \text{Irred}_k \\ i=1, \dots, n}} \mathbb{E} \left\{ i \xrightarrow{w} i \right\} = \sum_{\Gamma} \text{Pr}(\Gamma) \omega(\Gamma, k), \quad (2.2)$$

where  $\omega(\Gamma, k)$  denotes the number of irreducible words,  $w$ , of length  $k$  compatible with  $\Gamma$ , i.e. for which there are  $i$  and  $t$  with  $\Gamma_{w,i,t} = \Gamma$ .

Next we go up together generalized forms,  $\Gamma$ , whose underlying graphs have the same rough shape. Consider again, for a fixed word,  $w$ , and vertex,  $i$ , the process of determining the  $t_j$ 's. We begin by generating distinct vertices  $t_1, t_2, \dots$ , adding vertices and edges to  $\Gamma$ , until we encounter a coincidence. We may then encounter forced choices, which is to say walk in  $\Gamma$  without generating new vertices or edges, until we encounter a free choice,  $t_l$ . Starting with the edge from  $t_{l-1}$  to  $t_l$ , we depart from the "old"  $\Gamma$ , walking along new edges and vertices until we encounter our next

coincidence. We call a point such as  $t_{l-1}$  (more precise, the last vertex in a consecutive sequence of forced choices) a *departure*. We will sometimes also refer to the edge  $(t_{l-1}, t_l)$  as a departure, if  $t_{l-1}$  is. Clearly departures alternate with coincidences throughout the walk. We now go up  $\Gamma$ 's by forgetting any vertex which is not a coincidence or a departure. In other words, we forget all degree 2 vertices (except  $i_1$ ); nothing interesting happens at them.

We say that a vertex  $i_j$  of  $\Gamma$  is a coincidence (departure) if one of its corresponding  $t_l$ 's is a coincidence (departure); as with  $\text{coin}(\Gamma)$ , these notions are independent of  $(w, i, t)$ . In fact,  $i_1$  is always a coincidence,  $i_j$  for  $j > 1$  is a coincidence iff its indegree is  $> 1$ , and  $i_j$  for all  $j$  is a departure iff its outdegree is  $> 1$ . Any vertex of  $\Gamma$  which is not a coincidence or a departure has indegree and outdegree 1, and so the edges of  $\Gamma$  are a union of simple (directed) paths from pairs of the subset of vertices,  $S$ , consisting of all coincidences and departures. With  $\Gamma$  we associate its *type*,  $T_\Gamma$ , which is an *undirected* graph, possibly with multiple edges and self-loops. Its vertex set is  $\{v_1, \dots, v_l\}$  where  $l = |S|$ , and as before we think of  $v_j$  as representing the  $j$ -th vertex of  $S$  in the order  $i_1, \dots, i_m$ .  $T_\Gamma$  has an edge for each simple directed path from pairs of  $S$ .

Given an abstract type,  $T$ , which is just an undirected graph (corresponding to at least one  $\Gamma$ ), we order any go up of multiple edges to distinguish each edge from its copies. In other words, for each edge of multiplicity  $s$ , represented by distinct edges  $\{e_1, \dots, e_s\}$ , we label  $\{e_1, \dots, e_s\}$  with distinct integers  $\{1, \dots, s\}$  in some fashion. This labelling is equivalent to imposing an order on each group of multiple edges. We call a type together with such an ordering or labelling a *type with ordered multiple edges*, or simply an *ordered type*. The point is that given an abstract generalized form,  $\Gamma$ , any word  $w$  compatible with  $\Gamma$  gives rise to a unique *oriented* labelling with labels in  $\Pi^*$  of the edges of the corresponding (abstract) ordered type  $T_\Gamma$  (an oriented labelling being one where we orient the edge and give it a label, assigning the inverse label to the oppositely oriented direction), and to a unique walk in  $T_\Gamma$ . To see this, consider the walk from  $i_1$  to itself along  $w$ , which traces out a unique walk in  $\Gamma$ . This gives rise to possibly many walks in  $T_\Gamma$ , but only to one walk such that (1) the edges are labelled and oriented as they are traversed, labelled with the word in  $\Pi^*$  which is simultaneously traced out in  $\Gamma$ , and oriented in the direction traversed, (2) once an edge is labelled it is traversed only when it matches the word in  $\Pi^*$  traced out during the traversal, and (3) multiple edges are traversed in increasing order.

Conversely, let  $T$  be an (abstract) ordered type. By a *legal walk* in  $T$  we mean a walk from  $v_1$  to itself which traverses every edge at least once, and which traverses multiple edges in increasing order. By a *legal oriented labelling* we mean an oriented labeling of  $T$ 's edges by irreducible words in  $\Pi^*$  of length  $\geq 1$  such that for any vertex,  $v$ , in  $T$  the labels on the outwardly oriented edges from  $v$  begin with distinct letters. It is easy to see that we have a one-to-one correspondence

$$\{(\Gamma, w) \mid \text{General forms } \Gamma, w \in \text{Irred compatible with } \Gamma\} \xleftrightarrow{1-1} \{(T, y, \ell) \mid \text{Ordered types } T, \text{ legal walks } y \text{ and oriented labellings } \ell \text{ for } T\}.$$



To calculate the asymptotic expansion, we will sum over types, walks, and oriented labellings, rather than over generalized forms and words.

Given a legal walk,  $y$ , on an ordered type,  $T$ , we define the *multiplicity of the walk* to be the function which assigns to each edge of  $T$  the number of times it is traversed, disregarding orientation. For a legal oriented labelling,  $\ell$ , of  $T$ , we define the *lettering of the labelling* to be the function which takes a vertex,  $v$ , and an incident edge,  $e$ , and returns the first letter of  $e$ 's label with the outward orientation from  $v$ . We define the *weight of the labelling* to be the function which takes an edge and returns the length of the word with which it is labelled. The notion of  $\text{coin}(T)$  for types  $T$  carries over from that of generalized forms (i.e. as equal to  $\text{coin}(\Gamma)$  for one and therefore all  $\Gamma$ 's with  $T_\Gamma = T$ ). Our strategy will be to fix a type, a multiplicity, and a lettering, and then sum the corresponding terms in equation 2.2.

We are now ready for the nitty-gritty. Lemmas which are not followed by proofs are immediate or easy.

**Lemma 2.1** *For any type,  $T = (V, E)$  we have  $\text{coin}(T) = |E| - |V| + 1$ .*

**Lemma 2.2** *For a type,  $T$ , of coincidence  $r$  we have  $|V| \leq 2r$  and  $|E| \leq 3r - 1$ .*

**Proof** In any legal walk in a generalized form, the coincidences and departures alternate, starting and ending with coincidences. Any vertex,  $v_l$ , of the type, with  $l \geq 2$ , must be either a coincidence or a departure (or both). Thus  $|V| \leq 2r$ . This and lemma 2.1 yield  $|E| \leq 3r - 1$ .

□

**Lemma 2.3** *The number of types of coincidence  $r$  is less than  $(2r)^{6r-2}$ . The number of types of coincidence  $\leq r$  is less than  $(2r)^{6r-1}$ .*

**Lemma 2.4** *The number of letterings of a type of coincidence  $r$  is  $\leq (2d)^{6r} D_2$ .*

Let  $\mathcal{T}_r$  be the type with two vertices,  $v_1, v_2$ , with one edge from  $v_1$  to  $v_2$  and  $r$  edges from  $v_2$  to itself (i.e. self-loops).  $\mathcal{T}_r$  is of coincidence  $r$ .

**Lemma 2.5** *In a type,  $T$ , of coincidence  $r$  which is not equal to  $\mathcal{T}_r$ , each vertex has degree at most  $2r$ .*

**Proof** Again, consider any legal walk in a generalized form. When a vertex is first encountered it is entered and left, giving it indegree and outdegree 1 up to that point. Then it can be entered by coincidence edges up to  $r$  times, and left by departure edges up to  $r - 1$  times. The degree of the corresponding vertex in the type will therefore have degree  $\leq 2r + 1$ . We claim that equality can only occur in the type  $\mathcal{T}_r$ . For equality implies that the corresponding vertex,  $v$ , in the type must have  $r$  self-edges and that it is the only coincidence and departure in the type; this leaves room for only (possibly) one other vertex,  $v_1$ , which corresponds to  $i_1$ . If  $v$  is not  $v_1$ , then this type must be  $\mathcal{T}_r$ , and if  $v = v_1$ , then  $v$ 's degree is  $2r$ .

□

**Lemma 2.6** *In a fixed type of coincidence  $r$ , there are  $\leq (2r)^m$  legal walks whose sum of its multiplicities is  $m$ .*

**Proof** If  $T$  is not  $\mathcal{T}_r$ , then each vertex has degree  $\leq 2r$  and so there are  $\leq (2r)^m$  legal walks of length  $m$  in  $T$ . In  $\mathcal{T}_r$ , a legal walk consists of one step from  $v_1$  to  $v_2$ , then  $m-2$  steps each taking one of the  $r$  self-loops in either of its two directions, and finally returning to  $v_1$ , for a total of  $\leq (2r)^{m-2}$  legal walks.

