Non-Hermitian random matrices

Boris Khoruzhenko
School of Mathematical Sciences
Queen Mary, University of London

`http://www.maths.qmw.ac.uk/~boris`

The Diablerets Winter School
"Random Matrices"
(18-23 March 2001)

Plan:

- Non-Hermitan random matrices: Circular law
- Weakly non-Hermitian random matrices
- Asymmetric tridiagonal random matrices

Part I. Circular law

Consider random matrices M_n of size n.

Eigenvalue counting measure:

$$N(D, M_n) = \frac{1}{n} \#\{\text{eigvs. of } M_n \text{ in } D\}$$

What is the limit of $N(D, M_n)$ when $n \to \infty$?

If Hermitian (or real symmetric) matrices, then $dN(\lambda; M_n)$ is supported on $\mathbf{R}$. These tools work well:

(a) moments

$$\int \lambda^m dN(\lambda; M_n) = \frac{1}{n} \sum_{l=1}^{n} \lambda_l^m = \frac{1}{n} \operatorname{tr} M_n^m$$

(b) Stieltjes transform

$$\int \frac{dN(\lambda; M_n)}{\lambda - z} = \frac{1}{n} \sum_{l=1}^{n} \frac{1}{\lambda - z} = \frac{1}{n} \operatorname{tr}(M_n - zI)^{-1},$$

defined for all $\operatorname{Im} z \neq 0$

(c) orthogonal polynomials, etc ...

Not so many tools are available for complex eigenvalues!

Complex (or real asymmetric) matrices – availability of tools:

(a) moments fail;

(b) Stieltjes transform is difficult to use because of singularities; best hope - spectral boundary(ies);

(c) orthogonal polynomials (use of this method is essentially limited to Gaussian random matrices)

(d) potentials: if $p(z) = \int_{\mathbf{C}} \ln|z-\zeta| dN(\zeta)$, $z \in \mathbf{C}$, then

$$\frac{1}{2\pi}\Delta p = dN \qquad \text{(as distributions in } \mathcal{D}' \text{)}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian.

Potentials

$$
\begin{aligned}
p(z; M_n) &= \frac{1}{n}\sum_{l=1}^{n} \ln|z - z_l| \\
&= \frac{1}{2n} \ln\det(M_n - zI_n)(M_n - zI_n)^*
\end{aligned}
$$

Two strategies:

- Obtain

  $$\lim_{n\to\infty} p(z; M_n) = F(z) \quad (\text{in } \mathcal{D}').$$

  Then the limiting eigenvalue distribution is $\frac{1}{2\pi}\Delta F(z)$.

  Works well when have eigenvalue curves

- Regularize potentials

  $$
  \begin{aligned}
  p_\varepsilon(z; M_n) &= \frac{1}{n}\ln\det\left[(M_n - zI_n)(M_n - zI_n)^* + \varepsilon^2 I_n\right] \\
  &= \frac{1}{n}\int \ln(\lambda + \varepsilon) dN(\lambda; H_{n,z}),
  \end{aligned}
  $$

  where $H_{n,z} = (M_n - zI_n)(M_n - zI_n)$.

  Naive approach: let $n \to \infty$ and after $\varepsilon \to 0$; difficult to justify for non-normal matrices. The two limits commute for normal matrices and do not commute if M_n have orthogonal or almost orthogonal right and left eigenvectors.

Regularization of potentials

$$p_\varepsilon(z; M_n) = \frac{1}{2n} \log \det[(M_n - z)(M_n - z)^* + \varepsilon^2 I_n]$$

$$\begin{aligned} \frac{1}{2\pi}\Delta p_\varepsilon(z; M_n) &= \rho_\varepsilon(z; M_n) \\ &\underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{n}\sum \delta(z - z_j) \quad [n \text{ is finite}] \end{aligned}$$

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} p_\varepsilon(z; J_n) = \lim_{n\to\infty} \lim_{\varepsilon\to 0} p_\varepsilon(z; J_n) \quad ??$$

Yes, for normal matrices. Counterexamples for non-normal matrices

In the vicinity of z_j:

$$\begin{aligned} \rho_\varepsilon(z; M_n) &\doteq \frac{(\kappa_j\varepsilon)^2}{\pi} \frac{1}{[(\kappa_j\varepsilon)^2 + |z - z_j|^2]^2} \\ &\underset{\varepsilon\to 0}{\longrightarrow} \delta(z - z_j) \quad \text{if } \kappa_j \neq 0 \end{aligned}$$

where $\kappa_j = |(\psi_j^L, \psi_j^R)^{-1}|$ and $\psi_j^{L(R)}$ are normalized left (right) eigevectors at z_j.

Spectral condition numbers, pseudospectra, etc.

Consider complex matrices $J_n = ||J_{lm}||_{l.m=1}^n$

- $\{J_{ml}\}_{l,m=1}^n$ are indp. standard complex normals

(with normalization: $E(|J_{ml}|^2) = 1$).

Theorem *(Ginibre) If f is a symmetric functional of the eigenvalues of J_n then*

$$E(f) = \underbrace{\int \ldots \int}_{\mathbf{C}^n} f(z_1, \ldots z_n) p_n(z_1, \ldots z_n) d^2 z_1 \cdots d^2 z_n,$$

where

$$p_n(z_1, \ldots z_n) = \frac{1}{\pi^n \prod_{l=1}^n l!} \; e^{-\sum_{l=1}^n |z_l|^2} \prod_{1 \le l < m \le n} |z_l - z_m|^2$$

Notation: $N(D; J) = \#\{\text{eigvs. of } J \text{ in } D\}$

Corollary

$$E(N(D; J_n)) = \int_D E(|\det(J_{n-1} - zI_{n-1})|^2) \, \frac{e^{-|z|^2} d^2 z}{\pi (n-1)!},$$

where J_{n-1} is an $(n-1) \times (n-1)$ matrix of independent standard complex normals.

