Bounds and Approximations for Optimum Combining of Signals in the Presence of Multiple Cochannel Interferers and Thermal Noise

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***Abstract*—We derive an upper bound and investigate some approximations on the symbol error probability (SEP) for coherent detection of M-ary phase-shift keying, using an array of antennas with optimum combining in wireless systems in the presence of multiple uncorrelated equal-power cochannel interferers and thermal noise in a Rayleigh fading environment. Our results are general and valid for an arbitrary number of antenna elements as well as an arbitrary number of interferers. In particular, the exact SEP is derived for an arbitrary number of antennas and interferers; the computational complexity of the exact solution depends on the minimum number of antennas and interferers. Moreover, closed-form approximations are provided for the cases of dual optimum combining with an arbitrary number of interferers, and of two interferers with an arbitrary number of antenna elements. We show that our bounds and approximations are close to Monte Carlo simulation results for all cases considered in this paper.**

***Index Terms*—Adaptive arrays, antenna diversity, cochannel interference, eigenvalue distribution, optimum combining, Wishart matrices.**

I. Introduction

ADAPTIVE ARRAYS can significantly improve the performance of wireless communication systems by weighting and combining the received signals to reduce fading effects and suppress interference. In particular, with optimum combining, the received signals are weighted and combined to maximize the output signal-to-interference-plus-noise ratio (SINR). In the presence of interference, this technique provides substantial improvement in performance over maximal ratio combining where the received signals are combined to maximize the desired signal-to-noise ratio (SNR) only. However, determining the performance of optimum combining is more difficult than with maximal ratio combining.

In this regard, closed-form expressions for the bit-error probability (BEP) of binary phase-shift keying (BPSK) have been derived for the single-interferer case with Rayleigh fading of the desired signal in [1] and [2], and with Rayleigh fading of the desired signal and interferer in [3]. An exact BEP expression, which requires numerical integration, for BPSK and a single interferer is also given in [4].

With multiple interferers of arbitrary power, Monte Carlo simulation has been used to determine the BEP in [2]. In [5], upper bounds on the BEP of optimum combining were derived given the average powers of the interferers. However, these bounds are generally not tight.

To avoid Monte Carlo simulation, the exact BEP expression was derived in [6] for the case of equal-power interferers, which permits analytical tractability. However, the results are limited to the case of BPSK and no thermal noise. Approximations for the BEP have been presented in [7] and [8] for binary modulation in the presence of thermal noise. However, the approximation of [7] still requires Monte Carlo simulation to derive mean eigenvalues (a table is provided in [7] for some cases), and the approximation of [8] is valid only for the case when the number of interferers is less than the number of antenna elements.

In this paper, starting from the eigenvalues distribution of complex Wishart matrices, we first give the exact expression of the symbol-error probability (SEP) for coherent detection of M-ary phase-shift keying (MPSK) using optimum combining in the presence of multiple uncorrelated equal-power interferers, as well as thermal noise, in a Rayleigh fading environment. Evaluation of this expression involves multiple numerical integrals. Then, based on some new results on the eigenvalues distribution of complex Wishart matrices, we derive new closed-form upper bounds. We show that these bounds are generally tighter than those of [5]. Moreover, we extend the approaches in [7] and obtain new closed-form approximations of the SEP that do not require Monte Carlo simulation and are close to simulation results.

In Section II, we describe the system model, and in Section III, derive the exact SEP of optimum combining with multiple interferers. Upper bounds are derived in Section IV, and approximate formulas are given in Section V. In Section VI, we compare our analytical results with simulations, and in Section VII, we present a summary and conclusions.

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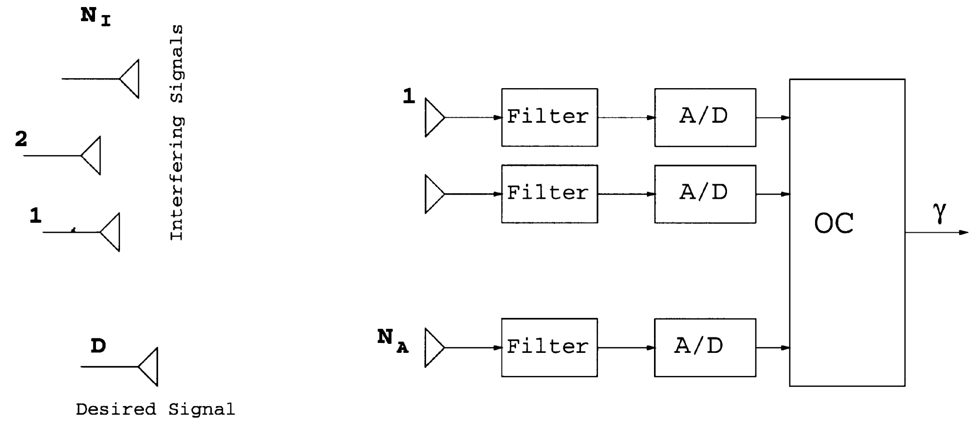

Fig. 1. Baseband model of optimum combining receiver.

II. System Model

We consider coherent demodulation with optimum combining of multiple received signals in a flat fading environment as in Fig. 1. The fading rate is assumed to be much slower than the symbol rate. Throughout the paper, $(\cdot)^T$ denotes the transposition operator, and $(\cdot)^\dagger$ stands for conjugation and transposition. The received signal at the N_A-element array output consists of the desired signal, N_I interfering signals, and thermal noise. After matched filtering and sampling at the symbol rate, the array output vector at time k can be written as

$$\mathbf{z}(k) = \sqrt{E_D}\boldsymbol{c}_D b_0 n(k) + \sum_{j=1}^{N_I} \sqrt{E_{\mathbf{I},j}}\boldsymbol{c}_{I,j} b_j(k) + \mathbf{n}(k), \quad (1)$$

where E_D and $E_{I,j}$ are the mean (over fading) energies of the desired signal and jth interferer, respectively; $\boldsymbol{c}_D = [\boldsymbol{c}_{D,1}, \ldots, \boldsymbol{c}_{D,N_A}]^T$ and $\boldsymbol{c}_{I,j} = [\boldsymbol{c}_{I,j,1}, \ldots, \boldsymbol{c}_{I,j,N_A}]^T$ are the desired and jth interference propagation vectors, respectively; $b_0(k)$ and $b_j(k)$ (both with unit variance) are the desired and interfering data samples, respectively; and $\mathbf{n}(k)$ represents the additive noise. We model $\boldsymbol{c}_D$ and $\boldsymbol{c}_{I,j}$ as multivariate complex-valued Gaussian vectors having $\mathbb{E}\{\boldsymbol{c}_D\} = \mathbb{E}\{\boldsymbol{c}_{I,j}\} = \mathbf{0}$ and $\mathbb{E}\left\{\boldsymbol{c}_D\boldsymbol{c}_D^\dagger\right\} = \mathbb{E}\left\{\boldsymbol{c}_{I,j}\boldsymbol{c}_{I,j}^\dagger\right\} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The additive noise is modeled as a white Gaussian random vector with independent and identically distributed (i.i.d.) elements with $\mathbb{E}\{\mathbf{n}(k)\} = \mathbf{0}$ and $\mathbb{E}\left\{\mathbf{n}(k)\mathbf{n}^\dagger(k)\right\} = N_0\mathbf{I}$, where $N_0/2$ is the two-sided thermal noise power spectral density per antenna element.

The SINR at the output of the N_A-element array with optimum combining can be expressed [1], [2] as

$$\gamma = E_D \boldsymbol{c}_D^\dagger \mathbf{R}^{-1} \boldsymbol{c}_D \quad (2)$$

where the short-term covariance matrix $\mathbf{R}$, conditioned to all interference propagation vectors, is

$$\mathbf{R} = \mathbb{E}_{\mathbf{n},b_j(k)}\left\{\left[\sum_{j=1}^{N_I}\sqrt{E_{I,j}}\boldsymbol{c}_{I,j}b_j(k) + \mathbf{n}(k)\right] \times \left[\sum_{j=1}^{N_I}\sqrt{E_{I,j}}\boldsymbol{c}_{I,j}b_j(k) + \mathbf{n}(k)\right]^\dagger\right\} \quad (3)$$

and $\mathbb{E}_X\{\cdot\}$ denotes expectation with respect to X. Therefore

$$\mathbf{R} = \sum_{j=1}^{N_I} E_{I,j}\boldsymbol{c}_{I,j}\boldsymbol{c}_{I,j}^\dagger + N_0\mathbf{I}. \quad (4)$$

It is important to remark that $\mathbf{R}$ and, consequently, also the SINR γ vary at the fading rate.