□

Consider the function

$$g(x) = (1 - b_1 x) \dots (1 - b_r x) \left( \frac{1}{1 - c_1 x} \right) \dots \left( \frac{1}{1 - c_s x} \right),$$

where  $b_1, \dots, b_r, c_1, \dots, c_s$  are positive constants. Its  $i$ -th derivative is

$$\sum_{\substack{i_1 + \dots + i_r + j_1 + \dots + j_s = i \\ i_k \leq 1 \text{ for } k=1, \dots, r}} \binom{i}{i_1, \dots, i_r, j_1, \dots, j_s} (-b_1)^{i_1} (1 - b_1 x)^{i_1 - 1} \dots (-b_r)^{i_r} (1 - b_r x)^{i_r - 1} \times \\ j_1! \dots j_s! \frac{c_1^{j_1} \dots c_s^{j_s}}{(1 - c_1 x)^{j_1} \dots (1 - c_s x)^{j_s}}.$$

For any  $0 \leq x \leq 1/c$ , where  $c$  is an upper bound on the  $b_k$ 's and  $c_k$ 's, we have

$$\begin{aligned} |g^{(i)}(x)| &\leq \left( \frac{1}{1 - xc} \right)^i i! \sum_{i_1 + \dots + i_r + j_1 + \dots + j_s = i} \binom{i}{i_1, \dots, i_r, j_1, \dots, j_s} b_1^{i_1} \dots b_r^{i_r} c_1^{j_1} \dots c_s^{j_s} \\ &= \left( \frac{1}{1 - xc} \right)^i i! \left( \sum b_k + \sum c_k \right)^i, \end{aligned} \quad (2.3)$$

by the multinomial theorem. As a consequence we have

**Lemma 2.7** *For any generalized form  $\Gamma$  of  $\leq k$  edges,  $\Pr(\Gamma)$  has an expansion*

$$\Pr(\Gamma) = \frac{1}{n^{\text{coin}(\Gamma)-1}} \left( p_0 + \frac{p_1}{n} + \frac{p_2}{n^2} + \dots + \frac{p_r}{n^r} + \frac{\epsilon}{n^{r+1}} \right) \quad (2.4)$$

(with  $p_0 = 1$ ), where  $\epsilon$  is an error term bounded by

$$e^{(r+1)k/n} k^{2r+2}$$

and the  $p_i$ 's are constants.

**Proof** We apply equation 2.3 to equation 2.1, where we have  $\sum b_j + \sum \epsilon_j \leq k^2$ . By Taylor's theorem,  $\epsilon$  is bounded by

$$\frac{1}{(r+1)!} \left( \frac{1}{1-k\xi} \right)^{r+1} (r+1)! (k^2)^{r+1},$$

for some  $\xi \in [0, 1/n]$  for each  $n$ .

□

The  $p_i$  of lemma 2.7 are polynomials in the  $a_j$  and  $|V_\Gamma|$  of equation 2.1, and we wish to bound their size and coefficients.

For any integer  $a$ , consider the function

$$g(x) = \frac{1}{1-x} \frac{1}{1-2x} \cdots \frac{1}{1-ax}.$$

A power series for  $g$  is given in terms of Stirling numbers. We wish to bound the size of their coefficients as polynomials in  $a$ .

**Lemma 2.8**  *$g(x)$ 's power series about  $x = 0$  is of the form*

$$1 + xR_1(a) + x^2R_2(a) + \cdots + x^iR_i(a) + \cdots,$$

where the  $R_i$ 's are polynomials of degree  $2i$ ,

$$R_i(a) = \sum_{j=0}^{2i} \epsilon_{i,j} \binom{a}{j},$$

where the  $\epsilon_{i,j}$ 's are non-negative integers with

$$\epsilon_{i,0} + \cdots + \epsilon_{i,2i} \leq 8^i i!.$$

**Proof** Writing

$$1 + xR_1(a) + x^2R_2(a) + \cdots = (1+ax)(1+xR_1(a-1) + x^2R_2(a-1) + \cdots)$$

and comparing terms yields

$$R_\ell(a) = R_\ell(a-1) + aR_{\ell-1}(a) \tag{2.5}$$

for  $\ell \geq 1$  (with  $R_0(a) \equiv 1$ ). We get a definition for the  $R_\ell$ 's recursive in  $r$ . Since  $R_0(a) \equiv 1$  it easily follows that  $R_r$  is a polynomial of degree  $2r$ . Since  $R_r(1) = 1$  for all  $r$ , setting  $a = 1$  in equation 2.5 yields  $R_r(0) = 0$ , where we identify  $R_\ell$  with the

polynomial which agrees with it on all positive integers. Thus  $c_{r,0} = 0$  for all  $r$ . For the other coefficients we have

$$\begin{aligned} R_r(a) - R_r(a-1) &= \sum_{k=0}^{2r-2} c_{r-1,k} a \binom{a}{k} \\ &= \sum_{k=0}^{2r-2} c_{r-1,k} \left( \binom{a}{k+1} (k+1) + \binom{a}{k} k \right) \\ &= \sum_{k=0}^{2r-2} c_{r-1,k} \left( \binom{a-1}{k+1} (k+1) + \binom{a-1}{k} (k+1) + \binom{a-1}{k} k + \binom{a-1}{k-1} k \right). \end{aligned}$$

Solving the above difference equation it follows by induction on  $r$  that all the  $c_{r,k}$ 's are non-negative integers, and that  $\sum_k c_{r,k} \leq (8r-6) \sum_k c_{r-1,k}$ .

□

Similarly, we consider

$$g(x) = (1-x)(1-2x)\dots(1-ax).$$

**Lemma 2.9**  $g(x)$ 's power series about  $x = 0$  is of the form

$$1 - xQ_1(a) + x^2Q_2(a) - \dots + (-1)^r x^r Q_r(a) + \dots,$$

where the  $Q_i$ 's are polynomials of degree  $2i$ ,

$$Q_i(a) = \sum_{j=0}^{2i} c_{i,j} \binom{a}{j},$$

where the  $c_{i,j}$ 's are non-negative integers with

$$c_{i,0} + \dots + c_{i,2i} \leq 4^i i!.$$

**Proof** The  $Q$ 's satisfy

$$Q_r(a) = Q_r(a-1) + aQ_{r-1}(a-1),$$

and a similar analysis yields the lemma.

**Corollary 2.10** In equation 2.4, we have

$$p_i(\Gamma) = \sum_{|I| \leq i} c_{i,I} \binom{|V_\Gamma| - 1}{I_0} \binom{a_1(\Gamma) - 1}{I_1} \dots \binom{a_d(\Gamma) - 1}{I_d}, \quad (2.6)$$

where we have used "tensor" notation with  $I = (I_0, \dots, I_d)$  (i.e.  $I_k$ 's are non-negative integers, and  $|I| = I_0 + \dots + I_d$ ), with constants  $c_{i,I}$  each of absolute value  $\leq 8^i i!$ .

Our goal is to estimate the terms of the sum in the right-hand-side of equation 2.6, when summed over all  $(\Gamma, w)$  pairs of a given type. This is easy once a lettering of the type is fixed.

For a fixed  $\sigma, \tau \in \Pi$ , let  $\text{Irred}_{k, \sigma, \tau}$  denote the set of irreducible words of length  $k$  beginning with  $\sigma$  and ending with  $\tau$ .

**Lemma 2.11** *For any  $\sigma, \tau \in \Pi$  and non-negative integers  $s_1, \dots, s_d$  we have*

$$f(k) \equiv \sum_{w \in \text{Irred}_{k, \sigma, \tau}} \binom{a_1(w)}{s_1} \dots \binom{a_d(w)}{s_d} = (2d-1)^k P(k) + (-1)^k Q(k) + R(k)$$

where  $P, Q, R$  are polynomials of degree  $s = s_1 + \dots + s_d$  with coefficients bounded by  $(cd)^{cs}$ .

**Proof** Consider the  $2d \times 2d$  matrix

$$M = \begin{pmatrix} x_1 & 0 & x_2 & x_2 & \cdots & x_d & x_d \\ 0 & x_1 & x_2 & x_2 & \cdots & x_d & x_d \\ x_1 & x_1 & x_2 & 0 & \cdots & x_d & x_d \\ x_1 & x_1 & 0 & x_2 & \cdots & x_d & x_d \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1 & x_1 & x_2 & x_2 & \cdots & x_d & 0 \\ x_1 & x_1 & x_2 & x_2 & \cdots & 0 & x_d \end{pmatrix}$$

i.e. the matrix  $M$  whose coefficients are

$$\begin{aligned} M_{2i-1, 2j-1} &= M_{2i, 2j} = x_j \\ M_{2i, 2j-1} &= M_{2i-1, 2j} = x_j \quad \text{if } i \neq j \\ M_{2i, 2j-1} &= M_{2i-1, 2j} = 0 \quad \text{if } i = j \end{aligned}$$

over all  $0 \leq i \leq d$  and  $0 \leq j \leq d$ . It is easy to see that

$$(M^k)_{i,j} = \sum_{w \in \text{Irred}_{k, \sigma_i, \sigma_j}} x_1^{a_1(w)} \dots x_d^{a_d(w)}$$

where  $\sigma_{2i-1}$  is  $\pi_i$  and  $\sigma_{2i}$  is  $\pi_i^{-1}$ . Hence

$$f(k) = \frac{1}{s_1! \dots s_d!} \left( \left( \frac{\partial}{\partial x_1} \right)^{s_1} \dots \left( \frac{\partial}{\partial x_d} \right)^{s_d} M^k \Big|_{x_1 = \dots = x_d = 1} \right)_{a,b}$$

for the  $a, b$  with  $\sigma_a = \sigma, \sigma_b = \tau$ . Let  $C_i = \frac{\partial}{\partial x_i} M$ , which is a matrix with  $4d - 2$  1's and the rest 0's. All second derivatives of  $M$  vanish, and so

$$\begin{aligned} \frac{1}{s_1! \dots s_d!} \left( \frac{\partial}{\partial x_1} \right)^{s_1} \dots \left( \frac{\partial}{\partial x_d} \right)^{s_d} M^k &= \\ \sum_{(i_1, \dots, i_s)} \sum_{j_0 + \dots + j_s = k-s} M^{j_0} C_{i_1} M^{j_1} C_{i_2} \dots C_{i_s} M^{j_s} & \end{aligned}$$

where the first summation is over all tuples  $(i_1, \dots, i_s)$  which contain  $s_1$  1's,  $s_2$  2's, etc. For fixed  $(i_1, \dots, i_s)$  we have

$$\sum_{j_0 + \dots + j_s = k-s} M^{j_0} C_{i_1} \dots M^{j_s} |_{x_1 = \dots = x_d = 1} = \sum_{j_0 + \dots + j_s = k-s} C^{j_0} C_{i_1} \dots C^{j_s}$$

where

$$C = M|_{x_1 = \dots = x_d = 1} = \begin{pmatrix} 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 \end{pmatrix}.$$