Ginibre's theorem: sketch of Dyson's proof

- p.d.f. of joint distribution of the matrix entries:

$$\left(\frac{1}{\pi}\right)^{n^2} \exp\left(-\sum_{l,m=1}^{n} |J_{lm}|^2\right) = \left(\frac{1}{\pi}\right)^{n^2} \exp\left(-\operatorname{tr} JJ^*\right)$$

- assign a label to each of the eigenvalues

- Schur decomposition $J_n = U(Z+T)U^*$, where
  U is unitary,
  T is strictly upper-triangular, complex,
  $Z = \operatorname{diag}(z_1, \ldots, z_n)$

- $J_n \to ([U], Z, T)$ is one-to-one,
  $[U] = \{UV : V = \operatorname{diag}(e^{i\phi_1}, \ldots, e^{i\phi_n})\}$
  Jacobian = $\prod_{1\le l<m\le n} |z_l - z_m|^2$

-

$$\begin{aligned} \operatorname{tr} J_n J_n^* &= \operatorname{tr}(Z+T)(Z+T)^* \\ &= \operatorname{tr} ZZ^* + \operatorname{tr} TT^* = \sum_{l=1}^{n} |z|^2 + \sum_{l<m} |T_{lm}|^2 \end{aligned}$$

- $[U]$ and T can easily be integrated out

- remove eigenvalues labelling

Proof of Corollary:

Use Ginibre's theorem for $f = \sum_{l=1}^{n} \chi_D(z_l)$:

$$E(N(D; J_n)) = n \int_D \Big\{ \int_{\mathbf{C}^{n-1}} \cdots \int p_n(z_1, \dots z_n) d^2 z_2 \cdots d^2 z_n \Big\} d^2 z_1$$

Now note that

$$p_n(z_1, z_2, \dots, z_n) = \frac{1}{\pi n!} \; e^{-|z_1|^2} \prod_{m=2}^{n} |z_1 - z_m|^2 p_{n-1}(z_2, \dots, z_n).$$

To complete the proof, use Ginibre's theorem (backwards) for $f = \prod_{m=2}^{n} |z_1 - z_m|^2 = |\det(J_{n-1} - z I_{n-1})|^2$.

Another (more direct) proof:

$$J_n = U \begin{pmatrix} z & \underline{w} \\ 0 & J_{n-1} \end{pmatrix} U^*$$

Here $\underline{w} \in \mathbf{C}^{n-1}$, z is an eigenvalue of J_{n-1}, and U is a unitary matrix that exchanges the corresponding eigenvector (normalized) and $(1, 0, \dots, 0)$.

Jacobian is $|\det(z I_{n-1} - J_{n-1})|^2$ and

$$\mathrm{tr}\, J_n J_n^* = |z|^2 + \underline{w w^*} + \mathrm{tr}\, J_{n-1} J_{n-1}^*.$$

The entries of $\underline{w}$ and J_{n-1} are independent complex normal variables.

$E(|\det(J_n - zI_n)|^2)$ is easy to compute using the independence of the entries of J_n.

Proposition *If* $A = ||A_{lm}||_{l,m=1}^n$ *and* A_{lm}, $l, m = 1, \ldots, n$, *are independent* real or complex *random variables such that* $E(A_{lm}) = 0$ *and* $E(|A_{lm}|^2) = 1$ *for all pairs* (l, m) *then*

$$E(|\det(A - zI)|^2) = n! \sum_{l=1}^{n} \frac{|z|^{2l}}{l!}.$$

Proof.

$$\det(zI - A) = z^n - z^{n-1} \sum_{l=1}^{n} A_{ll} + z^{n-2} \sum_{1 \le l < j \le n} \begin{vmatrix} A_{ll} & A_{lj} \\ A_{jl} & A_{jj} \end{vmatrix} - \ldots$$

$$= z^n - z^{n-1} m_{n-1}(A) + z^{n-2} m_{n-2}(A) - \ldots \pm m_n(A),$$

where $m_k(A)$ is the sum of all minors of A of order k (have C_n^k minors of order k). By the independence of the A_{lj}'s,

$$E(|\det(zI - A)|^2) = |z|^{2n} + |z|^{2(n-1)} E(|m_1(A)|^2) + \ldots$$

and, for every $k = 0, 1, \ldots, n$,

$$\begin{aligned} E(|m_k(A)|^2) &= C_n^k E(|\text{principal minor of order } (k-1)|^2) \\ &= \frac{n!}{k!(n-k)!} \times k! = \frac{n!}{(n-k)!} \end{aligned}$$

Therefore

$$E(|\det(zI - A)|^2) = n! \left(\frac{|z|^{2n}}{n!} + \frac{|z|^{2(n-1)}}{(n-1)!} + \ldots + 1 \right).$$

By Corollary and Proposition,

$$E(N(D; J_n)) = \int_D R_1^{(n)}(z) d^2 z,$$

where

$$R_1^{(n)}(z) = \frac{1}{\pi} e^{-|z|^2} \sum_{l=0}^{n-2} \frac{|z|^{2l}}{l!}$$

$R_1^{(n)}(z)$ is the mean density of eigenvalues of J_n

For large n, this density is approximately $\frac{1}{\pi}$ inside the circle $|z| = \sqrt{n}$ and it vanishes outside.

Consider matrices $\frac{J_n}{\sqrt{n}}$.

$$\begin{aligned} N\left(D; \frac{J}{\sqrt{n}}\right) &= \#\{\text{eigvs. of } \tfrac{J_n}{\sqrt{n}} \text{ in } D\} \\ &= \#\{\text{eigvs. of } J_n \text{ in } \sqrt{n} D\}. \end{aligned}$$

Then

$$E\left(N\left(D; \frac{J}{\sqrt{n}}\right)\right) = \int_D n R_1^{(n)}(\sqrt{n} z) d^2 z,$$

where

$$n R_1^{(n)}(\sqrt{n} z) = \frac{1}{\pi} e^{-n|z|^2} \sum_{l=0}^{n-1} \frac{n^l |z|^{2l}}{l!}$$

is the mean density of eigenvalues of $\frac{J_n}{\sqrt{n}}$.

A fact from analysis:

$$\frac{1}{\pi}e^{-n|z|^2}\sum_{l=0}^{n-1}\frac{n^l|z|^{2l}}{l!} \to \rho(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2<1 \\ 0 & \text{if } x^2+y^2>1. \end{cases}$$

Circular Law (uniform distr. of eigvs. of $\frac{J}{\sqrt{n}}$ in $|z| \leq 1$):

For any bounded $D \subset \mathbf{C}$,

$$E\left(N\left(D; \frac{J}{\sqrt{n}}\right)\right) = n \int\int_D \rho(x,y)\, dxdy + o(n).$$

Also,

the expected number of eigvs. of $\frac{J}{\sqrt{n}}$ outside $|z| \leq 1$ is

$$n\int_{|z|>1} R^{(n)}(\sqrt{n}z)d^2z \simeq \sqrt{\frac{n}{2\pi}}.$$

compare with $n^{1/6}$ for GUE.