The matrix $\mathbf{R}^{-1}$ can be written as $\mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\dagger$ where $\mathbf{U}$ is a unitary matrix and $\boldsymbol{\Lambda}$ is a diagonal matrix whose elements on the principal diagonal are the eigenvalues of $\mathbf{R}$, denoted by $(\lambda_1, \ldots, \lambda_{N_A})$. The vector $\mathbf{u} = \mathbf{U}^\dagger\boldsymbol{c}_D = [u_1, \ldots, u_{N_A}]^T$ has the same distribution as $\boldsymbol{c}_D$, since $\mathbf{U}$ represents a unitary transformation. The SINR given in (2) can be rewritten as

$$\gamma = E_D\boldsymbol{c}_D^\dagger\mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\dagger\boldsymbol{c}_D = E_D\sum_{i=1}^{N_A}\frac{|u_i|^2}{\lambda_i}. \quad (5)$$

Since $\mathbf{R}$ is a random matrix, its eigenvalues are random variables.

We now investigate the statistical properties of $(\lambda_1, \ldots, \lambda_{N_A})$. We will show later that this is related to problems arising in multivariate statistics, regarding the eigenvalue distribution of complex Wishart matrices. Let

$$\mathbf{C}_I \triangleq \begin{bmatrix} | & | & & | \\ \boldsymbol{c}_{I,1} & \boldsymbol{c}_{I,2} & \cdots & \boldsymbol{c}_{I,N_I} \\ | & | & & | \end{bmatrix} \quad (6)$$

be a $(N_A \times N_I)$ random matrix composed of N_I interference propagation vectors as columns. For equal-power interferers, i.e., $E_{I,j} = E_I$ for $j = 1, \ldots, N_I$, (4) can be rewritten as

$$\mathbf{R} = E_I\tilde{\mathbf{R}} + N_0\mathbf{I} \quad (7)$$

where $\tilde{\mathbf{R}} = \mathbf{C}_I\mathbf{C}_I^\dagger$ is a $(N_A \times N_A)$ random matrix. The eigenvalues of $\mathbf{R}$ can be written in terms of eigenvalues of $\tilde{\mathbf{R}}$, denoted by $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{N_A})$, as

$$\lambda_i = E_I\tilde{\lambda}_i + N_0, \qquad i = 1, \ldots, N_A \quad (8)$$

where the joint probability density function (pdf) of the N_A eigenvalues of $\tilde{\mathbf{R}}$ are given by the following theorem.

Theorem 1: The joint pdf of the first $N_{\min} \triangleq \min\{N_A, N_I\}$ *ordered* eigenvalues $\tilde{\boldsymbol{\lambda}} = [\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{N_{\min}}]^T$ of $\tilde{\mathbf{R}}$, with $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{N_{\min}}$, is

$$f_{\tilde{\boldsymbol{\lambda}}}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{N_{\min}}) = K \prod_{i=1}^{N_{\min}} e^{-\tilde{\lambda}_i} \tilde{\lambda}_i^{N_{\max}-N_{\min}} \times \prod_{i=1}^{N_{\min}-1} \left[\prod_{j=i+1}^{N_{\min}} (\tilde{\lambda}_i - \tilde{\lambda}_j)^2 \right] \quad (9)$$

where $N_{\max} \triangleq \max\{N_A, N_I\}$ and K is a normalizing constant given by

$$K = \frac{\pi^{N_{\min}(N_{\min}-1)}}{\tilde{\Gamma}_{N_{\min}}(N_{\max}) \tilde{\Gamma}_{N_{\min}}(N_{\min})} \quad (10)$$

with

$$\tilde{\Gamma}_{N_{\min}}(n) = \pi^{N_{\min}(N_{\min}-1)/2} \prod_{i=1}^{N_{\min}} (n-i)!. \quad (11)$$

The additional $N_A - N_{\min}$ eigenvalues of $\tilde{\mathbf{R}}$ are identically equal to zero.

Proof: See Appendix B. ■

As a consequence of *Theorem 1*, we have the following corollary.

Corollary 1 (Reciprocity Principle): The statistical distributions of the eigenvalues of $\tilde{\mathbf{R}}$, for the case of m antennas and p interferers with $m \leq p$, are equal to that of the (nonzero) eigenvalues of $\tilde{\mathbf{R}}$ for the case of p antennas and m interferers.[1]

Using the distribution theory for transformations of random vectors [9] together with (8), the joint pdf of $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_{N_{\min}}]^T$ with $\lambda_1 \geq \cdots \geq \lambda_{N_{\min}} \geq N_0$ is

$$f_{\boldsymbol{\lambda}}(\lambda_1, \dots, \lambda_{N_{\min}}) = \frac{1}{E_I^{N_{\min}}} \times f_{\tilde{\boldsymbol{\lambda}}}\left(\frac{\lambda_1 - N_0}{E_I}, \frac{\lambda_2 - N_0}{E_I}, \dots, \frac{\lambda_{N_{\min}} - N_0}{E_I} \right) \quad (12)$$

where $f_{\tilde{\boldsymbol{\lambda}}}(\cdot)$ is given by *Theorem 1*. The additional $N_A - N_{\min}$ eigenvalues of $\mathbf{R}$ are identically equal to N_0.

III. Evaluation of the Exact SEP

The SEP for optimum combining in the presence of multiple cochannel interferers and thermal noise in a fading environment is obtained by averaging the conditional SEP over the (desired and interfering signal) channel ensemble. This can be accomplished by

$$\begin{aligned} P_e &= \mathbb{E}_\gamma \left\{ \Pr\left\{ e \mid \gamma \right\} \right\} \\ &= \int_0^\infty \Pr\left\{ e \mid \gamma = x \right\} f_\gamma(x) dx \end{aligned} \quad (13)$$

where $\Pr\{e \mid \gamma\}$ is the SEP conditioned on the random variable γ, and $f_\gamma(\cdot)$ is the pdf of the combiner output SINR. Note that γ depends on the desired and interference propagation vectors. Although the evaluation of (13) involves a single integration for averaging over the channel ensemble, it requires the knowledge of the pdf of γ, which can be quite difficult to obtain. This is alleviated by using the chain rule of conditional expectation as

$$P_e = \mathbb{E}_{\boldsymbol{\lambda}} \left\{ \underbrace{\mathbb{E}_{\mathbf{u}} \left\{ \Pr\left\{ e \mid \gamma = E_D \sum_{i=1}^{N_A} \frac{|u_i|^2}{\lambda_i} \right\} \right\}}_{P_{e|\boldsymbol{\lambda}}} \right\} \quad (14)$$

where we first perform $\mathbb{E}_{\mathbf{u}}\{\cdot\}$ (i.e., average over the channel ensemble of the desired signal) to obtain the conditional SEP, conditioned on the random vector $\boldsymbol{\lambda}$, denoted by $P_{e|\boldsymbol{\lambda}}$. We then perform $\mathbb{E}_{\boldsymbol{\lambda}}\{\cdot\}$ to average out the channel ensemble of the interfering signals.