We claim that  $C$  has eigenvalues  $2d - 1$ ,  $-1$  with multiplicity  $d - 1$ , and  $1$  with multiplicity  $d$ . To see this, note that the map  $T: \mathbf{R}^{2d} \rightarrow \mathbf{R}^{2d}$  given by

$$T(y_1, \dots, y_{2d}) = (y_1 + y_2, y_1 - y_2, y_3 + y_4, y_3 - y_4, \dots, y_{2d-1} - y_{2d})$$

gives

$$TCT^{-1} = \begin{pmatrix} 1 & 0 & 2 & 0 & \dots & 2 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 2 & 0 & 1 & 0 & \dots & 2 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 2 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

By permuting the basis vectors this becomes the block matrix with  $d \times d$  blocks

$$\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}, \quad (2.7)$$

where  $I$  is the identity matrix, and

$$B = \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 1 & 2 & \dots & 2 \\ 2 & 2 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 1 \end{pmatrix}.$$

Since  $B + I$  is the  $d \times d$  matrix of all 2's, whose eigenvalues are  $2d$  (simple) and  $0$  with multiplicity  $d - 1$ , it follows that  $B$ 's eigenvalues are  $2d - 1$  and  $-1$  with multiplicity  $d - 1$ . Thus  $C$ 's eigenvalues, which by equation 2.7 are the union of those of  $B$  and those of  $I$ , are  $2d - 1$  (simple),  $-1$  with multiplicity  $d - 1$ , and  $1$  with multiplicity  $d$ .

Thus

$$\sum_{j_0+\dots+j_s=k-s} C^{j_0} C_{i_1} \dots C^{j_s} = \sum_j AD^{j_0} A^{-1} C_{i_1} AD^{j_1} A^{-1} \dots C_{i_s} AD^{j_s} A^{-1}$$

where  $D$  is the diagonal matrix with diagonal consisting of one  $2d-1$ ,  $d-1$   $-1$ 's, and  $d$   $1$ 's, and  $A$  is an orthogonal matrix diagonalizing  $C$ . We can write the sum on the right-hand-side of the above equation as

$$\sum_j AD^{j_0} E_1 D^{j_1} \dots E_s D_{j_s} A^{-1} \quad (2.8)$$

with

$$E_j = A^{-1} C_{i_j} A.$$

Since

$$\|E_j\|_2 = \|A\|_2 \|C_{i_j}\|_2 \|A^{-1}\|_2 = \|C_{i_j}\|_2 \leq \sqrt{4d-2},$$

(where  $\|\cdot\|_2$  denotes the  $L_2$  operator norm) we have that each of  $E_j$ 's entries are  $\leq \sqrt{4d-2}$ . Clearly each of  $A$  and  $A^{-1}$ 's entries are  $\leq 1$  in absolute value. Therefore, the  $a, b$ -th entry of the matrix in equation 2.8,

$$\sum_j \sum_{b_0, \dots, b_s \in [1, \dots, 2d]} (A)_{a, b_1} (D^{j_0})_{b_0, b_0} (E_1)_{b_0, b_1} (D^{j_1})_{b_1, b_1} \dots (E_s)_{b_{s-1}, b_s} (D_{j_s})_{b_s, b_s} (A^{-1})_{b_s, b},$$

is just

$$\sum_j \sum_{\substack{\epsilon_0, \dots, \epsilon_s \\ \epsilon_l \in \{2d-1, -1, 1\}}} c_{\epsilon_0, \dots, \epsilon_s} \epsilon_0^{j_0} \dots \epsilon_s^{j_s} \quad (2.9)$$

where the  $c_{\epsilon_0, \dots, \epsilon_s}$ 's are constants. Fixing  $\epsilon_0, \dots, \epsilon_s$  and letting  $\alpha, \beta, \gamma$  be the number of respective occurrences of  $2d-1, -1, 1$  among the  $\epsilon$ 's, we see that

$$|c_{\epsilon_0, \dots, \epsilon_s}| \leq (4d-2)^{s/2} (d-1)^\beta d^\gamma \leq 2^s d^{3s/2}. \quad (2.10)$$

Also,

$$\sum_j \epsilon_0^{j_0} \dots \epsilon_s^{j_s} = \sum_{u+v+w=k-s} (2d-1)^u (-1)^v \binom{u+\alpha-1}{\alpha-1} \binom{v+\beta-1}{\beta-1} \binom{w+\gamma-1}{\gamma-1}. \quad (2.11)$$

From

$$\sum_{u=0}^k R^u = \frac{R^{k+1} - 1}{R - 1}$$

we derive

$$\sum_{\substack{u+v \leq k \\ u, v \geq 0}} R^u S^v = \left( \frac{R^{k+1} - 1}{R - 1} \right) + \left( \frac{R^k - 1}{R - 1} \right) S + \dots + \left( \frac{R^1 - 1}{R - 1} \right) S^k$$

$$\begin{aligned}
&= \frac{R^{k+1}}{R-1} \left( 1 + \frac{S}{R} + \cdots + \left( \frac{S}{R} \right)^k \right) - \frac{1}{R-1} (1 + S + \cdots + S^k) \\
&= \frac{S^{k+2}(R-1) + R^{k+2}(1-S) + (S-R)}{(R-1)(S-1)(S-R)}
\end{aligned}$$

and thus

$$\sum_{\substack{u+v+w=k \\ u,v,w \geq 0}} R^u S^v P^w = \frac{S^{k+2}(R-P) + R^{k+2}(P-S) + P^{k+2}(S-R)}{(R-P)(S-P)(S-R)}.$$

Multiplying the above by  $R^{\alpha-1}S^{\beta-1}P^{\gamma-1}$ , then differentiating by  $(\frac{\partial}{\partial R})^{\alpha-1}(\frac{\partial}{\partial S})^{\beta-1}(\frac{\partial}{\partial P})^{\gamma-1}$ , the substituting  $k-s$  for  $k$  yields

$$\sum_{u+v+w=k-s} (2d-1)^u (-1)^v \binom{u+\alpha-1}{\alpha-1} \binom{v+\beta-1}{\beta-1} \binom{w+\gamma-1}{\gamma-1} \quad (2.12)$$

is equal to a sum of six terms, the first of which is

$$\frac{1}{(\alpha-1)!} \frac{1}{(\beta-1)!} \frac{1}{(\gamma-1)!} \left( \frac{\partial}{\partial R} \right)^{\alpha-1} \left( \frac{\partial}{\partial S} \right)^{\beta-1} \left( \frac{\partial}{\partial P} \right)^{\gamma-1} \left[ \frac{R^\alpha S^{k-s+\beta+1} P^{\gamma-1}}{(R-P)(S-P)(S-R)} \right]$$

evaluated at  $R = 2d-1$ ,  $S = -1$ ,  $P = 1$ ; the other five terms are similar. In the above term, the  $\frac{\partial}{\partial P}$ 's can be applied to any of the three terms  $P^{\gamma-1}$  (in the numerator) and  $(R-P)$  and  $(S-P)$  (in the denominator), and similarly for the other partials. We get a sum of  $3^{\alpha+\beta+\gamma-3}$  terms, each of the form

$$\binom{k-s+\beta+1}{l} S^k \theta,$$

where  $l$  is an integer between 0 and  $s$  and  $\theta$  is a polynomial in  $R, S, P$ , independent of  $k$ , which satisfies

$$|\theta(R, S, P)| \Big|_{R=2d-1, S=-1, P=1} \leq (2d-1)^s.$$

The coefficients of the polynomial

$$\binom{k-s+\beta+1}{l}$$

are clearly dominated by those of

$$\frac{(k+s)^l}{l!},$$



which are clearly less than

$$\max_{j=0,\dots,l} \frac{s^j \binom{l}{j}}{l!} \leq \max_{j=0,\dots,l} \frac{s^j}{j!} \leq e^s.$$

After doing a similar analysis for the other five terms whose sum equals the expression in equation 2.12, it follows that the expression in equation 2.11 is of the form

$$(2d-1)^k p(k) + (-1)^k q(k) + r(k)$$

with  $p, q, r$  polynomials of degree  $s$  whose coefficients are bounded by

$$3^{s-2} e^s (2d-1)^s.$$

Summing over the  $3^{s+1}$  possibilities for  $\epsilon_0, \dots, \epsilon_s$  in equation 2.9, using also equation 2.10, yields

$$f(k) \equiv \sum_{w \in \text{Irred}_{k,\sigma,\tau}} \binom{a_1(w)}{s_1} \dots \binom{a_d(w)}{s_d} = (2d-1)^k P(k) + (-1)^k Q(k) + R(k)$$

where  $P, Q, R$  are polynomials in  $k$  of degree  $s$  whose coefficients are bounded by

$$2^s d^{3s/2} 3^{s+1} 3^{s-2} e^s (2d-1)^s < (36e)^s d^{5s/2}.$$

□

**Corollary 2.12** *For any  $\sigma, \tau \in \Pi$  and non-negative integers  $s_1, \dots, s_d$  we have*

$$f(k) \equiv \sum_{w \in \text{Irred}_{k,\sigma,\tau}} (a_1(w))^{s_1} \dots (a_d(w))^{s_d} = (2d-1)^k P(k) + (-1)^k Q(k) + R(k)$$

where  $P, Q, R$  are polynomials of degree  $s = s_1 + \dots + s_d$  with coefficients bounded by  $(c d s)^{c s}$  for some absolute constant  $c$ .