Consider real matrices $J_n = ||J_{lm}||_{l.m=1}^n$

- $\{J_{ml}\}_{l,m=1}^n$ are independent $N(0,1)$ (real)

More difficult than complex matrices.

Non-real eigenvalues come in pairs $z_j, \overline{z}_j$.

Theorem *(Edelman) For any $D \subset \mathbf{C}_+$,*

$$E(N(D; J_n)) = \int\int_D R_1^{(n)}(x,y)dxdy,$$

$$R_1^{(n)}(x,y) = \sqrt{\frac{2}{\pi}} ye^{-(x^2-y^2)}\text{erfc } (y)\frac{E(|\det(J_{n-2} - zI_{n-2})|^2)}{(n-2)!}$$

where J_{n-2} is a matrix of independent $N(0,1)$ of size $n-2$ and

$$\text{erfc } (y) = \int_t^{+\infty} \frac{e^{-t^2/2}dt}{\sqrt{2\pi}}$$

Since $E(|\det(J_{n-2} - zI_{n-2})|^2) = (n-2)! \sum_{l=0}^{n-2} \frac{|z|^{2l}}{l!}$, we have

$$R_1^{(n)}(x,y) = \sqrt{\frac{2}{\pi}} ye^{2y^2}\text{erfc } (y)e^{-(x^2+y^2)} \sum_{l=0}^{n-2} \frac{(x^2+y^2)^l}{l!}$$

This is the mean density of eigenvalues of J_n in the upper half of the complex plane.

For matrices $\frac{J}{\sqrt{n}}$, the mean density of eigenvalues in $\mathbf{C}_+$ is $nR_1^{(n)}(\sqrt{n}x, \sqrt{}$

$$R_1^{(n)}(\sqrt{n}x, \sqrt{n}y) = g(y)e^{-n|z|^2} \sum_{l=0}^{n-2} \frac{n^l |z|^{2l}}{l!},$$

where $g(y) = \sqrt{\frac{2}{\pi}} \sqrt{n} y e^{2ny^2}$.

In the limit $n \to \infty$,

$$g(y) \to \frac{1}{\pi} \quad \text{and} \quad e^{-n|z|^2} \sum_{l=0}^{n-2} \frac{n^l |z|^{2l}}{l!} \to \begin{cases} 1 & \text{if } |z| < 1 \\ 0 & \text{if } |z| > 1 \end{cases}$$

and we have

Circular Law for real matrices

For any bounded $D \subset \mathbf{C}_+$,

$$E\left(N\left(D; \frac{J}{\sqrt{n}}\right)\right) = n \int\int_D \rho(x, y)\, dxdy + o(n).$$

where ρ is the density of the uniform distr. in $|z| \leq 1$.

Edelman proved his theorem using the following matrix decomposition:

If A_n is an $n \times n$ matrix with eigenvalue $x + iy$, $y > 0$, then there is an orthogonal O such that

$$A_n = O \begin{pmatrix} \begin{matrix} x & b \\ -c & x \end{matrix} & W \\ 0 & A_{n-2} \end{pmatrix} O^T$$

where A_{n-2} is $(n-2) \times (n-2)$, W is $2 \times (n-2)$, and b and c are such that $bc > 0$, $b \geq c$, and $y = \sqrt{bc}$.

Jacobian is $2(b-c)|\det(A_{n-2} - zI_{n-2})|^2$

$$\operatorname{tr} A_n A_n^T = 2x^2 + b^2 + c^2 + \operatorname{tr} WW^T + \operatorname{tr} A_{n-2} A_{n-2}^T,$$

if A_n is Gaussian then so is A_{n-2}.

Real eigenvalues of real asymmetric matrices

The expected number of real eigenvalues of J_n is proportional to $\sqrt{n}$. The limiting distribution of properly normalized real eigenvalues is Uniform($[-1, 1]$).

Theorem *(Edelman, Kostlan and Shub) If J_n is a matrix of independent standard normals, then, in the limit $n \to \infty$,*

(a) $E(N(\mathbf{R}, J_n)) = \sqrt{\frac{2n}{\pi}} + o(\sqrt{n})$,

(b) for any bounded $K \subset \mathbf{R}$,

$$E\left(N\Big(K, \frac{J_n}{\sqrt{n}}\Big)\right) = \sqrt{\frac{2n}{\pi}} \int_K f(x)dx + o(\sqrt{n}).$$

where is the density of Uniform($[-1, 1]$).

Two key elements of proof:

- $$E(N(K,J)) = C_n \int_K e^{-\frac{x^2}{2}} E(|\det(J_{n-1} - xI_{n-1})|)dx$$

  where J_{n-1} is a matrix of independent standard normals.

  This bit is based on the decomposition

  $$J_n = O \begin{pmatrix} x & \underline{w} \\ o & J_{n-1} \end{pmatrix} O^T$$

  where O is orthogonal and J_{n-1} is $(n-1) \times (n-1)$.

  Jacobian is $|\det(J_{n-1} - xI_{n-1})|$

- Computation of $E(|\det(J_{n-1} - xI_{n-1})|)$

  Difficult bit (because of the absolute value).

References

J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, *J. Math. Phys.* **6**, 440 – 449 (1965).

A. Edelman, The probability that a random real gaussian matrix has k real eigenvalues, related distributions, and the circular law, *Journ. Mult. Analysis.* **60**, 203 – 232 (1997).

A. Edelman, E. Kostlan, and M. Shub, How many eigenvalues of a random matrix are real?, *Journ. Amer. Math. Soc.* **7**, 247 – 267 (1994).

Also,

V.L. Girko, *Random determinats*, Kluver, 1991.

for a proof of the circular law for random matrices with i.i.d. entries.

Part II. Weakly Non-Hermitian Random Matrices

Consider random $n \times n$ matrices $\tilde{J} = A + ivB$

(i) A and B are independent *Hermitian*,
with i.i.d. entries
(ii) $E(A) = 0$, $E(B) = 0$
(iii) $E(\text{tr}\, A^2) = E(\text{tr}\, B^2) = \sigma^2 n^2$

Motivation: for any complex J
$J = X + iY$ where $X = \frac{J+J^*}{2}$ and $Y = \frac{J-J^*}{2i}$.