The jth interfering data samples, $b_j(k)$ $j = 1, \dots, N_I$, can be modeled as zero-mean, unitary variance Gaussian random variables. Note that the Gaussian assumption gives a good approximation when the interfering contribution is due to a large number of interferers sampled at a random time, and generally it represents a worst case [10]; here, it will be used regardless of the number of interferers. In the following, we assume that $b_0(k)$ is an MPSK data sample. With the previous assumption together with the Gaussianity of $\mathbf{n}(k)$, $\Pr\{e \mid \gamma\}$ for coherent detection of MPSK is given by [11], [12]

$$\Pr\left\{ e \mid \gamma \right\} = \frac{1}{\pi} \int_0^\Theta \exp\left(-\frac{c_{\text{MPSK}}}{\sin^2\theta} \gamma \right) d\theta \quad (15)$$

where $c_{\text{MPSK}} = \sin^2(\pi/M)$ and $\Theta = \pi(M-1)/M$. Using (15), $P_{e|\boldsymbol{\lambda}}$ can be written as

$$\begin{aligned} P_{e|\boldsymbol{\lambda}} &= \frac{1}{\pi} \int_0^\Theta \mathbb{E}_{\mathbf{u}} \left\{ \exp\left(-\frac{c_{\text{MPSK}}}{\sin^2\theta} E_D \sum_{i=1}^{N_A} \frac{|u_i|^2}{\lambda_i} \right) \right\} d\theta \\ &= \frac{1}{\pi} \int_0^\Theta \psi_{\gamma|\boldsymbol{\lambda}} \left(-\frac{c_{\text{MPSK}}}{\sin^2\theta} \right) d\theta \end{aligned} \quad (16)$$

where $\psi_{\gamma|\boldsymbol{\lambda}}(\cdot)$ is the characteristic function (cf) of γ, conditioned on $\boldsymbol{\lambda}$, given by

$$\psi_{\gamma|\boldsymbol{\lambda}}(j\nu) = \frac{1}{\prod_{i=1}^{N_A} \left(1 - \frac{j\nu E_D}{\lambda_i} \right)} \quad (17)$$

and we have used the fact that $\mathbf{u}$ is Gaussian with i.i.d. elements. Therefore, the conditional SEP, conditioned on $\boldsymbol{\lambda}$, in the general case of N_A antennas and N_I interferers, becomes

$$P_{e|\boldsymbol{\lambda}} = \frac{1}{\pi} \int_0^\Theta A(\theta) \prod_{i=1}^{N_{\min}} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \frac{E_D}{\lambda_i}} \right] d\theta \quad (18)$$

where

$$A(\theta) \triangleq \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \frac{E_D}{N_0}} \right]^{N_A - N_{\min}}. \quad (19)$$

Using (9), (12), (14), and (18), the unconditional SEP for optimum combining becomes

$$\begin{aligned} P_e &= \mathbb{E}_{\boldsymbol{\lambda}} \left\{ P_{e|\boldsymbol{\lambda}} \right\} \\ &= \int_{N_0}^\infty \cdots \int_{\lambda_3}^\infty \int_{\lambda_2}^\infty P_{e|\boldsymbol{\lambda}} \cdot f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) d\lambda_1 d\lambda_2 \dots d\lambda_{N_{\min}}. \end{aligned} \quad (20)$$

Equation (20) is exact and valid for arbitrary numbers of antennas and interferers; however, it requires the evaluation of

[1]This proves the equality, observed also numerically by Monte Carlo simulation in [7, Table I], of the expectations of the nonzero eigenvalues of $\tilde{\mathbf{R}}$ when the number of antennas is exchanged with the number of interferers.

nested $N_{\min}$-fold integrals, which can be cumbersome to evaluate for large $N_{\min}$. To give an idea of the amount of time needed for $N_{\min}=2$ (which allows us to investigate either dual combining with an arbitrary number of interferers or an arbitrary number of antennas with two interferers), the computation of (20) on a 450-MHz PC requires about 100 s.

Since the computation time for the numerical evaluation of (20) increases with the number of antennas and interferers, rigorous bounds, as in [5], or approximate expressions, as in [7], are useful; unfortunately, the bounds in [5] are generally not very tight, and the approximation in [7] requires Monte Carlo simulation. This motivates the need to derive simpler and tighter bounds or approximate expressions in closed form.

IV. Upper Bounds on SEP

In this section, we derive a new upper bound for the SEP based on the knowledge of the pdf of the trace of the covariance matrix $\mathbf{R}$.

Theorem 2: The SEP is upper bounded by

$$P_e \leq \frac{1}{\pi}\int_0^{\Theta} A(\theta)\mathbb{E}_Y\left\{B(Y,\theta)\right\}d\theta \tag{21}$$

where $A(\theta)$ is defined in (19), and Y is a chi-square distributed random variable with $2N_AN_I$ degrees of freedom (DOFs), having pdf given by

$$f_Y(y)=\begin{cases}\frac{1}{\Gamma(N_AN_I)}y^{(N_AN_I)-1}e^{-y}, & \text{if } y\geq 0\\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

In (22), $\Gamma(x)$ is the gamma function [13, eq.(8.310), p. 942], and

$$B(y,\theta)=\left[\frac{\left(y+\frac{N_{\min}N_0}{E_I}\right)\sin^2\theta}{\left(y+\frac{N_{\min}N_0}{E_I}\right)\sin^2\theta+\frac{c_{\mathrm{MPSK}}N_{\min}E_D}{E_I}}\right]^{N_{\min}}. \tag{23}$$

For a single-interferer scenario, (21) is an equality, i.e., it gives the exact SEP for $N_I=1$.

Proof: By applying the result in Appendix C to (18), we have

$$\begin{aligned}P_{e|\boldsymbol{\lambda}} =&\frac{1}{\pi}\int_0^{\Theta}A(\theta)\prod_{i=1}^{N_{\min}}\left[\frac{\sin^2\theta}{\sin^2\theta+\frac{c_{\mathrm{MPSK}}E_D}{\lambda_i}}\right]d\theta\\ \leq&\frac{1}{\pi}\int_0^{\Theta}A(\theta)\left[\frac{\sin^2\theta}{\sin^2\theta+\frac{c_{\mathrm{MPSK}}E_DN_{\min}}{\sum_{i=1}^{N_{\min}}\lambda_i}}\right]^{N_{\min}}d\theta\end{aligned} \tag{24}$$

where the equality is verified for $N_{\min}=1$, therefore

$$P_e\leq\frac{1}{\pi}\int_0^{\Theta}A(\theta)\mathbb{E}_{\lambda}\left\{\left[\frac{\sin^2\theta}{\sin^2\theta+\frac{c_{\mathrm{MPSK}}E_DN_{\min}}{\sum_{i=1}^{N_{\min}}\lambda_i}}\right]^{N_{\min}}\right\}d\theta. \tag{25}$$

Note, from (8), that

$$\begin{aligned}\sum_{i=1}^{N_{\min}}\lambda_i =&E_I\sum_{i=1}^{N_{\min}}\tilde{\lambda}_i+N_{\min}N_0\\ =&E_I\sum_{i=1}^{N_A}\tilde{\lambda}_i+N_{\min}N_0\end{aligned} \tag{26}$$

where we have used the fact that $N_A-N_{\min}$ eigenvalues of $\tilde{\mathbf{R}}$ are identically equal to zero by *Theorem 1*, and hence

$$\sum_{i=1}^{N_{\min}}\lambda_i=E_Itr\left[\tilde{\mathbf{R}}\right]+N_{\min}N_0. \tag{27}$$

In order to evaluate the expectation in (25), we observe that $tr\left[\tilde{\mathbf{R}}\right]=tr\left[\mathbf{C}_I\mathbf{C}_I^{\dagger}\right]=\sum_{i,j}|c_{I,j,i}|^2$. Hence, the random variable $Y\triangleq tr\left[\tilde{\mathbf{R}}\right]$ is chi-square distributed with $2N_AN_I$ DOFs, with pdf given by (22). This completes the proof of the theorem. ■

The expectation $\mathbb{E}_Y\{B(Y,\theta)\}$ is evaluated in Appendix D as shown in (28) at the bottom of the next page, where $E_1(x)$ is the exponential integral defined by (57) in Appendix D.

The bound (21) allows the evaluation of SEP for coherent detection of MPSK modulation with optimum combining; the numerical evaluation of it only requires a fraction of a second on a PC. Note that the inequality in (24) becomes equality for the case of single interferer (as well as for single antenna), and our bound gives the exact results.

V. Approximations on the SEP

In this section, some new results on the SEP approximations will be presented. Here, we start from the approximation proposed in [7], and we derive a methodology which allows us to eliminate the need for Monte Carlo simulation in the cases of dual optimum combining with an arbitrary number of interferers, and of two interferers with an arbitrary number of antenna elements. We prove that the approximation proposed in [8] is an upper bound of [7]; furthermore, we generalize the result of [8], and the generalized results are now applicable for the case $N_I\geq N_A$ in addition to $N_I<N_A$.

A. Approximation via Expected Eigenvalues

In [7], it is proposed to approximate the unconditional cf of γ as $\mathbb{E}_{\boldsymbol{\lambda}}\left\{\psi_{\gamma|\boldsymbol{\lambda}}(j\nu)\right\}\approx\psi_{\gamma|\mathbb{E}_{\boldsymbol{\lambda}}\{\boldsymbol{\lambda}\}}(j\nu)$.