**Proof** We use the fact that the constants  $c_m$  in

$$x^n = \sum_{m=0}^n c_m \binom{x}{m}$$

are always positive (see, for example, [Knu73] page 65); it follows that  $c_m \leq n^n$  by substituting  $n$  for  $x$  in the above. Applying this to each  $a_i(w)^{s_i}$ , expanding the product, and applying the previous lemma yields the desired result.

□

For a type  $T = (V_T, E_T)$  with  $E_T = \{e_1, \dots, e_t\}$ , let  $\mathcal{L}_{T, k_1, \dots, k_t}$  denote the set of oriented labelling with weights  $(k_1, \dots, k_t)$ . For a labelling,  $\ell$ , we define  $a_i(\ell)$  to be the number of occurrences of  $\pi_i$  and  $\pi_i^{-1}$  in  $\ell$ . Any generalized form,  $\Gamma$ , compatible with  $T$  and a labelling in  $\mathcal{L}_{T, k_1, \dots, k_t}$  has

$$|V_\Gamma| = |V_T| + \sum (k_j - 1) = |V_T| + |\ell| - t = |\ell| + 1 - r ,$$

by lemma 2.1, where  $|\ell| = \sum a_i(\ell)$ . We can therefore define the  $p_i$  of equation 2.4 as  $p_i(T, \ell)$ , for  $T$  and  $\ell$  determine the value of  $p_i(\Gamma)$  for any compatible  $\Gamma$ .

**Lemma 2.13** *For any fixed type,  $T$ , of coincidence  $\leq r$ , and  $k_1, \dots, k_t$ , let  $\mathcal{L} = \mathcal{L}_{T, k_1, \dots, k_t}$ . For any  $i \leq r + 1$  we have*

$$\sum_{\ell \in \mathcal{L}} p_i(T, \ell) = \sum_{K_1, K_2, K_3} (2d - 1)^{|K_1|} (-1)^{|K_2|} P_{K_1, K_2, K_3}(k_1, \dots, k_t) , \quad (2.13)$$

where the right-hand-side sum is over all partitions of  $K = \{k_1, \dots, k_t\}$  into three disjoint sets  $K_1, K_2, K_3$ , where

$$|K_j| = \sum_{k_s \in K_j} k_s ,$$

and where  $P_{K_1, K_2, K_3}$  are polynomials of degree  $\leq 2i$  whose coefficients are bounded by  $(cdr)^{cr}$  for some absolute constant  $c$ .

**Proof** By equation 2.6 we have that the sum on the left-hand-side of equation 2.13 is

$$\sum_{|I| \leq i} c_I \binom{k_1 + \dots + k_t - r}{I_0} \binom{a_1(\ell) - 1}{I_1} \dots \binom{a_d(\ell) - 1}{I_d} \quad (2.14)$$

with  $|c_I| \leq 8^i i!$ . Let  $a_{jl}(\ell)$  denote the number of  $\pi_j, \pi_j^{-1}$  occurrences in  $e_l$ . Since

$$a_j(\ell) = a_{j1}(\ell) + \dots + a_{jt}(\ell) ,$$

we can expand equation 2.14 as a sum of  $(t + 1)I_0 2^{I_1} \dots 2^{I_d}$  terms of the form

$$c_s \left( \prod_{l=1}^t k_l^{s_l} \right) \left( \prod_{j=1}^d \prod_{l=1}^t (a_{jl}(\ell))^{s_{jl}} \right)$$

with coefficients  $c_s$  bounded by  $(cdr)^{cr}$  and with

$$\sum_l s_l + \sum_{j,l} s_{j,l} \leq i .$$

Fix a lettering, and let  $\overline{\mathcal{L}}$  denote those  $\ell \in \mathcal{L}$  of that lettering. We can write

$$\sum_{\ell \in \overline{\mathcal{L}}} \prod_{j,l} (a_{jl}(\ell))^{s_{jl}} = \prod_{l=1}^t \sum_{w \in \text{Irred}_{k_l, \sigma_l, \tau_l}} \prod_{j=1}^d (a_{jl}(w))^{s_{jl}} ,$$

with  $\sigma_1, \dots, \sigma_t, \tau_1, \dots, \tau_t$  given by the lettering. By corollary 2.12, the above is

$$= \prod_{l=1}^t [(2d-1)^{k_l} P_l(k_l) + (-1)^{k_l} Q_l(k_l) + R_l(k_l)],$$

with  $P_l, Q_l, R_l$  polynomials of degree  $\leq 2 \sum_l s_{j,l}$  and coefficients bounded by  $(cdr)^{cr}$  for some absolute constant  $c$ . Hence the above is

$$\sum_{\substack{K_1, K_2, K_3 \\ \text{partition } \{k_1, \dots, k_t\}}} (2d-1)^{|K_1|} (-1)^{|K_2|} Q_{K_1, K_2, K_3}(k_1, \dots, k_t),$$

for polynomials  $Q_{K_1, K_2, K_3}$  of degree  $\leq 2 \sum_{j,l} s_{j,l}$  with coefficients bounded by  $(cdr)^{cr}$  for some absolute constant  $c$ . Summing the above over the  $\leq (2d)^{6r-2}$  letterings yields the lemma.  $\square$

Our goal is to sum  $p_i(T, \ell)$  over all triples  $(T, y, \ell)$  corresponding to pairs  $(\Gamma, w)$  with  $w \in \text{Irred}_k$ . So far we can estimate

$$\sum_{\ell \in \mathcal{L}} p_i(T, \ell)$$

with  $\mathcal{L} = \mathcal{L}_{T, k_1, \dots, k_t}$ . We start summing over walks. So fix multiplicities  $m_1, \dots, m_t$ , and let  $k = k_1 m_1 + \dots + k_t m_t$ . Clearly  $k$  is the length of  $w$  in any  $(\Gamma, w)$  corresponding to a  $(T, y, \ell)$  with multiplicity  $m_1, \dots, m_t$  and weights  $k_1, \dots, k_t$ .

**Lemma 2.14** *For fixed type,  $T$ , with  $t$  edges and of coincidence  $\leq r$ , fixed multiplicities  $m_l$ , and  $i \leq r+1$  we have for any  $k \geq m = m_1 + \dots + m_t$*

$$\sum_{\substack{k_1, \dots, k_t \geq 1 \\ k_1 m_1 + \dots + k_t m_t = k}} \sum_{\ell \in \mathcal{L}_{T, k_1, \dots, k_t}} p_i(T, \ell) = (2d-1)^{k+t-m} P_i(k) + \epsilon$$

where  $\epsilon$  vanishes if  $m_1 = \dots = m_t = 1$  and otherwise

$$|\epsilon| \leq (2d-1)^{(k-m)/2} k^{t+2i} (cdr+m)^{cr^2},$$

and  $P_i$  a polynomial of degree  $t+2i$  with coefficient bounded by  $(cdr+m)^{cr^2}$  for some absolute constant  $c$ .

**Proof** Applying lemma 2.13 and exchanging summations yields

$$\sum_{k_1, \dots, k_t} \sum_{\ell} p_i(T, \ell) = \sum_{K_1, K_2, K_3} \sum_{\substack{k_1, \dots, k_t \geq 1 \\ k_1 m_1 + \dots + k_t m_t = k}} (2d-1)^{|K_1|} (-1)^{|K_2|} P_{K_1, K_2, K_3}(k_1, \dots, k_t).$$

Fix a partition,  $K_1, K_2, K_3$ . We shall need some sublemmas.

**Sublemma 2.15** *For any integer  $r$ , integer  $t \leq 3r - 1$ , and polynomial  $P$  of degree  $\leq r + t$ , we have for any  $k \geq t$*

$$\sum_{\substack{k_1+k_2+\dots+k_t=k \\ k_i \geq 1}} P(k_1, \dots, k_t) = Q(k)$$

with  $Q$  of degree  $\deg(P) + t - 1$  with

$$|Q| \leq |P|(cdr)^{ct},$$

where  $||$  for a polynomial denotes the largest absolute value among its coefficients.