Since A and B are Hermitian, have $\tilde{J}_{kl}$ and $\tilde{J}_{lk}$ correlated for all $1 \leq k < l \leq n$:

$$E(\tilde{J}_{kl}\tilde{J}_{lk}) = E(|A_{kl}|^2) - v^2 E(|B_{kl}|^2) = \sigma^2(1 - v^2).$$

All other pairs are independent.

Have central matrix distribution with two parameters:

$$\sigma^2(1 + v^2) = E(|\tilde{J}_{kl}|^2)$$

and

$$\tau = \text{corr}(\tilde{J}_{kl}\tilde{J}_{lk}) = \frac{E(\tilde{J}_{kl}\tilde{J}_{lk})}{\sqrt{E(|\tilde{J}_{kl}|^2)E(|\tilde{J}_{lk}|^2)}} = \frac{1 - v^2}{1 + v^2}.$$

Without loss of generality, assume $\sigma^2 = 1/(1 + v^2)$, so that

$$E(|\tilde{J}_{kl}|^2) = 1 \text{ and } E(\tilde{J}_{kl}\tilde{J}_{lk}) = \tau$$

Typical eigenvalues of $\tilde{J}$ are of the order of $\sqrt{n}$, so introduce $J = \tilde{J}/\sqrt{n} = (A + ivB)/\sqrt{n}$.

Eigenvalue correlation functions $R_k^n(z_1,\dots z_k)$:

$R_1^n(z)$ is the probability *density* of finding an eigenvalue of $J = \frac{\tilde{J}}{\sqrt{n}}$, *regardless of label*, at z.

E.g., if D_0 is an infinitesimal circle covering z_0, then the probability of finding an eigenvalue of J in D_0 is approximately $R_1^n(z_0) \times \text{area}(D_0)$.

Similarly, $R_k^n(z_1,\dots z_k)$ is the *probability density* of finding an eigenvalue J, *regardless of labeling*, at each of the points $z_1,\dots z_k$.

Have k slots $z_1,\dots z_k$ and n eigenvalues of J to fill these slots, hence normalization:

$$\int \dots \int R_k^n(z_1,\dots z_k) d^2 z_1 \cdots d^2 z_k = n(n-1)\cdots(n-k+1).$$

$R_1^{(n)}(z)$ gives the mean density of eigenvalues at z, i.e.

$$R_1^{(n)}(z) = E\Big(\sum \delta^{(2)}(z-\lambda_j)\Big)$$

where the summation is over all eigenvalues λ_j of J and $\delta^{(2)}(x+iy) = \delta(x)\delta(y)$.

If N_D is the number of eigenvalues in D, then

$$E(N_D) = \int_D R_1^{(n)}(z)\, d^2 z = \int\int_D R_1^{(n)}(x,y)\, dxdy$$

Convention: $z = x + iy \equiv (x,y)$ and $d^2 z = dxdy$.

From now on, replace (i)-(iii) by

(iv) Hermitian A and B are drawn independently from the normal matrix distribution with density

$$\frac{1}{Q}\,\text{exp}\left(-\frac{1}{2\sigma^2}\,\text{tr}\,X^2\right) = \frac{1}{Q}\,\text{exp}\left(-\frac{1}{2\sigma^2}\sum_{k,l=1}^{n}|X_{kl}|^2\right),$$

where $\sigma^2(1+v^2) = 1$ (with no loss of generality).

Have

$$\begin{aligned} X_{kl} &\sim N\Big(0, \frac{1}{2}\sigma^2\Big) + i \times \text{indp.}\, N\Big(0, \frac{1}{2}\sigma^2\Big), \qquad k < l \\ X_{kk} &\sim N(0, \sigma^2) \end{aligned}$$

and the $\{X_{kl}\}$, $1 \le k \le l \le n$ are independent.

The entries of $\tilde{J} = A + ivB$ have multivariate complex normal distribution with density

$$\text{exp}\left[-\frac{1}{1-\tau^2}\left(\text{tr}\,\tilde{J}\tilde{J}^* - \frac{\tau}{2}\,\text{Re tr}\,\tilde{J}^2\right)\right], \quad \tau = \frac{1-v^2}{1+v^2}.$$

Have $E(\tilde{J}_{kl}) = 0$ and $E(|\tilde{J}_{kl}|^2) = 1$ for all (k, l) and

$$\begin{aligned} E(\tilde{J}_{kl}\tilde{J}_{mj}) &= \tau \quad \text{when}\, k = j \ \text{ and } \ l = m \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

- If $\tau = 0$, then $\tilde{J}$ has independent entries (Ginibre's ensemble); have maximum asymmetry.

- If $\tau = 1$ or $\tau = -1$, then $\tilde{J} = \tilde{J}^*$ (GUE) or $\tilde{J} = -\tilde{J}^*$, have no asymmetry at all.

Hermite polynomials:

$$H_n(z) = (-1)^n \exp\left(\frac{z^2}{2}\right) \frac{d^n}{dz^n} \exp\left(-\frac{z^2}{2}\right)$$

Generating function: $$\exp\left(zt - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}.$$

By making use of generating function,

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) \exp\left(-\frac{x^2}{2}\right) dx = \delta_{n,m} n! \sqrt{2\pi} \qquad (1)$$

and, for all $0 < \tau < 1$,

$$\frac{\tau^n}{\sqrt{1-\tau^2}} \int H_n\left(\frac{z}{\sqrt{\tau}}\right) H_n\left(\frac{\bar{z}}{\sqrt{\tau}}\right) w_\tau^2(z, \bar{z}) d^2 z = \delta_{n,m} \pi n! \qquad (2)$$

$$\begin{aligned} w_\tau^2(z, \bar{z}) &= \exp\left\{-\frac{1}{1-\tau^2}\left[|z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2)\right]\right\} \\ &= \exp\left(-\frac{x^2}{1+\tau} - \frac{y^2}{1-\tau}\right) \end{aligned}$$

Since

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) \to \delta(y), \quad \text{as } \sigma \to 0,$$

(1) can be obtained from (2) by letting $\tau \to 1$.