By adopting this approximation in (20), the SEP for MPSK is approximated as follows:

$$P_e\approx F\left(\boldsymbol{\beta}_A\right) \tag{29}$$

where $F\left(\boldsymbol{\beta}\right)$ is given by

$$F\left(\boldsymbol{\beta}\right)=\frac{1}{\pi}\int_0^{\Theta}A(\theta)\prod_{i=1}^{N_{\min}}\left[\frac{\sin^2\theta}{\sin^2\theta+c_{\mathrm{MPSK}}\frac{E_D}{\beta_i}}\right]d\theta \tag{30}$$

and the ith element of $\boldsymbol{\beta}_A$ is

$$\beta_{A,i}=E_I\mathbb{E}\left\{\tilde{\lambda}_i\right\}+N_0,\qquad i=1,\ldots,N_{\min}. \tag{31}$$

Discussion

Since $\mathbf{R}$ is semidefinite positive, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{N_A}$ are real and nonnegative. Therefore, for each θ, it is easy to verify that the function

$$g(\lambda_1, \lambda_2, \ldots, \lambda_{N_{\min}}; \theta) \triangleq A(\theta) \prod_{i=1}^{N_{\min}} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \frac{E_D}{\lambda_i}} \right] \tag{32}$$

is $\cap$-concave in *each* λ_i when the other variables are fixed, but, despite this, the function is neither globally convex nor concave.

Approximation (29) is obtained by replacing the expected value of $g(\cdot)$ with the function evaluated at the expected values of the λ_i's, i.e.,

$$\mathbb{E}\{g(\lambda_1, \lambda_2, \ldots, \lambda_{N_{\min}}; \theta)\} \approx g(\mathbb{E}\{\lambda_1\}, \mathbb{E}\{\lambda_2\}, \ldots, \mathbb{E}\{\lambda_{N_{\min}}\}; \theta). \tag{33}$$

Now, if the function $g(\cdot)$ were concave (convex), applying Jensen's inequality will produce an upper (lower) bound, but, since (32) is neither concave nor convex, Jensen's inequality [14] cannot be applied. However, (29) gives good agreement with the exact SEP expression (20) for typical parameters of interest. This may be due to the fact that, in the region where the pdf of the eigenvalues λ is not negligible, (32) behaves essentially as an affine function.

Integrating both sides of (33) over θ and scaling by $1/\pi$, we obtain

$$\begin{aligned} P_e &= \mathbb{E}_{\boldsymbol{\lambda}}\left\{P_{e|\boldsymbol{\lambda}}\right\} \\ &\approx \frac{1}{\pi}\int_0^{\Theta} g(\mathbb{E}\{\lambda_1\}, \mathbb{E}\{\lambda_2\}, \ldots, \mathbb{E}\{\lambda_{N_{\min}}\}; \theta)\, d\theta \\ &= \frac{1}{\pi}\int_0^{\Theta} A(\theta) \prod_{i=1}^{N_{\min}} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\text{MPSK}} \frac{E_D}{\beta_{A,i}}} \right] d\theta \\ &= F(\boldsymbol{\beta}_A). \end{aligned} \tag{34}$$

Note that, given the expectation of the eigenvalues $\mathbb{E}\left\{\tilde{\lambda}_i\right\}$, the last integral can be also derived in closed form by using a canonical decomposition method [15], [16]. In the following, (34) will be denoted as approximation A, and we will show in Section VI that it is in good agreement with the exact analysis of (20) as well as simulation results. In general, approximation A requires knowledge of $\mathbb{E}\left\{\tilde{\lambda}_i\right\}$. In [7], the expectation of the eigenvalues for some specific cases were calculated via Monte Carlo simulation. For the case of dual optimum combining ($N_A = 2$) with arbitrary N_I, or the case of two interferers ($N_I = 2$) with an arbitrary number of antenna elements, $\mathbb{E}\left\{\tilde{\lambda}_i\right\}$ is obtained easily in a closed form using the reciprocity principle given in *Corollary 1*, together with the results of Appendix E.

B. Approximation via Equal Expected Eigenvalues

The determination of $\mathbb{E}\left\{\tilde{\lambda}_i\right\}$, in general, requires the evaluation of multiple integrals for each of the $(N_{\min}-1)$ eigenvalues. This can be alleviated, at the expense of tightness, by the following bound.

Theorem 3: $F(\boldsymbol{\beta}_A)$ is upper bounded as follows:

$$F(\boldsymbol{\beta}_A) \le F(\boldsymbol{\beta}_B) \tag{35}$$

where $F(\boldsymbol{\beta})$ is given in (30) and the ith element of $\boldsymbol{\beta}_B$ is

$$\beta_{B,i} = E_I N_{\max} + N_0, \qquad i = 1, \ldots, N_{\min}. \tag{36}$$

Proof: The integrand of (34) can be written as

$$g(y_1, \ldots, y_{N_{\min}}; \theta) = A(\theta) \prod_{i=1}^{N_{\min}} \left[\frac{\sin^2\theta}{\sin^2\theta + \frac{c_{\text{MPSK}} E_D}{y_i}} \right] \tag{37}$$

where $y_i = \mathbb{E}\{\lambda_i\}$ for $i = 1, \ldots, N_{\min}$. By using (45) of Appendix C, with $n = N_{\min}$, we get

$$g(y_1, \ldots, y_{N_{\min}}; \theta) \le A(\theta) \left[\frac{\sin^2\theta}{\sin^2\theta + \frac{c_{\text{MPSK}} E_D N_{\min}}{\sum_{i=1}^{N_{\min}} y_i}} \right]^{N_{\min}}. \tag{38}$$

$$\begin{aligned} \mathbb{E}_Y\{B(Y,\theta)\} = \frac{1}{\Gamma(N_A N_I)} \times \Bigg\{ &\sum_{n=0}^{N_{\min}} \sum_{m=n}^{N_A N_I - 1} \binom{N_{\min}}{n} \binom{N_A N_I - 1}{m} (-1)^{N_A N_I - 1 - (m-n)} \left(\frac{\frac{c_{\text{MPSK}} N_{\min} E_D}{E_I}}{\sin^2\theta} \right)^n (m-n)! \\ &\times \sum_{l=0}^{m-n} \frac{1}{l!} \left(\frac{\frac{c_{\text{MPSK}} N_{\min} E_D}{E_I}}{\sin^2\theta} + \frac{N_{\min} N_0}{E_I} \right)^{N_A N_I - 1 - m + l} \\ &+ \sum_{n=1}^{N_{\min}} \sum_{m=0}^{n-1} \binom{N_{\min}}{n} \binom{N_A N_I - 1}{m} \frac{(-1)^{N_A N_I}}{(n-m-1)!} \left(\frac{\frac{c_{\text{MPSK}} N_{\min} E_D}{E_I}}{\sin^2\theta} \right)^n \\ &\times \Bigg\{ \sum_{l=1}^{n-m-1} (-1)^l (l-1)! \left(\frac{\frac{c_{\text{MPSK}} N_{\min} E_D}{E_I}}{\sin^2\theta} + \frac{N_{\min} N_0}{E_I} \right)^{N_A N_I - 1 - m - l} \\ &+ \Bigg[\left(\frac{\frac{c_{\text{MPSK}} N_{\min} E_D}{E_I}}{\sin^2\theta} + \frac{N_{\min} N_0}{E_I} \right)^{N_A N_I - 1 - m} \\ &\times e^{+\left(c_{\text{MPSK}} N_{\min} E_D / E_I / \sin^2\theta + N_{\min} N_0 / E_I\right)} \\ &\times E_1 \left(\frac{\frac{c_{\text{MPSK}} N_{\min} E_D}{E_I}}{\sin^2\theta} + \frac{N_{\min} N_0}{E_I} \right) \Bigg] \Bigg\} \Bigg\} \end{aligned} \tag{28}$$

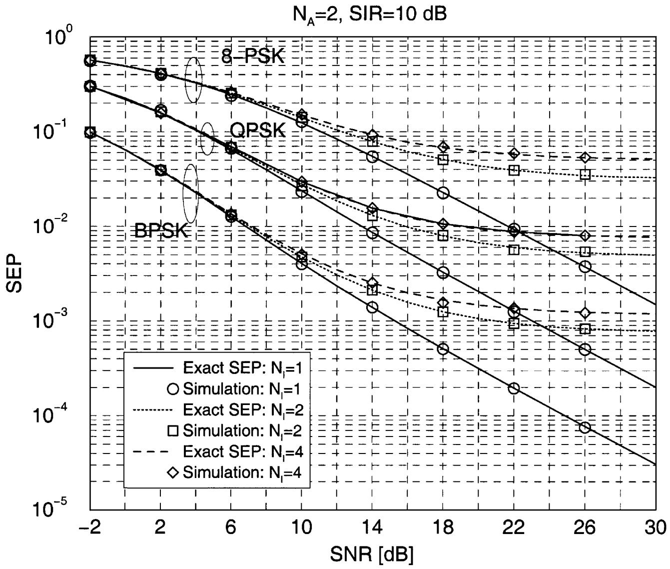

Fig. 2. SEP for coherent detection of BPSK, quaternary PSK, and 8-PSK using dual optimum combining ($N_A = 2$) for $N_I = 1$, 2, and 4 and $\mathrm{SIR} = 10$ dB. Excellent agreement between exact analysis and simulation can be observed.