**Proof** We proceed by induction on  $t$ . For  $t = 1$  there is nothing to prove. We claim that if the sublemma holds for  $t$  replaced by  $t - 1$ , then it also holds for  $t$ . To see this, we write

$$P(k_1, \dots, k_t) = \sum_{l=0}^{\deg(P)} \bar{P}_l(k_1, \dots, k_{t-1}) k_t^l,$$

with  $\deg(\bar{P}_l) \leq \deg(P) - l$ , and  $|\bar{P}_l| \leq |P|$ . Applying the sublemma to each  $\bar{P}_l$  we get

$$\sum_{\substack{k_1+k_2+\dots+k_t=k \\ k_i \geq 1}} P(k_1, \dots, k_t) = \sum_{l=0}^{\deg(P)} \sum_{j=1}^{k-t+1} Q_l(k-j) j^l,$$

with  $Q_l$  of degree  $\leq \deg(P) - l + (t - 2)$  and with coefficients bounded by  $|P|(cdr)^{c(t-1)}$ . For each fixed  $l$ , the sum over  $j$  on the left-hand-side of the above is clearly a polynomial of degree  $1 + l + \deg(Q_l) \leq \deg(P) + t - 1$  and coefficients bounded by  $|P|(cdr)^{ct}$ , assuming  $c$  is sufficiently large. Summing over  $l$ , using  $\deg(P) \leq r + t \leq 4r - 1$  proves the inductive step. □

**Sublemma 2.16** *For any  $m_1 = 1, m_2, \dots, m_t$ , partition  $K_1, K_2, K_3$  of  $\{k_1, \dots, k_t\}$  with  $k_1 \in K_1$ ,  $t \leq 3r - 1$ , and polynomial  $P$  of degree  $\leq r + t$ , we have for any  $k \geq m = 1 + m_2 + \dots + m_t$*

$$\sum_{\substack{k_1 m_1 + k_2 m_2 + \dots + k_t m_t = k \\ k_i \geq 1}} (2d - 1)^{|K_1|} (-1)^{|K_2|} P(k_1, \dots, k_t) = (2d - 1)^{k+t-m} Q(k) + \epsilon \tag{2.15}$$

where  $\epsilon$  vanishes if  $m_i = 1$  and  $k_i \in K_1$  for all  $i$ , and otherwise

$$|\epsilon| \leq (2d - 1)^{(k-m)/2} k^{\deg(P)+t-1} |P|(cdr + m)^{ct}, \tag{2.16}$$

and with  $Q$  of degree  $\deg(P) + t - 1$  with

$$|Q| \leq |P|(cdr + m)^{ct}. \tag{2.17}$$

**Proof** We can assume  $k_1, \dots, k_s$  are all in  $K_1$ , that  $m_1 = \dots = m_s = 1$ , and that  $K_1 = \{k_1, \dots, k_u\}$  with  $m_{s+1}, \dots, m_u$  all greater than 1. Writing

$$P(k_1, \dots, k_t) = \sum_{l=0}^{\deg(P)} \bar{P}_l(k_1, \dots, k_s) \bar{R}_l(k_{s+1}, \dots, k_t)$$

with  $\bar{R}_l$  of degree  $l$ , by sublemma 2.15 we can write the sum in equation 2.15 as

$$\sum_{l=0}^{\deg(P)} \sum_{j=m-s}^{k-s} \sum_{\substack{k_{s+1}m_{s+1} + \dots + k_t m_t = j \\ k_l \geq 1}} (2d-1)^{k_{s+1} + \dots + k_u} (-1)^{|K_2|} (2d-1)^{k-j} Q_l(k-j) R_l(k_{s+1}, \dots, k_t).$$

For each fixed  $l$  we write the above sum over  $j$  as

$$\sum_{j=m-s}^{k-s} \binom{\quad}{\quad} = \sum_{j=m-s}^{\infty} \binom{\quad}{\quad} - \sum_{k-s+1}^{\infty} \binom{\quad}{\quad}$$

where  $\binom{\quad}{\quad}$  denotes the summand (for  $j$ ) in the preceding equation. Note that the number of solutions to  $k_{s+1}m_{s+1} + \dots + k_t m_t = j$  is clearly  $\leq \binom{j+t-s-1}{t-s-1}$ , and that

$$\begin{aligned} k_{s+1} + \dots + k_u &\leq \frac{1}{2}(2k_{s+1} + \dots + 2k_u) \\ &\leq \frac{1}{2}(j - m_{u+1} - \dots - m_t - (m_{s+1} - 2)k_{s+1} - \dots - (m_u - 2)k_u) \\ &\leq \frac{1}{2}(j - m_{u+1} - \dots - m_t - (m_{s+1} - 2) - \dots - (m_u - 2)) \\ &\leq \frac{1}{2}(j - m + 2(u - s) + s) \\ &\leq \frac{1}{2}(j - m - s) + t. \end{aligned}$$

It follows from the identity

$$\sum_{n=0}^{\infty} \binom{n}{\tau} \rho^n = \frac{\rho^{\tau+1}}{(1-\rho)^\tau}$$

that

$$\sum_{j=m-s}^{\infty} \binom{\quad}{\quad} = (2d-1)^{k-m} \tilde{Q}_l(k)$$

where  $\tilde{Q}$  is a polynomial of degree  $\leq \deg(Q_l)$ , and  $|\tilde{Q}_l| \leq |Q_l| |R_l| (crd + m)^{crt} \leq |P| (crd + m)^{crt}$ , and that

$$\left| \sum_{k-s+1}^{\infty} \binom{\quad}{\quad} \right| \leq (2d-1)^{(k-m)/2} k^{\deg(P)+t-s-1} |P| (cdr + m)^{crt}.$$

Summing over  $l$  completes the proof.

□

**Sublemma 2.17** For any  $m_1, m_2, \dots, m_t$ , partition  $K_1, K_2, K_3$  of  $\{k_1, \dots, k_t\}$  with  $k_1 \in K_1$ ,  $t \leq 3r - 1$ , and polynomial  $P$  of degree  $\leq r + t$ , we have for any  $k \geq m = m_1 + m_2 + \dots + m_t$

$$\sum_{\substack{k_1 m_1 + k_2 m_2 + \dots + k_t m_t = k \\ k_i \geq 1}} (2d - 1)^{|K_1|} (-1)^{|K_2|} P(k_1, \dots, k_t) = \epsilon$$

where

$$|\epsilon| \leq (2d - 1)^{(k-m)/\mu} k^{\deg(P)+t-1} |P|(cdr + m)^{cr t},$$

where  $\mu$  is the smallest integer among the  $m_i$  with  $k_i \in K_1$ .

**Proof** We do the same calculation as in the last sublemma, the difference being that this time we have

$$k_1 + \dots + k_u \leq \frac{1}{\mu} (\mu k_1 + \dots + \mu k_u) \leq \frac{1}{\mu} (j - m) + u$$

and therefore get the  $1/\mu$  factor in the exponent of  $(2d - 1)$ .

□

To finish the proof of lemma 2.14, we sum over the  $3^t$  partitions of  $\{k_1, \dots, k_t\}$  into three sets,  $K_1, K_2, K_3$ , applying one of the sublemmas (noting that  $rt \leq 3r^2$ ).

□

Let

$$P_{i,T,\vec{m}}(k)$$

denote the polynomial  $P_i$  in lemma 2.14, which depends (only) on  $T$  and  $m_1, \dots, m_t$ , where we use  $\vec{m}$  to abbreviate the sequence  $(m_1, \dots, m_t)$ . For an ordered type  $T$  and multiplicities  $\vec{m}$ , let  $W(T; \vec{m})$  be the number of legal walks in  $T$  with multiplicities  $\vec{m} = (m_1, \dots, m_t)$ . By lemma 2.6,

$$\sum_{|\vec{m}|=m} W(T; \vec{m}) \leq (2r)^m$$

for types of coincidence  $\leq r$ , where  $|\vec{m}| = m_1 + \dots + m_t$ . It follows from this and lemma 2.14 that for any type of coincidence  $\leq d - 1$  the infinite sum

$$\sum_{m_1=1}^{\infty} \dots \sum_{m_t=1}^{\infty} W(T; \vec{m}) (2d - 1)^{t-m} P_{i,T,\vec{m}}(k)$$

converges for each  $i$  to a polynomial of degree  $2i + t$ , which we denote  $f_{i,T}$ . For  $0 \leq i \leq d - 2$ , let

$$f_i(k) = \sum_{\text{coin}(T) \leq i+1} f_{i+1-\text{coin}(T),T}(k),$$

which is a polynomial in  $k$  of degree  $\leq 2i + (3i + 2) = 5i + 2$ , since  $t \leq 3i + 2$  for any  $T$  of coincidence  $\leq i + 1$ .

**Theorem 2.18** *Let  $d$  and  $r \leq \sqrt{2d-1}/2$  be fixed positive integers. For any fixed  $v \in \{1, \dots, n\}$  we have*

$$E \left\{ \sum_{w \in \text{Irred}_k} v \xrightarrow{w} v \right\} = 2d(2d-1)^{k-1} \left( \frac{1}{n} f_0(k) + \frac{1}{n^2} f_1(k) + \dots + \frac{1}{n^r} f_{r-1}(k) + \text{error} \right),$$

where

$$\text{error} \leq \frac{k^{2r+2}}{n^{r+1}} (1 + 2k^{2r} e^{(r+1)k/n}) + (2d-1)^{-k/2} \sum_{i=0}^{r-1} \frac{(ckid)^{ci^2}}{n^{i+1}}$$

and where the  $f_i$  are polynomials of degree  $\leq 5i+2$  whose coefficients are bounded by  $(cdr)^{cr^2}$ , where  $c$  is an absolute constant (independent of  $d$  and  $r$ ). For any

$$k \leq \frac{1}{cd^{3/2}} n^{1/(c\sqrt{d})}$$

we have

$$\text{error} \leq \frac{k^{4r+2}}{n^{r+1}} c + (2d-1)^{-k/2} \frac{(ckd)^c}{n}.$$

**Proof** By virtue of the one-to-one correspondence between pairs  $(\Gamma, w)$  and triples  $(T, y, \ell)$  we have

$$E \left\{ \sum_{w \in \text{Irred}_k} i \xrightarrow{w} i \right\} = \frac{1}{n} \sum_T \sum_{m_1, \dots, m_t} \left[ W(T; \vec{m}) \sum_{\substack{k_1, \dots, k_t \geq 1 \\ k_1 m_1 + \dots + k_t m_t = k}} \sum_{\ell \in \mathcal{L}_{T, k_1, \dots, k_t}} \text{Pr}(T, \ell) \right],$$

where  $\text{Pr}()$ , as with  $p_i$ , extends from a function on general forms,  $\Gamma$ , to a function on pairs  $(T, \ell)$ . The probability that the walk along  $v$  starting at  $i$  has more than  $r$  coincidences is no more than the number of ways of choosing  $r+1$  points of the  $k$  unknown points times the probability that each of these points is indeed a coincidence. Thus we can replace the summation over all types,  $T$ , above, with a summation over all types  $T$  of coincidence  $\leq r$  while incurring an error of no more than

$$2d(2d-1)^{k-1} \binom{k}{r+1} \left( \frac{k}{n-k} \right)^{r+1} \leq 2d(2d-1)^{k-1} \frac{k^{2r+2}}{n^{r+1}}$$

if  $k \leq n/2$ . Obviously the second summation has non-zero terms only when  $m = m_1 + \dots + m_t \leq k$ ; we shall restrict the second sum to being over such  $m$ .