Useful integral representation:

$$H_n(z) = \frac{(\pm i)^n}{\sqrt{2\pi}} \exp\left(\frac{z^2}{2}\right) \int_{-\infty}^{+\infty} t^n \exp\left(-\frac{t^2}{2} \mp izt\right) dt.$$

Finite matrices

Theorem* *Under assumption (iv), for any finite* n *and any* $0 \leq \tau \leq 1$,

$$R_k^{(n)}(z_1, \dots z_k) = \det ||K_\tau^{(n)}(z_m, \bar{z}_l)||_{m,l=1}^k,$$

where

$$K_\tau^{(n)}(z_1, \bar{z}_2) \quad = \quad \frac{n}{\pi\sqrt{1-\tau^2}} \sum_{j=0}^{n-1} \frac{\tau^n}{j!} H_j\Big(\sqrt{\frac{n}{\tau}} z_1\Big) H_j\Big(\sqrt{\frac{n}{\tau}} \bar{z}_2\Big) \times$$
$$\exp\Big[-\frac{n}{2(1-\tau^2)} \sum_{j=1}^{2} (|z_j|^2 - \tau \operatorname{Re} z_j^2)\Big]$$

Special cases: $\tau = 0$ (Ginibre's ens.) and $\tau = 1$ (GUE).

When $\tau = 0$ (in the limit $\tau \to 0$, to be more precise):

$$K_0^{(n)}(z_1, \bar{z}_2) = \frac{n}{\pi} \sum_{j=0}^{n-1} \frac{n^j}{j!} z_1^j \bar{z}_2^j \exp\Big[-\frac{n}{2}(|z_1|^2 + |z_2|^2)\Big].$$

Can be seen from

$$\sqrt{\tau^j} H_j\Big(\frac{z}{\sqrt{\tau}}\Big) = z^j + \sqrt{\tau} \times (...)$$

Sketch of proof: obtain induced density of eigenvalues and use the orthogonal polynomial technique; the required orthogonal polynomials are Hermite polynomials $H_j\Big(\sqrt{\frac{1}{\tau}} z\Big)$, they are orthogonal in $\mathbf{C}$ with weight function $w_\tau^2(z, \bar{z})$

Mean eigenvalue density for finite matrices

By Theorem (*), $R^{(n)}(z) = K_\tau^{(n)}(z, \bar{z})$, and

(a) if $0 < \tau < 1$ then

$$R_1^{(n)}(z) = \frac{n}{\pi\sqrt{1-\tau^2}}\, e^{-n\frac{|z|^2 - \tau \operatorname{Re} z_j^2}{2(1-\tau^2)}} \sum_{j=0}^{n-1} \frac{\tau^n}{j!} \left| H_j\Big(\sqrt{\frac{n}{\tau}} z\Big)\right|^2 .$$

By letting $\tau \to 0$ in (a):

(b) If $\tau = 0$ (Ginibre's ensemble) then

$$R_1^{(n)}(z) = \frac{n}{\pi}\, e^{-n|z|^2} \sum_{j=0}^{n-1} \frac{n^j |z|^{2j}}{j!}.$$

By letting $\tau \to 1$ in (a):

(c) if $\tau = 1$ (GUE) then

$$R_1^{(n)}(z) \equiv R^{(n)}(x,y) = \delta(y) \sqrt{\frac{n}{2\pi}} e^{-\frac{n}{2}x^2} \sum_{j=0}^{n-1} \frac{1}{j!} \left| H_j(\sqrt{n}x)\right|^2 .$$

Limit of infinitely large matrices

Consider matrices $\tilde{J} = X + iY$.

Can have two regimes when $n \to \infty$:

- strong non-Hermiticity $E(\mathrm{tr}\, Y^2) = O(E(\mathrm{tr}\, X^2))$,
- weak non-Hermiticity $E(\mathrm{tr}\, Y^2) = o(E(\mathrm{tr}\, X^2))$.

If $v^2 > 0$ stays constant as $n \to \infty$, have strongly non-Hermitian $J = \frac{1}{\sqrt{n}}(A + ivB)$.

Recall $\tau = \frac{1-v^2}{1+v^2}$. The following result is a corollary of Theorem (*):

Theorem *(Girko's Elliptic Law) For any $\tau \in (-1, 1)$ and any bounded $D \subset \mathbf{C}$*

$$E(N_D) = n \int\int_D \rho(x, y)\, dxdy + o(n)$$

where N_D is the number of eigenvalues of J in D and

$$\rho(x,y) = \begin{cases} \frac{1}{\pi(1-\tau^2)}, & \textit{when } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1 \\ 0, & \textit{otherwise} \end{cases}$$

(Girko considered matrices J with symmetric pairs (J_{12}, J_{21}), (J_{13}, J_{31}), ... drawn independently from a bivariate distribution (not necessarily normal))

Local scale: area is measured in units of mean density of eigenvalues, i.e. unit area contains, on average, 1 eigenvalue.

Unit area on the global scale is n times unit area on the local scale.

Limit distribution of eigvs of J: uniform in the ellipse

$$\mathcal{E} = \left\{ z : \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1 \right\}$$

of area $|\mathcal{E}| = \pi(1-\tau^2)$. That is

$$E(N_D) \doteqdot \frac{|D \cap \mathcal{E}|}{|\mathcal{E}|}.$$

E.g. if $z_0 = x_0 + iy_0 \in \mathcal{E}$ and

$$D = \{z : |x - x_0| \leq \frac{\alpha}{2\sqrt{n|\mathcal{E}|}}, |y - y_0| \leq \frac{\beta}{2\sqrt{n|\mathcal{E}|}}\}$$

then $E(N_{D_0}) \doteqdot \alpha\beta$.

But also

$$E(N_D) = \iint_D R_1^{(n)}(z) d^2 z = \iint \frac{1}{n|\mathcal{E}|} R_1^{(n)}\left(z_0 + \frac{w}{\sqrt{n|\mathcal{E}|}}\right) d^2 w$$

Rescaled mean density of eigenvalues (around z_0):

$$\frac{1}{n|\mathcal{E}|} R_1^{(n)}\left(z_0 + \frac{w}{\sqrt{n|\mathcal{E}|}}\right)$$

Similarly, rescaled eigenvalue correlation functions:

$$\widehat{R}_k^{(n)}(w_1,\dots,w_k) := \frac{1}{(n|\mathcal{E}|)^k} R_k^{(n)}\Big(z_0+\frac{w_1}{\sqrt{n|\mathcal{E}|}},\ \dots,\ z_0+\frac{w_k}{\sqrt{n|\mathcal{E}|}}\Big)$$

The following result is a corollary of Theorem (*):

Theorem *For any* $\tau \in (-1,1)$ *and* $z_0 \in$ *int* $\mathcal{E}$

$$\lim_{n\to\infty} \widehat{R}_k^{(n)}(w_1,\dots,w_k) = \det ||K(w_m,\bar{w}_l)||_{m,l=1}^k,$$

where

$$K(w_1,\bar{w}_2) = \exp\Big[-\frac{\pi}{2}(|w_1|^2+|w_2|^2-2w_1\bar{w}_2)\Big]$$