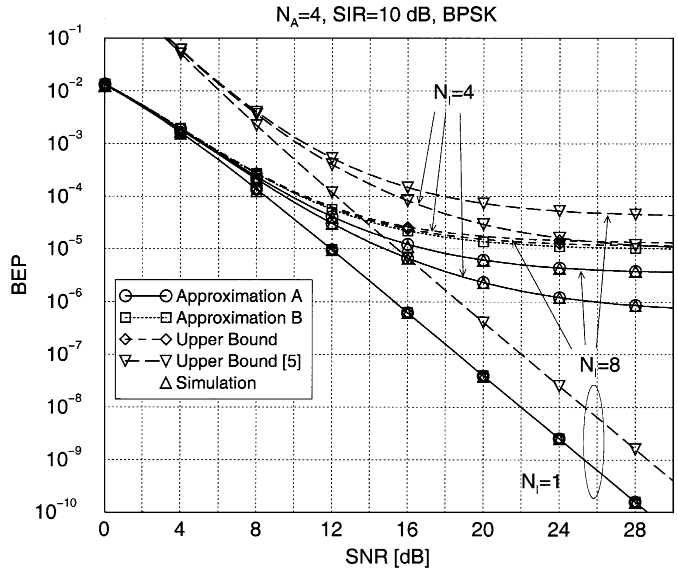

Fig. 3. Comparison between upper bound derived in Section IV with the only previously known upper bound given by [5, eq. (13)] for the case of BPSK, $\mathrm{SIR} = 10$ dB, $N_A = 4$, $N_I = 1$, 4, and 8. Note that our upper bound is 4.8 and 5.3 dB (at BEP of 10^{-3}) tighter and 4.8 and 7.4 dB (at BEP of 10^{-4}) tighter than [5, eq. (13)] for $N_I = 4$ and 8, respectively.

Using (8)

$$\sum_{i=1}^{N_{\min}} y_i = E_I \sum_{i=1}^{N_{\min}} \mathbb{E}\left\{\tilde{\lambda}_i\right\} + N_{\min} N_0 \tag{39}$$

$$= E_I N_{\min} N_{\max} + N_{\min} N_0 \tag{40}$$

where we have used (66) from Appendix E in deriving (40). Therefore

$$g(y_1, \ldots, y_{N_{\min}}; \theta) \leq A(\theta) \left(\frac{\sin^2\theta}{\sin^2\theta + \frac{c_{\mathrm{MPSK}} E_D}{E_I N_{\max} + N_0}} \right)^{N_{\min}}. \tag{41}$$

Finally, by using (30), (36), and (41), it is straightforward to show that (34) is upper bounded by $F(\boldsymbol{\beta}_B)$. ∎

The above theorem provides a rigorous proof that the approximate solution for $N_A > N_I$ proposed in [8], based on heuristic assumptions, represents an upper bound of the solution proposed in [7]. It also provides the generalization of the approximation of [8], which is now valid for arbitrary numbers of antennas and interferers. In the following, we will denote (30) together with (36) as the approximation B. Note that approximation B does not require knowledge of $\mathbb{E}\left\{\tilde{\lambda}_i\right\}$.

VI. Numerical Results

In this section, we evaluate the exact SEP [given by (20)], the upper bound [given by (21) together with (28)], the approximation A [given by (30) together with (31)] and the approximation B [given by (30) together with (36)] derived in previous sections, and compare them with Monte Carlo simulation results. The simulations were performed over 10 000 trials. We investigate the effect of SNR defined as E_D/N_0, signal-to-interference ratio (SIR) defined as $E_D/(N_I \cdot E_I)$, the number of interferers, and the number of antenna branches on the SEP. Unless otherwise stated, we consider the coherent detection of 8-PSK with optimum combining.

We first consider coherent detection of BPSK, quaternary PSK and 8-PSK using dual optimum combining ($N_A = 2$).

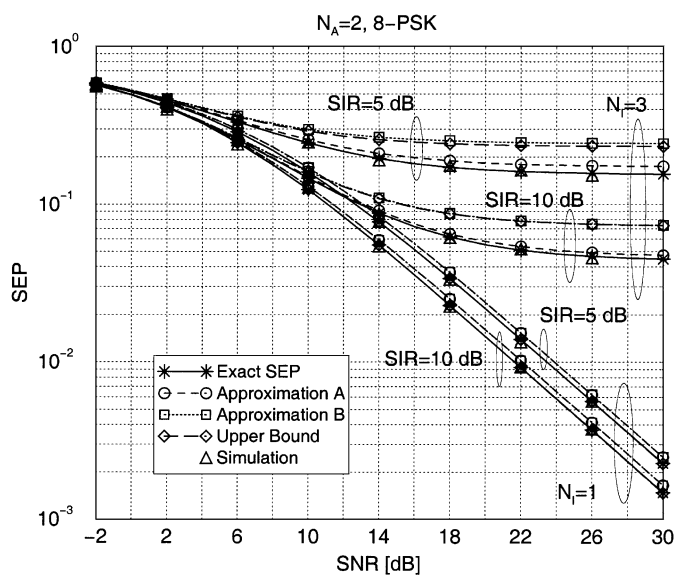

Fig. 4. SEP as a function of SNR for coherent detection of 8-PSK using dual optimum combining ($N_A = 2$) for the case of $N_I = 1$ and 3 with SIR = 5 and 10 dB.

Fig. 2 shows the SEP as a function of SNR, for $N_I = 1$, 2, and 4, and $\mathrm{SIR} = 10$ dB. The results show excellent agreement between exact analysis and simulation. The curves also exhibit an error floor when the number of interferers N_I is greater than the array DOFs, i.e., $N_A - 1$. Next, we compare in Fig. 3 the upper bound derived in Section IV with the only previously known upper bound given by [5, eq. (13)]. Note that our upper bound is 4.8 and 5.3 dB (at BEP of 10^{-3}) tighter and 4.8 and 7.4 dB (at BEP of 10^{-4}) tighter than [5, eq (13)] for $N_I = 4$ and 8, respectively.

Fig. 4 shows the SEP with dual optimum combining for the case of $N_I = 1$ and 3 with SIR = 5 and 10 dB. Note that there is the error floor for the case of $N_I = 3$ which decreases as

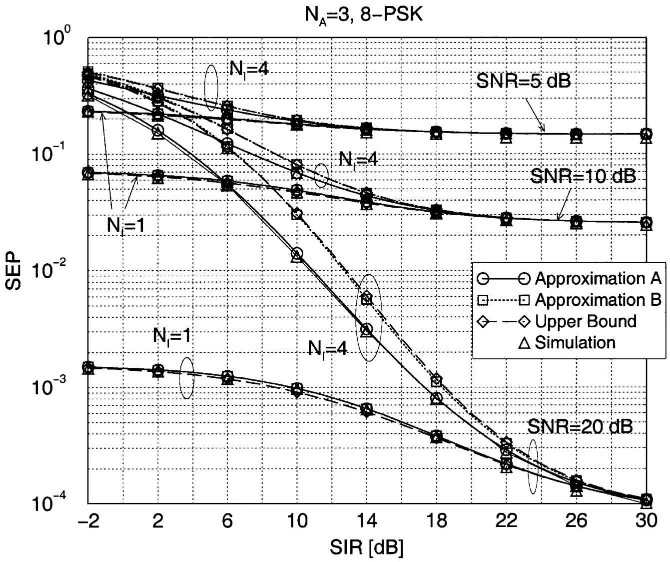

Fig. 5. SEP as a function of SIR for 8-PSK, $N_A = 3$, $N_I = 1$ and 4, SNR $=$ 5, 10, and 20 dB.