The total number of  $\text{Pr}(T, \ell)$ 's occurring in the above quadruple summation is the same as the total number of  $(\Gamma, w)$  pairs (and less when we restrict ourselves to types of coincidence  $\leq r-1$ ). For each  $w \in \text{Irred}_k$  there are no more than

$$\sum_{j=0}^r \binom{k}{j} k^j \leq 2k^{2r}$$

compatible  $\Gamma$ 's of coincidence  $\leq r$ , because  $\Gamma$  there is at most one  $\Gamma$  for any  $j \leq r$  specified points of coincidence (among the  $k$  unknown points  $t_1, \dots, t_k$  of the walk) once we specify at which of the previous vertices each of the  $j$  coincidences arrive. By lemma 2.7 we have for each  $T$  of coincidence  $\leq r$

$$\Pr(T, \ell) = \frac{p_0}{n^{\text{coin}(T)-1}} + \frac{p_1}{n^{\text{coin}(T)}} + \dots + \frac{p_{r-1}}{n^{r-1}} + \frac{\epsilon}{n^r},$$

where  $\epsilon$  is bounded by

$$e^{(r+1)k/n} k^{2r+2}.$$

It follows that

$$\frac{1}{n} \sum_{\text{coin}(T) \leq r} \sum_{m \leq k} W(T; \vec{m}) \sum_{\vec{k}, \ell} \Pr(T, \ell) = \frac{1}{n} \left( \tilde{f}_0(k) + \frac{\tilde{f}_1(k)}{n} + \dots + \frac{\tilde{f}_{r-1}(k)}{n^{r-1}} + \frac{\delta}{n^r} \right) \quad (2.18)$$

(we have abbreviated the inner two summations by  $\sum_{\vec{k}, \ell}$ ) where

$$\delta \leq k^{2r+2} (1 + 2k^{2r} e^{(r+1)k/n}), \quad (2.19)$$

and

$$\tilde{f}_i(k) = \sum_{\text{coin}(T) \leq i+1} \sum_{|\vec{m}| \leq k} W(T; \vec{m}) \sum_{\vec{k}, \ell} p_{i+1-\text{coin}(T)}(T, \ell).$$

We claim that  $\tilde{f}_i(k)$  do not differ much from the infinite sum

$$f_i(k) = \sum_{\text{coin}(T) \leq i+1} \sum_{\vec{m}} W(T; \vec{m}) (2d-1)^{t-m} P_{i+1-\text{coin}(T), T, \vec{m}}(k). \quad (2.20)$$

By lemma 2.14 we have for any type of coincidence  $\leq i+1$ , for  $i \leq r-1$ ,

$$\left| (2d-1)^{t-m} P_{i+1-\text{coin}(T), T, \vec{m}}(k) - \sum_{\vec{k}, \ell} p_{i+1-\text{coin}(T)}(T, \ell) \right| \leq (2d-1)^{(k-m)/2} k^{(3(i+1)-2)+2(i+1)} (crd+m)^{ci^2}.$$

Since there are less than  $(2i+2)^{6i+5}$  types of coincidence  $\leq i+1$ , and since for each such type there are  $\leq (2i+2)^m \leq (2d-1)^{m/2}$  legal walks with  $|\vec{m}| = m$ , we have

$$\sum_{\text{coin}(T) \leq i+1} \sum_{m \leq k} W(T, \vec{m}) \left| (2d-1)^{t-m} P_{i+1-\text{coin}(T), T, \vec{m}}(k) - \sum_{k_i} \sum_{\ell} p_i(T, \ell) \right| \leq (2d-1)^{k/2} (ckid)^{ci^2}. \quad (2.21)$$

It remains to estimate

$$\left| f_i(k) - \sum_{\text{coin}(T) \leq i+1} \sum_{|\vec{m}| \leq k} W(T, \vec{m}) (2d-1)^{t-m} P_{i+1-\text{coin}(T), T}(k) \right| \leq \sum_{\text{coin}(T) \leq i+1} \sum_{|\vec{m}| > k} W(T, |\vec{m}|) (2d-1)^{t-m} \left| P_{i+1-\text{coin}(T), T}(k) \right|. \quad (2.22)$$



By lemma 2.14, each  $P_{i,T,\vec{m}}$  has degree  $\leq t + 2i$ , and coefficients bounded by  $(cid + m)^{ci^2}$ . Thus we have

$$\left| P_{i+1-\text{coin } T, T, \vec{m}}(k) \right| \leq k^{t+2i} (cid + m)^{ci^2}$$

and thus for a type,  $T$ , of coincidence  $\leq i + 1$  we have that the expression in equation 2.22 is bounded by

$$k^{5i+2} (2d - 1)^{3i+2} \sum_{m=k+1}^{\infty} \left( \frac{2i+2}{2d-1} \right)^m (cid + m)^{ci^2} \leq (ckid)^{ci^2} \left( \frac{2i+2}{2d-1} \right)^k$$

(for  $i \leq d - 2$ ). Combining the above with equations 2.18, 2.19, and 2.21 yields the first estimate on the error term. For

$$k \leq \frac{1}{c'd^{3/2}} n^{1/c'\sqrt{d}},$$

assuming  $i \leq \sqrt{2d-1}/2$ , the sum

$$\sum_{i=0}^{r-1} \frac{(ckid)^{ci^2}}{n^{i+1}}$$

is clearly bounded by the first term. Finally,  $f_i$ 's coefficients are bounded by

$$(2i+2)^{6i+5} \sum_{m=1}^{\infty} (cid + m)^{ci^2} \left( \frac{2i+2}{2d-1} \right)^m \leq (c'id)^{c'i^2}$$

(for  $i \leq d - 2$ ).

□

### 3 Consequences of the Expansion

We start by computing  $f_0, f_1, \dots$  in theorem 2.18. Without too much work, one can use methods akin to those of section 2 to argue that  $f_0(k) = 1$  and if  $f_1(k)$  is non-zero, then its leading coefficient is negative (which is as good as proving  $f_1(k) = 0$  as far as eigenvalue estimates are concerned). Direct arguments for other  $f_i$ 's seem harder to come by. We will argue by using previously known facts about the second eigenvalue, which will give the desired values for the  $f_i$ 's. These values for  $f_i$  will in turn give much sharper information on the second eigenvalues, as we will show later in this section. We warn the reader that other literature on this subject often works with  $d$  regular, not  $2d$  regular, graphs; to maintain consistency when quoting results or using techniques from previous work, we will sometimes state theorems in terms of  $d$  regular graphs.

**Theorem 3.1** *For every  $d > 2$  there is a constant  $\alpha > 0$  such that for a random graph  $G \in \mathcal{G}_{n,2d}$  we have*

$$\begin{aligned} \Pr \{|\lambda_2(G)| \leq 2d - \alpha\} &= 1 - \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right), \\ \Pr \{\lambda_2(G) = 2d\} &= \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right). \end{aligned}$$

**Proof** A graph,  $G$ , with  $n$  vertices is said to be a  $\gamma$ -magnifier if for all subsets of vertices,  $A$ , of size  $\leq n/2$  we have

$$|\Gamma(A) - A| \geq \gamma|A|,$$

where  $\Gamma(A)$  denotes those vertices connected to some member of  $A$  by an edge. It is known that any  $d$  regular  $\gamma$ -magnifier has

$$\lambda_2(G) \leq d - \frac{\gamma^2}{4 + 2\gamma^2}$$

(see [Alo86]). We say that  $G$  is a  $\gamma$ -expander if for all subsets of vertices,  $A$ , of size  $\leq n/2$  we have

$$|\Gamma(A)| \geq (1 + \gamma)|A|.$$

A standard counting argument gives the following.

**Lemma 3.2**  *$G \in \mathcal{G}_{n,2d}$  is not a  $\gamma$ -expander with probability*

$$\leq \sum_{m=1}^{n/2} \binom{n}{m} \binom{n}{\lfloor (1 + \gamma)m \rfloor} \left(\frac{m + \lfloor m\gamma \rfloor}{n}\right)^{md},$$

and more precisely

$$\leq \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right) + \sum_{m=3}^{n/2} \binom{n}{m} \binom{n}{\lfloor (1 + \gamma)m \rfloor} \left(\frac{m + \lfloor m\gamma \rfloor}{n}\right)^{md},$$

for  $\gamma < 1/3$ .