E.g., the first two correlation fncs:

$$\begin{aligned}\widehat{R}_1(w) &= K(w,\bar{w}) = 1\\ \widehat{R}_2(w,w_2) &= \widehat{R}_1(w_1)\widehat{R}_1(w_2) - |K(w_1,\bar{w}_2)|^2\\ &= 1-\exp\Big(-\pi|w_1-w_2|^2\Big).\end{aligned}$$

No dependence on z_0, and, remarkably, no dependence on τ.

$$\lim_{\tau\to 1}\lim_{n\to\infty} \neq \lim_{n\to\infty}\lim_{\tau\to 1}.$$

Regime of weak non-Hermiticity

Now consider matrices $J = \frac{A}{\sqrt{n}} + iv\frac{B}{\sqrt{n}}$ in the limit when

$$n \to \infty \quad \text{and} \quad v^2 n \to \text{ const.} \tag{3}$$

May think of eigenvalues of J as of perturbed eigenvalues of $\frac{A}{\sqrt{n}}$. The eigenvalues of $\frac{A}{\sqrt{n}}$ are all real and are distributed in $[-2, 2]$ with density

$$\nu_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2} \qquad \text{(Wigner's semicircle law!)}$$

When perturbed they move off $[-2, 2]$ into $\mathbf{C}$ on the distance of the order $\frac{1}{n}$ (first order perturbations). Correspondingly, consider

$$D = \{(x, y) : x \in I \subset [-2, 2], \frac{s}{n} \leq y \leq \frac{t}{n}\}.$$

Then

$$E(N_D) = \int\int_D R_1^{(n)}(x, y) dx dy = \int_I dx \int_s^t d\hat{y}\, \frac{1}{n} R_1^{(n)}\left(x, \frac{\hat{y}}{n}\right),$$

where

$$\hat{y} = ny.$$

Hence

$$\hat{\rho}^{(n)}(x, \hat{y}) := \frac{1}{n^2} R_1^{(n)}\left(x, \frac{\hat{y}}{n}\right)$$

is the mean density of rescaled (distorted) eigenvalues $\hat{z} = x + i\hat{y} = x + iny$.

The following result is a corollary of Theorem (*).

Theorem *(Fyodorov, Khoruzhenko and Sommers)*

Let $\tau = 1 - \frac{\alpha^2}{2n}$*. Then, under assumption (iv),*

$$\lim_{n\to\infty} \widehat{\rho}^{(n)}(x, \widehat{y}) = \widehat{\rho}(x, \widehat{y}),$$

where

$$\widehat{\rho}(x, \widehat{y}) = \frac{1}{\pi\alpha} \exp\left(-\frac{2\widehat{y}^2}{\alpha^2}\right) \int_{-\pi\nu_{sc}(x)}^{\pi\nu_{sc}(x)} \exp\left(-\frac{\alpha^2 u^2}{2} - 2u\widehat{y}\right) \frac{du}{\sqrt{2\pi}}.$$

In the limit when $\alpha \to 0$

$$\frac{1}{\sqrt{2\pi}\pi\alpha} \exp\left(-\frac{2\widehat{y}^2}{\alpha^2}\right) \to \frac{1}{2\pi}\delta(\widehat{y})$$

and

$$\widehat{\rho}(x, \widehat{y}) \to \delta(\widehat{y})\nu_{sc}(x) \qquad \text{Wigner's semicircle law}$$

Introduce curvilinear coordinates in the $(x, \widehat{y})$ plane:

$$(x, \tilde{y}) = \Big(x, \frac{\widehat{y}}{\pi \nu_{sc}(x)}\Big).$$

If

$$\tilde{\rho}(x, \tilde{y}) = \frac{1}{\pi \nu_{sc}(x)} \widehat{\rho}\Big(x, \frac{\widehat{y}}{\pi \nu_{sc}(x)}\Big)$$

then

$$\tilde{\rho}(x, \tilde{y}) = \nu_{sc}(x) p_x(\tilde{y}),$$

where

$$p_x(\tilde{y}) = \frac{1}{\sqrt{2\pi} a} \exp\Big(-\frac{a^2 \tilde{y}^2}{2}\Big) \int_{-1}^{1} \exp\Big(-\frac{a^2 \tilde{y}^2}{2} - 2t\widehat{y}\Big) \frac{dt}{\sqrt{2\pi}}$$

and $a = \pi \nu_{sc}(x) \alpha$.

- Interpretation of $p_x(\tilde{y})$.
- Universality of $p_x(\tilde{y})$.

In the limit when $a \to \infty$ obtain uniform density

$$\tilde{\rho}(x, \tilde{y}) \doteq \begin{cases} \frac{1}{\pi a^2}, & \text{when } |\tilde{y}| \leq \frac{a^2}{2} \\ 0, & \text{otherwise} \end{cases}$$

Eigenvalue correlation functions:

have a crossover from Wigner-Dyson to Ginibre

Other types of weakly non-Hermitian matrices:

- Dissipative matrices:
  $J = A + i\Gamma$, $\Gamma \geq 0$ and is of finite rank m

Weakly non-unitary matrices:

- Submatrices of size m of unitary matrices of size n, in the limit $n \to \infty$ and $m = n - a$, a is a constant.

- Contractions: random matrices $J = U\sqrt{I - T}$, where $U \in U(n)$ and $0 \leq T \leq I$ in the limit when $n \to \infty$ and the rank of T remains finite. (Note that $J^*J = I - T$)

Weakly asymmetric matrices

- $J = A + vB$, where A and B are real and $A^T = A$, $B^T = -B$.

References

V. L. Girko, Elliptic law, *Theor. Probab. Appl.* **30**, 677 – 690 (1986).

Y.V. Fyodorov, B.A. Khoruzhenko, and H.-J. Sommers, Universality in the random matrix spectra in the regime of weak non-Hermiticity, *Ann. Inst. Henri Poincaré: Physique théorique*, **68**, 449 – 489 (1998).

Other types of weakly non-Hermitian matrices:

Y.V. Fyodorov and H.-J. Sommers, Statistic of resonance poles, phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time-reversal invariance, *J. Math. Phys.* Vol. 38 (1997) p. 1918 – 1981.