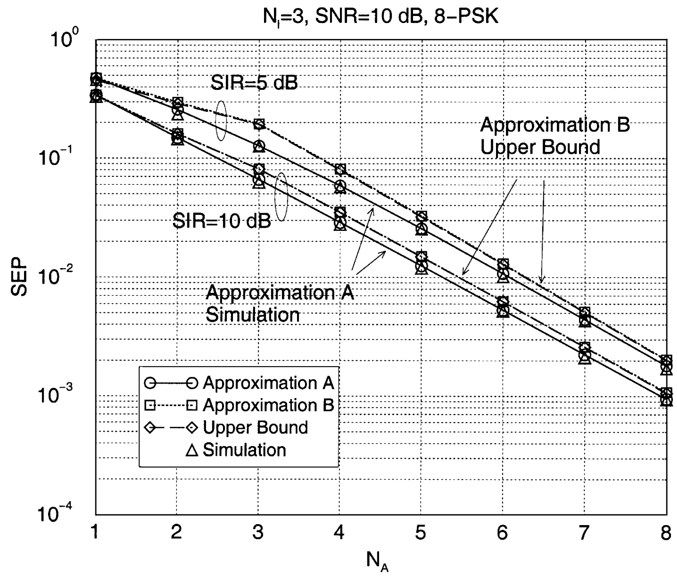

Fig. 7. SEP versus the number of antenna branches N_A for 8-PSK, $N_I = 3$, SNR $=$ 10 dB, SIR $=$ 5 and 10 dB.

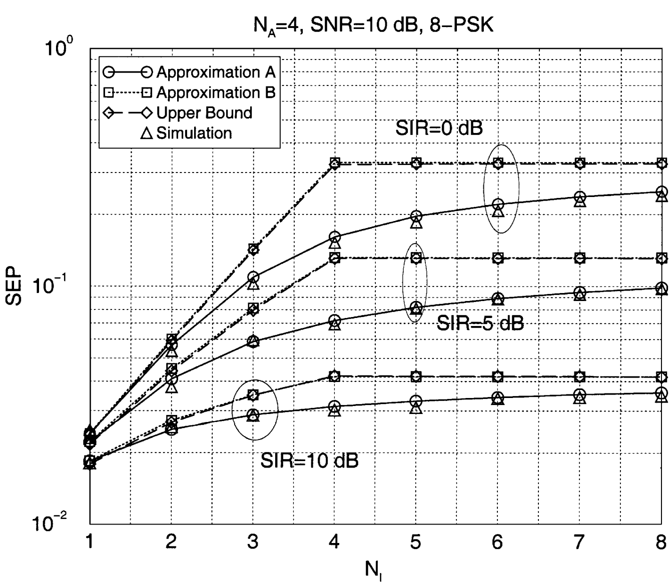

Fig. 6. SEP versus the number of interferers N_I for case of 8-PSK, $N_A = 4$, SNR $=$ 10 dB, SIR $=$ 0, 5, and 10 dB.

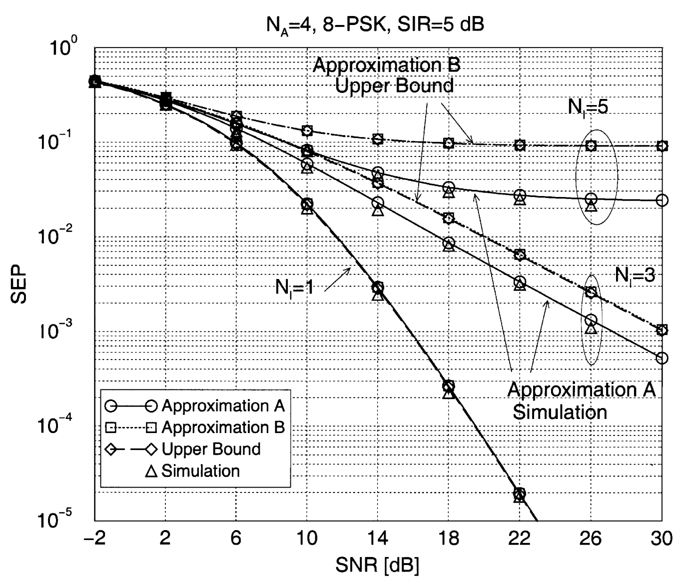

Fig. 8. SEP versus SNR for 8-PSK, $N_A = 4$, $N_I = 1$, 3, and 5; and SIR $= 5$ dB.

SIR increases. In order to further investigate the dependence of SEP on SIR, the SEP is plotted as a function of SIR in Fig. 5 for the case of $N_A = 3$, with $N_I = 1$ and 4, and SNR $=$ 5, 10, and 20 dB. Note that when the SIR is comparable with the SNR, the number of interferers plays a marginal role. Finally, the asymptotic SEP is limited by the thermal noise.

The SEP versus the number of interferers is plotted in Fig. 6 for $N_A = 4$, SNR $=$ 10 dB, and three different values of SIR (0, 5, and 10 dB). It can be seen that, when the array is overloaded, the performance does not depend significantly on the number of interferers; this behavior is accentuated for small values of SIR. The SEP versus the number of antenna branches is plotted in Fig. 7 for SNR $=$ 10 dB, SIR $=$ 5 and 10 dB, and $N_I = 3$. The figure shows that the system is able to exploit the spatial diversity provided by the increasing number of antennas (the SEP in logarithmic scale is a approximately linear in N_A). Note that our upper bound is quite close to the simulation results.

Fig. 8 shows the SEP as a function of SNR for $N_A = 4$, SIR $=$ 5 dB, and $N_I = 1$, 3, and 5. As expected, we note the presence of error floor in the overloaded case ($5 = N_I > N_A - 1 = 3$). Moreover, when $N_I < N_A$, the remaining DOFs (diversity order) is $L_{\rm Div} = N_A - N_I$ and we expect an asymptotic behavior for SEP proportional to $1/(\mathrm{SNR})^{L_{\rm Div}}$. This implies that the curve of the SEP versus SNR approaches, for large SNR, a straight line on a semilogarithmic scale with slope $-(N_A - N_I)/10$ decade/dB. Indeed, slopes of 3/10 decade/dB for $N_I = 1$, and $1/10$ decade/dB for $N_I = 3$ can be observed from Fig. 8. Similar results are shown in Fig. 9 for $N_A = 4$,

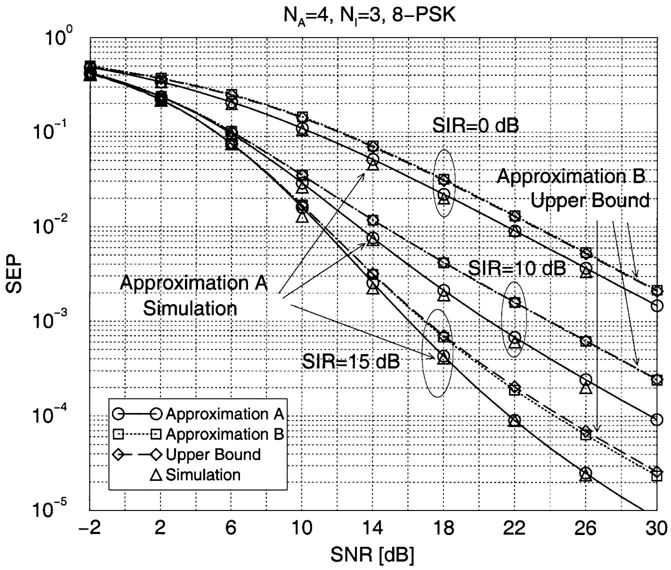

Fig. 9. SEP versus SNR for 8-PSK, $N_A = 4$, $N_I = 3$, SIR = 0, 10, and 15 dB.

$N_I = 3$, and SIR = 0, 10, and 15 dB, and the asymptotic behavior of SEP for large SNR can be seen for all values of SIR.

VII. Conclusion

In this paper, we derived the exact SEP for optimum combining of signals in the presence of multiple equal-power interferers and thermal noise. Both cases $N_A \le N_I$ and $N_A > N_I$ were investigated and, to validate the analysis, results were compared to Monte Carlo simulation results. The exact analytical SEP requires the solution of a multiple integral whose complexity depends on the smaller of N_A and N_I. This led us to derive upper bounds and approximations for reduced computational complexity.

For the case of a single interferer (as well as for a single antenna) our bound becomes the exact result, and agrees with known results for the single-interferer case given in [3] and [4]. Finally, the performance of the upper bound and the approximate formulas have been assessed by comparison with simulations.