**Proof** For fixed subsets of  $V$ ,  $A$  and  $B$ , we have

$$\begin{aligned} \Pr \{\Gamma(A) \subset B\} &\leq \Pr \{\pi_i(a) \in B \quad \forall a \in A, \quad \forall i \in \{1, \dots, d\}\} \\ &\leq \left(\frac{|B|}{n} \frac{|B| - 1}{n - 1} \cdots \frac{|B| - |A| + 1}{n - |A| + 1}\right)^d \leq \left(\frac{|B|}{n}\right)^{d|A|}. \end{aligned}$$

If  $G$  is not a  $\gamma$ -expander, then there exist some  $m \in \{1, \dots, n/2\}$ ,  $|A| = m$ , and  $|B| = \lfloor (1 + \gamma)m \rfloor$  with  $\Gamma(A) \subset B$ . This yields the first estimate. To refine this

estimate, notice that the bound given for  $\Pr\{\Gamma(A) \subset B\}$  is only tight when  $A = B$ . Indeed, if  $A$  and  $B$  are disjoint, then for  $\Gamma(A)$  to be contained in  $B$  we need both  $\pi_i$  and  $\pi_i^{-1}$  to map  $A$  into  $B$  for each  $\pi_i$ , which roughly doubles the amount of conditioning on each  $\pi_i$ . In particular, for  $|A| = |B| = 1$ ,  $A \neq B$  we have

$$\Pr\{\Gamma(A) \subset B\} \leq \left(\frac{1}{n(n-1)}\right)^d,$$

and a similar analysis for the case  $|A| = 2$  and  $|A \cap B| = 1$  or  $0$ , giving the second estimate.  $\square$

Assuming  $\gamma < 1/3$  and using the estimate  $\binom{n}{m} \leq (ne/m)^m$  we can bound the probability that  $G \in \mathcal{G}_{n,2d}$  is not a  $\gamma$ -expander by

$$\frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right) + \sum_{m=3}^{n/2} \binom{m}{n}^{m^{d-2}} \left((1+\gamma)^{2d+1} e^{2+\gamma}\right)^m = \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right)$$

if  $\gamma$  is sufficiently close to 0 and  $d > 2$ . If  $G$  is a  $\gamma$ -expander, then clearly  $G^2$ , the graph whose adjacency matrix is the square of  $G$ 's, is also a  $\gamma$ -expander, and thus also a  $\gamma$ -magnifier. Therefore

$$(\lambda_2(G))^2 \leq 4d^2 - \frac{\gamma^2}{4 + 2\gamma^2}$$

with probability at least

$$1 - \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right).$$

On the other hand,  $\lambda_2(G) = d$  if  $G$  has an isolated vertex. By the inclusion-exclusion principle, this happens with probability

$$\leq n \binom{1}{n^d}$$

and

$$\geq n \binom{1}{n^d} - \binom{n}{2} \left(\frac{1}{n^d}\right)^2.$$

$\square$

Let  $G = (V, E)$  be an undirected  $d$ -regular graph. By a *non-backtracking walk* in  $G$  we mean a walk that at no point in the walk traverses an edge and then on the next step traverses the same edge in the reverse direction. For vertices  $v, w$ , and integer  $k$ , let  $F_k(v, w)$  be the number of non-backtracking walks of length  $k$  from  $v$  to  $w$ . Let

$$W_G(k) = \sum_{v \in V} F_k(v, v).$$

Theorem 2.18 gives us the expected value of  $W_G(k)$  for a random  $G \in \mathcal{G}_{n,2d}$ . In the spirit of [LPS86] we have:

**Lemma 3.3** *Let the eigenvalues of  $G$ 's adjacency matrix be  $\lambda_1, \dots, \lambda_n$ . Then*

$$W_G(k) = \sum_{i=1}^n q_k(\lambda_i),$$

where  $q_k$  is a polynomial of degree  $k$ . It is given by

$$q_k(2\sqrt{d-1} \cos \theta) = (\sqrt{d-1})^k \left( \frac{2}{d-1} \cos k\theta + \frac{d-2}{d-1} \frac{\sin(k+1)\theta}{\sin \theta} \right).$$

**Proof** Regarding  $F_k$  as an  $n \times n$  matrix, we easily see that  $F_0 = I$ ,  $F_1 = A$ , where  $A$  is the adjacency matrix of  $G$ ,

$$F_2 = A^2 - dI,$$

and for any  $t \geq 3$  we have

$$F_t = AF_{t-1} - (d-1)F_{t-2}.$$

It follows that

$$F_k = q_k(A)$$

where  $q_k$  is some  $k$  degree polynomial independent of  $A$ , and that

$$W_G(k) = \text{Trace}(F_k) = \text{Trace}(q_k(A)) = \sum_{i=1}^n q_k(\lambda_i).$$

One can solve for  $q_k$  explicitly by noting that for fixed  $\lambda$ ,  $q_k = q_k(\lambda)$  satisfies the simple recurrence

$$q_k = \lambda q_{k-1} - (d-1)q_{k-2}.$$

One therefore has

$$q_k = c_1(r_1)^k + c_2(r_2)^k$$

for some  $c_1, c_2$ , where  $r_1, r_2$  are the roots of

$$r^2 - \lambda r + (d-1) = 0.$$

One can directly solve for  $r_1, r_2, c_1, c_2$ , but it is easier to substitute  $\lambda = 2\sqrt{d-1} \cos \theta$ . One gets

$$r_{1,2} = \sqrt{d-1} e^{\pm i\theta},$$

and then after a few more calculations and simplifications one gets

$$q_k(2\sqrt{d-1} \cos \theta) = (\sqrt{d-1})^k \left( \frac{2}{d-1} \cos k\theta + \frac{d-2}{d-1} \frac{\sin(k+1)\theta}{\sin \theta} \right).$$

□

The  $q_k(\lambda)$ 's are quite easy to use. For example, if  $\lambda = d$ , the first eigenvalue, then clearly

$$q_k(d) = d(d-1)^{k-1}.$$

For  $|\lambda| \leq 2\sqrt{d-1}$  one has  $\lambda = 2\sqrt{d-1} \cos \theta$  for some real  $\theta$ , and therefore

$$|q_k(\lambda)| \leq (\sqrt{d-1})^k (k+2) \quad (3.1)$$

since  $|\sin(k+1)\theta/\sin\theta|$  is always  $\leq k+1$ . Otherwise, say for  $\lambda > 2\sqrt{d-1}$ , one can rewrite the identity in lemma 3.3 as

$$q_k(2\sqrt{d-1} \cosh x) = (\sqrt{d-1})^k \left( \frac{2}{d-1} \cosh kx + \frac{d-2}{d-1} \frac{\sinh(k+1)x}{\sinh x} \right)$$

and solve  $\lambda = 2\sqrt{d-1} \cosh x$ . For  $\lambda < -2\sqrt{d-1}$  one solves for  $\lambda = -2\sqrt{d-1} \cosh x$  and proceeds similarly. For  $|\lambda|$  near  $2\sqrt{d-1}$ ,  $\sin \theta$  or  $\sinh x$  is very close to 0, and it is convenient to write

$$\frac{\sin(k+1)\theta}{\sin\theta} = \cos k\theta + \cos\theta \cos(k-1)\theta + \cos^2\theta \cos(k-2)\theta + \dots + \cos^k\theta$$

for estimating. In particular, since as  $x \geq 0$  increases  $\lambda = 2\sqrt{d-1} \cosh x$  increases, we have that for all  $\lambda$  with  $2\sqrt{d-1} \leq \lambda \leq d - \alpha$ ,

$$x = \cosh^{-1} \left( \frac{\lambda}{2\sqrt{d-1}} \right)$$

must be

$$\leq \cosh^{-1} \left( \frac{d}{2\sqrt{d-1}} \right) - \epsilon$$

for some positive  $\epsilon$ . Doing the same for  $\lambda$  negative we therefore get

**Lemma 3.4** *For any  $\alpha > 0$  there is a  $\delta > 0$  such that if  $|\lambda| \leq d - \alpha$ , then*

$$|q_k(\lambda)| \leq (k+2)(d-1-\delta)^k.$$

**Proof** Using

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

it follows that

$$\cosh kx \geq \cosh x \cosh(k-1)x \geq \cosh^2 x \cosh(k-2)x \geq \dots$$

Also,  $\cosh kx \leq e^{kx} \leq (d-1-\delta)^k$  for all  $x$  corresponding to  $|\lambda| \leq d - \alpha$ ,  $|\lambda| \geq 2\sqrt{d-1}$ . For  $|\lambda| \leq 2\sqrt{d-1}$ , the lemma follows from equation 3.1

□

**Theorem 3.5** *In theorem 2.18, with  $r \leq \lfloor \sqrt{2d-1}/2 \rfloor$ , we have  $f_0(k) = 1$  and  $f_j(k) = 0$  for  $j \geq 1$ , if  $d-2 > \sqrt{2d-1}/2$  (i.e.  $d \geq 4$ ).*

**Proof** By theorem 3.1 we have that (remember, the degree is now  $2d$ )

$$\begin{aligned} \mathbb{E} \left\{ \sum_{w \in \text{Irred}_k} v \xrightarrow{w} v \right\} &= \left( 1 - \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right) \right) \left( 2d(2d-1)^{k-1} + O((k+2)(2d-1-\delta)^k) \right) + \\ &\quad \left( \frac{1}{n^{d-1}} + O\left(\frac{1}{n^{2d-2}}\right) \right) O(n2d(2d-1)^{k-1}). \end{aligned}$$

Taking  $k$  even and  $k = \lfloor (\log n)^2 \rfloor$  or  $k = \lfloor (\log n)^2 \rfloor + 1$ , letting  $n \rightarrow \infty$ , and noticing that  $(2d-1-\delta)^k/(2d-1)^k$  and  $(2d-1)^{-k/2}$  are less than any polynomial in  $n$  as  $n \rightarrow \infty$  yields  $f_0(k) = 1$  and  $f_j(k) = 0$  for any  $j \geq 1$ . □

To restate the results so far:

**Corollary 3.6** *For any fixed  $v$ ,  $k \geq 1$ , and  $d-2 > \sqrt{2d-1}/2$  (i.e.  $d \geq 4$ ) we have*

$$\mathbb{E} \left\{ \sum_{w \in \text{Irred}_k} v \xrightarrow{w} v \right\} = 2d(2d-1)^{k-1} \left( \frac{1}{n} + \text{error}_{n,k} \right),$$

where

$$\text{error}_{n,k} \leq (ckd)^c \left( \frac{k^2 \sqrt{2d}}{n^{1+\lfloor \sqrt{2d-1}/2 \rfloor}} + \frac{(2d-1)^{-k/2}}{n} \right).$$

□

Now we use this corollary to estimate the expected sum of the  $k$ -th powers of the eigenvalues. Any word in  $\Pi^k$  can be *reduced* by repeatedly cancelling all consecutive occurrences of  $\pi, \pi^{-1}$  in the word, until we get an irreducible word; this irreducible word is independent of how the reducing was done. Notice that

$$\frac{1}{(2d)^k} \mathbb{E} \left\{ \sum_{w \in \Pi^k} v \xrightarrow{w} v \right\} = p_{k,0} + \sum_{s=1}^k p_{k,s} \frac{1}{2d(2d-1)^{s-1}} \mathbb{E} \left\{ \sum_{w \in \text{Irred}_s} v \xrightarrow{w} v \right\},$$

where  $p_{k,s}$  is the probability that a random word in  $\Pi^k$  reduces to an irreducible word of size  $s$ . Since  $\sum_s p_{k,s} = 1$ , we have

$$\frac{1}{(2d)^k} \sum_v \mathbb{E} \left\{ \sum_{w \in \Pi^k} v \xrightarrow{w} v \right\} = 1 + (n-1)p_{k,0} + \sum_{s=1}^k n p_{k,s} \text{error}_{n,s}$$

and therefore

$$\mathbb{E} \left\{ \sum_{i=2}^n \lambda_i^k \right\} = (2d)^k (n-1) p_{k,0} + (2d)^k \sum_{s=1}^k n p_{k,s} \text{error}_{n,s}. \quad (3.2)$$

To estimate the above, first notice that  $p_{k,s} = 0$  if  $k$  and  $s$  have different parity. In [BS87], the following estimate is given:

**Lemma 3.7**

$$p_{2k,2s} \leq \frac{2s+1}{2k+1} \binom{2k+1}{k-s} \left(\frac{1}{2d}\right)^{k-s} \left(1 - \frac{1}{2d}\right)^{2s-1}$$

**Proof** See [BS87]; for an exact formula and sharper bounds, see [McK81]. □

Incidentally, from the proof of the above lemma in [BS87] it is clear that also

$$p_{2k,0} \geq \frac{1}{2k+1} \binom{2k+1}{k} \frac{(2d-1)^k}{(2d)^{2k}}.$$

It follows that for *any* graph of degree  $2d$ ,

$$\sum_{i=1}^n \lambda_i^{2k} \geq (2d)^{2k} (n-1) p_{k,0} \approx (n-1) 2^{2k} (2d-1)^k,$$

so that taking  $2k$  slightly less than  $2 \log_d n$  yields the lower bound mentioned in the introduction,

$$|\lambda_2| \geq 2\sqrt{2d-1} + O\left(\frac{1}{\log_d n}\right).$$

Now we take  $k = 2 \lfloor \log n \lfloor \sqrt{2d-1}/2 \rfloor / \log d \rfloor$ , so that  $k$  is even, and calculate using the simplified bound

$$p_{2k,2s} \leq 2^{2k} \left(\frac{1}{2d}\right)^{k-s}.$$

It is easy to see that the dominant terms of the summation over  $s$  in equation 3.2 are  $s = 1$  and  $s = k$ , and therefore

$$\mathbb{E} \left\{ \sum_{i=2}^n \lambda_i^k \right\} \leq n^{1 + \frac{\log 2}{\log d}} (ckd)^c k^{2\sqrt{2d}} \left( 2\sqrt{2d} \sqrt{\frac{2d}{2d-1}} \right)^k.$$

Taking  $k$ -th roots, applying Hölder's (or Jensen's) inequality, and noticing that

$$\left( n^{1 + \frac{\log 2}{\log d}} \right)^{1/k} = 1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right),$$

that

$$(ckd)^{c/k} k^{2\sqrt{2d}/k} = 1 + O\left(\frac{\log d \log \log n}{\log n}\right),$$

and that

$$k \leq \frac{1}{cd^{3/2}} n^{\frac{1}{c\sqrt{d}}}$$

for

$$\frac{\log n}{\log \log n} \geq c'\sqrt{d}$$

yields:

**Theorem 3.8** For  $G \in \mathcal{G}_{n,2d}$  we have

$$E\{|\lambda_2(G)|\} \leq 2\sqrt{2d-1} \left(1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right)\right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right)$$

(with an absolute constant in the  $O(\cdot)$  notation), and more generally

$$E\{|\lambda_2(G)|^m\} \leq \left(2\sqrt{2d-1} \left(1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right)\right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right)\right)^m$$

for any  $m \leq 2\lfloor \log n \lfloor \sqrt{2d-1}/2 \rfloor / \log d \rfloor$ .

□

As a corollary we also get:

**Theorem 3.9** For any  $\beta > 1$  we have

$$|\lambda_2(G)| \geq \left(2\sqrt{2d-1} \left(1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right)\right) + O\left(\frac{d^{3/2} \log \log n}{\log n}\right)\right) \beta$$

with probability

$$\leq \frac{\beta^2}{n^{2\lfloor \sqrt{2d-1}/2 \rfloor \log \beta / \log d}}.$$

## 4 Concluding Remarks

An interesting question would be to see if an expansion in theorem 2.18 exists for  $r > \sqrt{2d-1}/2$ . In section 2 we defined  $f_i(k)$  really for any  $i$  such that

$$\sum_{m=0}^{\infty} \left(\frac{2i+2}{2d-1}\right)^m$$



converges, which is to say any  $i \leq d - 2$ . We only used  $f_i(k)$  for  $i \leq (\sqrt{2d - 1}/2) - 1$ , because in equation 2.21, we needed to use

$$(2i + 2)^m \leq (2d - 1)^{m/2}$$

in order to get a reasonable bound on the error term. If there were some way to bound this error term by a reasonable quantity for larger  $i$ , one could extend the expansion, with the  $n^{-j}$  coefficient being  $f_{j-1}(k)$ . Whether or not  $f_i(k)$  for  $i \geq (\sqrt{2d - 1}/2) - 1$  is involved in an asymptotic expansion, it might be interesting to evaluate them, since they seem to be a fairly naturally defined quantity.

If one were able to get an expansion up to  $r$ -th order for  $r$  close to  $d$ , one might notice  $\mathcal{G}_{n,2d}$ 's weakness, that according to theorem 3.1 a graph in  $\mathcal{G}_{n,2d}$  has a  $d$  as a multiple eigenvalue with probability on the order of  $n^{-d-1}$ . From the arguments in section 3, it would follow that the expansion, if it existed, would fail to vanish by the  $n^{-d+1}$  term. However, one might hope for a better second eigenvalue estimate through various approaches. First of all, one might be able to find the graphs which give the non-vanishing terms, such as graphs with isolated vertices for the  $n^{-d+1}$  term, and thus show that upon removing them (which does not affect the probability measure very much) one has a better moment estimate. Secondly, one might do better with a different probability distribution, say  $\mathcal{H}_{n,2d}$ , which is constructed like  $\mathcal{G}_{n,2d}$  but only allows permutations,  $\pi$ , which consist of one cycle. There are  $(n - 1)!$  such permutations, and once  $a$  values of such a  $\pi$  have been determined (which do not already give  $\pi$  a cycle), there are

$$\frac{(n - 1)!}{(n - 1)(n - 2) \dots (n - a)}$$

ways of completing  $\pi$  to be a permutation of one cycle. It follows that over  $\mathcal{H}_{n,2d}$  one has

$$\Pr(\Gamma) = n(n - 1) \dots (n - |V_\Gamma| + 1) \prod_{j=1}^d \frac{1}{(n - 1)(n - 2) \dots (n - a_j(\Gamma))},$$

and therefore the asymptotic expansion and second eigenvalue theorems hold for  $\mathcal{H}_{n,2d}$  as well. However, in  $\mathcal{H}_{n,2d}$  one can never have an isolated vertex or similar phenomena, and so one would have a much better theorem 3.1 for  $\mathcal{H}_{n,2d}$ , and a possibility of having a further than  $d$ -th order expansion for which all lower order terms vanish.

## References

- [Alo86] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.
- [BS87] Andrei Boppana and Eli Shamir. On the second eigenvalue of random regular graphs. In *28th Annual Symposium on Foundations of Computer Science*, pages 286–294, 1987.

- [FK81] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.
- [Gem80] S. Geman. A limit theorem for the norm of random matrices. *Ann. of Prob.*, 8(2):252–261, 1980.
- [Knu73] Donald E. Knuth. *The Art of Computer Programming*, volume 1: Fundamental Algorithms. Addison-Wesley, Reading, Massachusetts, second edition, 1973.
- [KS] J. Kahn and E. Szemerédi. Personal communication, and to appear.
- [LPS86] A. Lubotzky, R. Phillips, and P. Sarnak. Explicit expanders and the ramanujan conjectures. In *18th Annual ACM Symposium on Theory of Computing*, pages 240–246, 1986.
- [Mar87] G. Margulis. Manuscript in Russian on graphs with large girth, 1987.
- [McK81] B. McKay. The expected eigenvalue distribution of a large regular graph. *Lin. Alg. Appl.*, 40:203–216, 1981.
- [Tan84] R.M. Tanner. Explicit concentrators from generalized N-gons. *SIAM J. Alg. Disc. Methods*, 5:287–293, 1984.
- [Wig55] E. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Math.*, 62(3):548–564, 1955.