Y.V. Fyodorov and B. A. Khoruzhenko, Systematic analytical approach to correlation functions of resonances in quantum chaotic scattering, *Phys. Rev. Lett.* Vol. 83 (1999) 65 – 68.

K. Zyczkowski and H.-J. Sommers, Truncations of random unitary matrices, *J. Phys. A: Math. and Gen.* Vol. 33 (2000) 2045 – 2057.

Y.V. Fyodorov and H.-J. Sommers, Spectra of random contractions and scattering theory for discrete-time systems, *JETF Letters* Vol. 72 (2000) 422 – 426

Part III Asymmetric Tridiagonal Random Matrices

Imposing periodic boundary conditions:

$$\begin{array}{cccccc} \not{a}_1 & q_1 & b_1 & & a_1 & \\ & a_2 & q_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & b_n & & a_n & q_n & \not{b}_n \end{array}$$

Problem: Fix a rectangle $K \subset \mathbf{C}$ and let $n \to \infty$. **What proportion of eigenvalues of J_n are in K?** [Eigenvalue distribution].

Example: $a_j = a$, $b_j = b$, $q_j = q$ for all k and $a, b, q \in \mathbf{R}$. The limit eigenvalue distribution is supported by the ellipse

$$\{(x, y) : x = q + (a + b)\cos p,\ y = (a - b)\sin p,\ \ p \in [0, 2\pi]\}$$

How will this picture change if allow random fluctuations of a_k, b_k and q_k? Answer depends on the sign of $a_k b_{k-1}$.

Consider

$$J_n = \mathrm{tridiag}(a_k, q_k, b_k) \quad + \quad \text{p.b.c.}$$

with positive sub- and super-diagonals:

$$a_k = \exp(\xi_{k-1}), \quad b_k = \exp(\eta_k)$$

Assumptions:

(I) (ξ_k, η_k, q_k), $k = 0, 1, 2, \dots$, are independent samples from a probability distribution in $\mathbf{R}^3$.

(II) $E(\ln(1 + |q|))$, $E(\xi)$ and $E(\eta)$ are finite.

E.g. (ξ_k, η_k, q_k), $k = 0, 1, 2, \dots$, are independent samples from a 3D prob. distr. with a compact supp. in $\mathbf{R}^3$.

By making use of the similarity transformation $W_n = \mathrm{diag}(w_1, \dots w_n)$, $w_k = \exp\left[\frac{1}{2}\sum_{j=0}^{k-1}(\xi_j - \eta_j)\right]$,

$$W_n^{-1} J_n W_n = H_n + V_n,$$

where

$$H_n = \begin{pmatrix} q_1 & c_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ 0 & & c_{n-1} & q_n \end{pmatrix} V_n = \begin{pmatrix} 0 & 0 & \dots & 0 & u_n \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ v_n & 0 & \dots & 0 & 0 \end{pmatrix}$$

$c_k = \sqrt{a_{k+1} b_k} = e^{\frac{1}{2}(\xi_k + \eta_k)}$ and

$$u_n / v_n = e^{n[\mathbf{E}(\xi_0 - \eta_0) + o(1)]} \qquad \text{as} \quad n \to \infty$$

rank 2 asymmetric perturb. of symmetric H_n!

"Rank 2" $\Rightarrow$ eignv. distbs. of H_n and $H_n + V_n$ are related

"Strongly asymmetric" $\Rightarrow$ non-trivial relation.

Facts from theory of Hermitian random operators (e.g. in Pastur and Figotin, Spectra of random and almost periodic operators):

• Empirical distribution fnc. of eigvs. of H_n

$$\begin{aligned} N(I, H_n) &= \frac{1}{n}\#\{\text{eigvs. of } H_n \text{ in } I \subset \mathbf{R}\} \\ &= \int_I dN_n(\lambda), \quad N_n(\lambda) = N((-\infty,\lambda],H_n) \end{aligned}$$

dN_n assigns mass $\frac{1}{n}$ to each of eigvs. of H_n.

Proposition $\exists$ *nonrandom* $N(\lambda)$ $\forall$ $I \subset \mathbf{R}$*:*

$$\lim_{n\to\infty} N(I, H_n) \overset{a.s.}{=} \int_I dN(\lambda)$$

• Potentials: $p(z; H_n) = \int \log|z-\lambda| dN_n(\lambda)$

$$\Phi(z) = \int \log|z-\lambda| dN(\lambda)$$

• Lyapunov exponent $\gamma(z) = \lim_{n\to\infty} \frac{1}{n} E(\ln ||S_n(z)||)$

Proposition *(Thouless formula)*

$$\begin{aligned} \lim_{n\to\infty} p(z; H_n) &\overset{a.s.}{=} \Phi(z) \text{ unif. in } z \text{ on } K \subset \mathbf{C}\backslash\mathbf{R} \\ &= \gamma(z) + \mathbf{E}\log c_0 \end{aligned}$$

Corollaries:

$\Phi(z)$ continuous in z;

$\Phi(x+iy) > \mathbf{E}\log c_0 \quad \forall y \neq 0;$ etc.

Consider

$$\mathcal{L} = \{z \in \mathbf{C} : \quad \Phi(z) = \max[E(\xi_0), E(\eta_0)]\}$$

This curve is an equipotential line of limiting eigenvalue distribution of H_n.

If the probability law of (ξ_k, η_k, q_k) has bounded support then $\mathcal{L}$ is confined to a bounded set in $\mathbf{C}$ and is a union of closed contours:

There are $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots$ such that

$$\mathcal{L} = \cup \mathcal{L}_j, \qquad \mathcal{L}_j = \{x \pm i y_j(x) : x \in [\alpha_j, \beta_j]\}$$

Notation:

$$N(K, J_n) = \frac{1}{n} \#\{\text{eigvs. of } J_n \text{ in } K\}, \qquad K \subset \mathbf{C}$$

(describes distribution of eigenvalues of J_n)

Theorem *(Goldsheid and Khoruzhenko) Assume (I-II). Then, with probability one,*

(a) $\forall K \subset \mathbf{C} \setminus \mathbf{R}$*:* $\quad N(K, J_n) \underset{n\to\infty}{\longrightarrow} \int_{K\cap\mathcal{L}} \rho(z(s)) ds$

where $\rho(z) = \frac{1}{2\pi} \left| \int \frac{dN(\lambda)}{z-\lambda} \right|$ *and* ds *is the arc-length measure on* $\mathcal{L}$*.*

(b) $\forall I \subset \mathbf{R}$*:* $\quad N(I, J_n) \underset{n\to\infty}{\longrightarrow} \int_{I_W} dN(\lambda)$

where $I_W = I \cap \{\lambda : \ \Phi(\lambda + i0) > \max[E(\xi_0), E(\eta_0)]\}$

Sketch of proof: Let

$$p(z;J_n)=\frac{1}{n}\sum_{j=1}^{n}\log|z-z_j|=\frac{1}{n}\log|\det(J_n-z)|$$

where $z_1,\dots,z_n$ are the eigenvalues of J_n.