The results show that, for typical cases considered in this paper, our new upper bound is at least 4.8 dB tighter than the only other available bound in the literature. The results also show that the approximation based on the knowledge of the expectation of the eigenvalues is close to Monte Carlo simulation results; to this end, we derived a closed-form expression for the expectation of the eigenvalues in the cases of dual optimum combining with an arbitrary number of interferers, and of two interferers with an arbitrary number of antennas. Finally, the results show that the upper bound and approximation B provide similar accuracy.

Appendix A
Distribution of Eigenvalues of the Wishart Matrix

Let us define $\mathbf{A} \in M_{m,p}$, with $m \le p$, where $M_{m,p}$ is the set of the $(m \times p)$ complex matrices, and $\tilde{\mathbf{W}}(m,p) = \mathbf{A}\mathbf{A}^\dagger$. If all the ijth elements of $\mathbf{A}$, a_{ij}, are complex values with real and imaginary part each belonging to a normal distribution $\mathcal{N}(0, 1/2)$, then the $(m \times m)$ Hermitian matrix $\tilde{\mathbf{W}}(m,p)$ is called Wishart. Moreover, the joint pdf of the ordered eigenvalues $\tilde{\boldsymbol{\lambda}} = [\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_m]^T$ of $\tilde{\mathbf{W}}(m,p)$, with $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \cdots \ge \tilde{\lambda}_m$, can be found in [17] as

$$f_{\tilde{\boldsymbol{\lambda}}}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_m) = \frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(p)\tilde{\Gamma}_m(m)} \prod_{i=1}^{m} e^{-\tilde{\lambda}_i} \tilde{\lambda}_i^{p-m} \prod_{i=1}^{m-1} \left[\prod_{j=i+1}^{m} (\tilde{\lambda}_i - \tilde{\lambda}_j)^2 \right] \tag{42}$$

with

$$\tilde{\Gamma}_m(p) = \pi^{m(m-1)/2} \prod_{i=1}^{m} (p-i)! \tag{43}$$

Appendix B
Proof of *Theorem 1*

In this appendix, we will prove *Theorem 1* using the results of Appendix A, and derive the distribution of the eigenvalues of the matrix $\tilde{\mathbf{R}}$ of (7) for arbitrary N_A and N_I. Let us consider the cases $N_A \le N_I$ and $N_A > N_I$, separately. The proof for the former case is straightforward application of Appendix A, but to prove the latter case, we need the following theorem.

Theorem 4: Suppose that $\mathbf{A} \in M_{m,p}$ and $\mathbf{B} \in M_{p,m}$ with $m \le p$, the $(p \times p)$ matrix $\mathbf{BA}$ has the same m eigenvalues as the $(m \times m)$ matrix $\mathbf{AB}$, counting multiplicity, together with an additional $p - m$ eigenvalues identically equal to zero.

Proof of Theorem 4: See [18, p. 53]. ■

Proof of Theorem 1: [Case I. $N_A \le N_I$]: When $N_A \le N_I$, $\tilde{\mathbf{R}}$ can be related directly to a Wishart matrix, since the entries of the random matrix $\mathbf{C}_I$ are i.i.d. Gaussian random variables with zero-mean, independent real and imaginary parts, each with variance 1/2. So, we can write

$$\tilde{\mathbf{R}} = \sum_{j=1}^{N_I} \boldsymbol{c}_{I,j} \boldsymbol{c}_{I,j}^\dagger = \mathbf{C}_I \mathbf{C}_I^\dagger = \tilde{\mathbf{W}}(N_A, N_I) \tag{44}$$

where $\tilde{\mathbf{W}}(N_A, N_I)$ is a $(N_A \times N_A)$ complex Wishart matrix. Thus, the joint pdf of the eigenvalues of $\tilde{\mathbf{R}}$ is given by (42) and (43) with $m = N_A$ and $p = N_I$. ■

Proof of Theorem 1: [Case II. $N_A > N_I$]: When $N_A > N_I$, $\tilde{\mathbf{R}}$ can still be related to the Wishart matrix, by means of *Theorem 4*. In fact, by introducing the $(N_I \times N_A)$ matrix $\mathbf{A} \triangleq \mathbf{C}_I^\dagger$ and the $(N_A \times N_I)$ matrix $\mathbf{B} \triangleq \mathbf{C}_I$, then the $(N_A \times N_A)$ matrix $\mathbf{BA} = \mathbf{C}_I \mathbf{C}_I^\dagger = \tilde{\mathbf{R}}$ has the same N_I eigenvalues as the $(N_I \times N_I)$ matrix $\mathbf{AB} = \mathbf{C}_I^\dagger \mathbf{C}_I$, and the additional $N_A - N_I$ eigenvalues are equal to zero. Moreover, since $N_A > N_I$, $\mathbf{C}_I^\dagger \mathbf{C}_I = \tilde{\mathbf{W}}(N_I, N_A)$ is a $(N_I \times N_I)$ complex Wishart matrix, and therefore, $\tilde{\mathbf{R}}$ has total of N_A eigenvalues, where N_I eigenvalues have the joint pdf given by (42) with $m = N_I$ and $p = N_A$, and the additional $N_A - N_I$ eigenvalues are identically equal to zero.

APPENDIX C
AN INEQUALITY

Here we prove the following inequality:

Theorem 5: For any $K_1 \geq 0$, $K_2 \geq 0$, and $x_i \in \mathbb{R}^+$, where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$

$$\prod_{i=1}^{n} \left[\frac{K_1}{K_1 + \frac{K_2}{x_i}} \right] \leq \left[\frac{K_1}{K_1 + \frac{K_2 n}{\sum_{i=1}^{n} x_i}} \right]^n . \tag{45}$$

Proof: We find the maximum of the function

$$h(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \left[\frac{K_1}{K_1 + \frac{K_2}{x_i}} \right] \tag{46}$$

subject to the constraint

$$C(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i - S = 0. \tag{47}$$

To this aim, by using the Lagrange's multipliers, we introduce a parameter ϵ and the function

$$H(x_1, \ldots, x_n) = h(x_1, \ldots, x_n) + \epsilon\, C(x_1, \ldots, x_n) \tag{48}$$

and set the partial derivatives of $H(x_1, \ldots, x_n)$ to zero. We can observe that the condition $\partial H(.)/\partial x_j = 0$ requires that

$$\epsilon = -h(x_1, \ldots, x_n) \left[\frac{K_2}{K_1 x_j^2 + K_2 x_j} \right] \qquad \forall j = 1, \ldots, n. \tag{49}$$

This equation is satisfied by choosing $x_j = S/n \;\; \forall j$, that, for $x_i \in \mathbb{R}^+$, provides a maximum of the function in (46). To see that this cannot be a minimum, it is sufficient to let one x_i going to zero, keeping finite S. In this case (46) goes to zero, whereas the right member of (45) remains finite. ■

APPENDIX D
CALCULATION OF $\mathbb{E}_y \{B(Y, \theta)\}$

Let $K_3 = N_{\min} N_0 / E_I$, and $K_4 = (c_{\text{MPSK}} N_{\min} E_D / E_I) / \sin^2 \theta$, and then (23) can be written as

$$B(y, \theta) = \left[\frac{y + K_3}{y + K_3 + K_4} \right]^{N_{\min}} . \tag{50}$$