Claim *(convergence of potentials)*

With probability one,

$$p(z;J_n)\underset{n\to\infty}{\longrightarrow}F(z)=\max[\Phi(z),E(\xi_0),E(\eta_0)]\qquad\forall z\notin\mathbf{R}\cup\mathcal{L}$$

The convergence is uniform in $z\in K\subset\mathbf{C}\setminus(\mathbf{R}\cup\mathcal{L})$.

Consider measures $d\nu_{J_n}$ assigning mass $\frac{1}{n}$ to each of the eigenvalues of J_n. Then

$$\frac{1}{2\pi}\Delta p(z;J_n)=d\nu_{J_n}$$

in the sense of distribution theory. By Claim, the potentials $p(z;J_n)$ converge for almost all $z\in\mathbf{C}$. This implies convergence in the sense of distribution theory. Since the Laplacian is continuous in $\mathcal{D}'$,

$$\frac{1}{2\pi}\Delta p(z;J_n)\to\frac{1}{2\pi}\Delta F(z)$$

in $\mathcal{D}'$. But then

$$d\nu_{J_n}\to d\nu\equiv\frac{1}{2\pi}\Delta F(z)$$

in the sense of of weak convergence of measures, hence Theorem.

Proof of Claim

$$\begin{aligned}\det(J_n - zI_n) &= \det(H_n + V_n - z) \\ &= \det(H_n - zI_n)\det(I_n + V_n(H_n - z)^{-1})\end{aligned}$$

Therefore

$$p(z; J_n) = p(z; H_n) + \frac{1}{n}\log|d_n(z)|.$$

V_n is rank 2. $V_n = A^T B$, where

$$A = \begin{pmatrix} u_n & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ v_n & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}\therefore\ d_n(z) &= \det(I_n + A^T B(H_n - z)^{-1}) \\ &= \det(I_2 + B(H_n - z)^{-1}A^T) \quad 2 \times 2 \det \\ &= (1 + u_n G_{1n})(1 + v_n G_{n1}) - u_n v_n G_{11} G_{nn}\end{aligned}$$

where G_{lk} is the (k, l) entry of $(H_n - z)^{-1}$.

Now use

$$\begin{aligned}|u_n G_{1n}| &= e^{n[E(\xi_0) - \Phi(z) + o(1)]} \\ |v_n G_{n1}| &= e^{n[E(\eta_0) - \Phi(z) + o(1)]}\end{aligned}$$

and $|1 - u_n v_n G_{11} G_{nn}| \geq \alpha(z) > 0$, $z \notin \mathbf{R}$ to complete the proof.

Exactly solvable model

Consider $J_n = \mathrm{tridiag}(e^g,\ \text{Cauchy}(0,b),\ e^{-g}) \quad + \quad$ p.b.c.,

$$\xi_k \equiv g, \quad \eta_k \equiv -g \quad P(q_k \in I) = \frac{1}{\pi}\int_I dq \frac{b}{q^2+b^2}$$

In this case $J_n = W_n^{-1}(H_n + V_n)W_n$, where

$$H_n = \mathrm{tridiag}(1,\ \text{Cauchy}(0,b),\ 1) \qquad \text{Lloyd's model}$$

For Lloyd's model an explicit expression for $\Phi(z)$ is available:

$$4\cosh\Phi(z) = \sqrt{(x+2)^2 + (b+|y|)^2} + \sqrt{(x-2)^2 + (b+|y|)^2}$$

By making use of it,

- If $K = 2\cosh g \le K_{cr} = \sqrt{4+b^2}$ then $\mathcal{L}$ is empty.
- If $K > K_{cr}$ then $\mathcal{L}$ consists of two symmetric arcs

$$y(x) = \pm\Big[\sqrt{\frac{(K^2-4)(K^2-x^2)}{K^2}} - b\Big] \quad -x_b \le x \le x_b$$

x_b is determined by $y(x_b) = 0$.

Corollaries

$g = \frac{1}{2}E(\xi_0 - \eta_0)$ is a measure of asymmetry of J_n.

(1) Special case: Suppose that $q_k \equiv Const$ all k. Then $\gamma(0) = 0$ and $\gamma(z) > 0 \; \forall z \neq 0$. Since

$$\Phi(0) = \gamma(0) + \frac{1}{2}E(\xi_0 + \eta_0) < \max[E(\xi_0), E(\eta_0)]$$

the equation for $\mathcal{L}$, $\Phi(z) = \max[E(\xi_0), E(\eta_0)]$, has continuum of solutions for any $g \neq 0$.

For any $g \neq 0$ we have a bubble of complex eigv. around $z = 0$, i.e. no matter how small the perturb. V_n is, it moves a finite proportion of eigvs. of H_n off the real axis!

(2) Suppose now that the diagonal entries q_k are random. Then $\gamma(x) > 0 \; \forall x \in \mathbf{R}$ (Furstenberg) and

$$0 < \min_{x \in \Sigma} \gamma(x) = g_{\text{cr}}^{(1)} < g_{\text{cr}}^{(2)} = \max_{x \in \Sigma} \gamma(x) \leq +\infty$$

where Σ is the support of $dN(\lambda)$. Therefore

(a) If $|g| < g_{\text{cr}}^{(1)}$, J_n has zero proportion of non-real eigenvalues

(b) If $g_{\text{cr}}^{(1)} < |g| < g_{\text{cr}}^{(2)}$, J_n has finite proportions of real and non-real eigenvalues.

(c) $|g| > g_{\text{cr}}^{(2)}$, J_n has zero proportion of real eigenvalues.

References

N. Hatano and D. R. Nelson, Vortex pinning and non-Hermitian quantum mechanics, *Phys. Rev.* Vol. B56 (1997) 8651 – 8673.

I.Ya Goldsheid and B.A. Khoruzhenko, Eigenvalue curves of asymmetric tri-diagonal random matrices, *Electronic Journal of Probability*, Vol. 5(2000), Paper 16, 28 pages.