Using (22) and letting $z = y + K_3 + K_4$ gives results as shown in (51) at the bottom of the page. Note (52) and (53) at the bottom of the page. Multiplying (52) and (53) gives (54) at the bottom of the page. Substituting (54) into (51), and noting that $K_3 + K_4 > 0$ we obtain (55) at the bottom of the page,

$$\mathbb{E}_Y\{B(Y, \theta)\} = \int_0^{\infty} B(y, \theta) f_Y(y) dy = \frac{1}{\Gamma(N_A N_I)} \int_{K_3 + K_4}^{\infty} \left(1 - \frac{K_4}{z} \right)^{N_{\min}} (z - K_3 - K_4)^{N_A N_I - 1} e^{-(z - K_3 - K_4)} dz \tag{51}$$

$$\left(1 - \frac{K_4}{z} \right)^{N_{\min}} = \sum_{n=0}^{N_{\min}} \binom{N_{\min}}{n} (-K_4)^n z^{-n} \tag{52}$$

$$(z - K_3 - K_4)^{N_A N_I - 1} = \sum_{m=0}^{N_A N_I - 1} \binom{N_A N_I - 1}{m} (-K_3 - K_4)^{N_A N_I - 1 - m} z^m \tag{53}$$

$$\begin{aligned} &\left(1 - \frac{K_4}{z} \right)^{N_{\min}} (z - K_3 - K_4)^{N_A N_I - 1} \\ &= \sum_{n=0}^{N_{\min}} \sum_{m=n}^{N_A N_I - 1} \binom{N_{\min}}{n} \binom{N_A N_I - 1}{m} (-1)^{N_A N_I - 1 - (m-n)} (K_4)^n (K_3 + K_4)^{N_A N_I - 1 - m} z^{m-n} \\ &+ \sum_{n=1}^{N_{\min}} \sum_{m=0}^{n-1} \binom{N_{\min}}{n} \binom{N_A N_I - 1}{m} (-1)^{N_A N_I - 1 - (m-n)} (K_4)^n (K_3 + K_4)^{N_A N_I - 1 - m} z^{-(n-m)} \end{aligned} \tag{54}$$

$$\begin{aligned} \mathbb{E}_Y \{B(Y, \theta)\} &= \frac{e^{+(K_3 + K_4)}}{\Gamma(N_A N_I)} \\ &\times \left\{ \sum_{n=0}^{N_{\min}} \sum_{m=n}^{N_A N_I - 1} \binom{N_{\min}}{n} \binom{N_A N_I - 1}{m} (-1)^{N_A N_I - 1 - (m-n)} (K_4)^n (K_3 + K_4)^{N_A N_I - 1 - m} \times \Gamma_c (m - n + 1,\, K_3 + K_4) \right. \\ &+ \left. \sum_{n=1}^{N_{\min}} \sum_{m=0}^{n-1} \binom{N_{\min}}{n} \binom{N_A N_I - 1}{m} (-1)^{N_A N_I - 1 - (m-n)} (K_4)^n (K_3 + K_4)^{N_A N_I - 1 - m} \times E_{n-m} (K_3 + K_4) \right\} \end{aligned} \tag{55}$$

where $\Gamma_c(k+1,x)$ is the complementary incomplete gamma function defined by [13 (8.350.2), p. 949]

$$\Gamma_c(k+1,x) \triangleq \int_x^{\infty} u^k e^{-u} du, \quad x>0, \quad k=0,1,2,\ldots. \tag{56}$$

and $E_k(x)$ is given by

$$E_k(x) \triangleq \int_x^{\infty} \frac{e^{-u}}{u^k} du, \quad x>0, \quad k=1,2,3,\ldots. \tag{57}$$

The special case of $k=1$ is known as the exponential integral.

Integrating (56) and (57) by parts, the recurrence relations can be obtained as

$$\Gamma_c(k+1,x) = x^k e^{-x} + k\,\Gamma_c(k,x), \qquad k=1,2,3,\ldots \tag{58}$$

$$E_k(x) = x^{-(k-1)}\frac{e^{-x}}{(k-1)} - \frac{1}{(k-1)}E_{k-1}(x), \quad k=2,3,4,\ldots. \tag{59}$$

By solving the recurrence relations (58) and (59), we get

$$\Gamma_c(k+1,x) = k! e^{-x} \sum_{l=0}^{k} \frac{x^l}{l!}, \quad x>0, \quad k=0,1,2,\ldots \tag{60}$$

$$E_k(x) = \frac{(-1)^{k-1}}{(k-1)!} \times \left\{ e^{-x} \sum_{l=1}^{k-1} (-1)^l (l-1)!\, x^{-l} + E_1(x) \right\}, \quad x>0, \quad k=1,2,3,\ldots \tag{61}$$

Using (60) and (61) results in (62) as shown at the bottom of the page.

APPENDIX E
SOME RESULTS ON MEAN EIGENVALUES OF $\tilde{\mathbf{W}}(m,p)$

Since (42) is a product of several terms in the form $e^{-a\lambda}\lambda^n$, the expected value of the eigenvalues of $\tilde{\mathbf{W}}(m,p)$ can be written in a closed form for all values of m and p.

As an example, we found the following results:

- $m=1$, $p \geq 1$: in this case it is straightforward to verify that $\mathbb{E}\left\{\tilde{\lambda}_1\right\} = p$;
- $m=2$, $p \geq 2$: after some algebra, we get (63) as shown at the bottom of the page.

It can be shown that $\mathbb{E}\left\{\tilde{\lambda}_1\right\}$ can be further simplified to

$$\mathbb{E}\left\{\tilde{\lambda}_1\right\} = \frac{p(p^2+3)}{2^p} + \sum_{k=2}^{p-1} \frac{1}{2^{2p-k-1}} \binom{2p-k-1}{p-2}\binom{k}{2}. \tag{64}$$

To derive $\mathbb{E}\left\{\tilde{\lambda}_2\right\}$, we first observe that

$$\sum_{i=1}^{m} \tilde{\lambda}_i = tr\left[\tilde{\mathbf{W}}(m,p)\right] \tag{65}$$

where $tr[\cdot]$ is the trace of the matrix. Then

$$\begin{aligned}\sum_{i=1}^{m} \mathbb{E}\left\{\tilde{\lambda}_i\right\} &= \mathbb{E}\left\{\sum_{i=1}^{m} \tilde{\lambda}_i\right\} \\ &= \mathbb{E}\left\{tr\left[\tilde{\mathbf{W}}(m,p)\right]\right\} \\ &= \mathbb{E}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} |a_{i,j}|^2\right\} = m\cdot p\end{aligned} \tag{66}$$

$$\begin{aligned}\mathbb{E}_Y\{B(Y,\theta)\} =& \frac{1}{\Gamma(N_A N_I)} \\ &\times \left\{ \sum_{n=0}^{N_{\min}} \sum_{m=n}^{N_A N_I -1} \binom{N_{\min}}{n}\binom{N_A N_I - 1}{m} (-1)^{N_A N_I - 1 - (m-n)} (K_4)^n \times (m-n)! \sum_{l=0}^{m-n} \frac{1}{l!} (K_3+K_4)^{N_A N_I - 1 - m + l} \right. \\ &+ \sum_{n=1}^{N_{\min}} \sum_{m=0}^{n-1} \binom{N_{\min}}{n}\binom{N_A N_I - 1}{m} \frac{(-1)^{N_A N_I}}{(n-m-1)!} (K_4)^n \\ &\left. \times \left[\sum_{l=1}^{n-m-1} (-1)^l (l-1)!\, (K_3+K_4)^{N_A N_I - 1 - m - l} + (K_3+K_4)^{N_A N_I - 1 - m}\, e^{+(K_3+K_4)}\, E_1(K_3+K_4) \right] \right\}\end{aligned} \tag{62}$$

$$\begin{aligned}&\mathbb{E}\left\{\tilde{\lambda}_1\right\} \\ &= \frac{1}{(p-1)!(p-2)!} \\ &\times \left[\sum_{k=1}^{p-1} \left(\frac{(p-1)!}{(p-1-k)!} + \frac{(p+1)!}{(p+1-k)!} - \frac{2(p)!}{(p-k)!} \right) \frac{(2p-1-k)!}{2^{2p-k}} - \frac{p!(p-1)!}{2^{p-1}} + (p+1)! \left(\frac{(p-1)!}{2^p} + \frac{(p-2)!}{2^{p-1}} \right) \right]\end{aligned} \tag{63}$$

where $a_{i,j}$ is the ijth element of $\mathbf{A}$ with $\tilde{\mathbf{W}}(m,p) = \mathbf{A}\mathbf{A}^{\dagger}$, and the last equality is due to the following normalization (see Appendix A):

$$\mathbb{E}\left\{|a_{i,j}|^2\right\} = 1, \quad i = 1, \dots, m \quad j = 1, \dots, p.$$

Finally, by using (66), we get

$$\mathbb{E}\left\{\tilde{\lambda}_2\right\} = 2p - \mathbb{E}\left\{\tilde{\lambda}_1\right\} \tag{67}$$

where $\mathbb{E}\left\{\tilde{\lambda}_1\right\}$ is given by (64).

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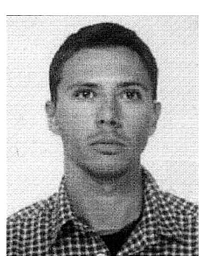

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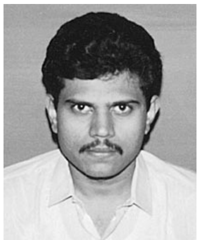

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