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Defn Let a & b be tuples of the same length & let A be a set of parameters.

We say that a & b have the same Lascar strong type over A, denoted $a \equiv^{LS}_A b$, if for each A-invariant equivalence relation E on tuples of same length as a (& b) that has a bounded number of equiv classes, then we have $E(a,b)$.

Note that equality of Lascar strong types over A is the finest bounded A-invariant equivalence relation on tuples of the relevant length.

Notation: Define $d_A(a,b) :=$ least $n<\omega$ st. there are $a=a_0, a_1, \dots, a_n = b$ st. for each $i<n$ there is an infinite A-indiscernible sequence starting with (a_i, a_{i+1}), or ∞ if n doesn't exist.

Lemma $a \equiv^{LS}_A b$ iff $d_A(a,b) < \infty$.

Proof $\Leftarrow$: First assume that $d_A(a,b) \leq 1$ ie $\exists\, a_2, a_3, a_4, \dots$ st. $\underset{a}{a_0}, \underset{b}{a_1}, a_2, a_3 \dots$ is an A-indiscernible sequence.

If $a \equiv^{LS}_A b$, OK.
If not, extend the sequence to an arbitrary length $(a_i : i<\lambda)$. By invariance, $a_i \not\equiv^{LS}_A a_j$ $\forall i<j<\lambda$, contradicting boundedness.

If $d_A(a,b) = n < \omega$, then $\exists\, a=a_0, a_1, \dots, a_n = b$ st $\forall i<n$ $d_A(a_i, a_{i+1}) = 1$
so $a = a_0 \equiv^{LS}_A a_1 \equiv^{LS}_A a_2 \equiv^{LS}_A \dots \equiv^{LS}_A a_n = b$.

$\Rightarrow$: Clearly "$d_A(x,y) < \infty$" is an A-invariant equiv reln.

(symmetric since we can always extend an indiscernible sequence to negative indices $\ldots a_{-2}\ a_{-1}\ \underset{a}{\overset{\shortparallel}{a_0}}\ \underset{b}{\overset{\shortparallel}{a_1}}\ a_2 \ldots$).

If it is bounded, then by defn, $d_A(x,y) < \infty \Rightarrow x \equiv_A^{ls} y$.

If not, then for λ sufficiently big, $\exists (a_i : i < \lambda)$ s.t.

$\forall i < j < \lambda$, $d_A(a_i, a_j) = \infty$.

Extract an A-indiscernible sequence $(a_i' : i < \omega)$
$\exists i < j < \lambda$ s.t. $a_0' a_1' \equiv_A^{is} a_i a_j$

$\Rightarrow d_A(a_0' a_1') = \infty$. But $d_A(a_0', a_1') \leq 1$. $\square$.

Lemma (Extension for strong types)

(T simple) Let $A \subseteq B$ & let a be a tuple. Then there exists a tuple a' s.t. $a' \equiv_A^{ls} a$ & $a' \underset{A}{\downarrow} B$

Proof Let $(a_i : i < \lambda)$ be a large Morley sequence in $tp(a/A)$ with $a_0 = a$.
(need ~~[illegible]~~ cof $\lambda > |Cr(T)| + |a|^+$)

By simplicity, there exists $I_0 \subseteq \lambda$ s.t. $B \underset{a_{i \in I_0}}{\downarrow} a_{i<\lambda}$.

Let $j = \sup I_0$. So $a_{i<\lambda} \underset{a_{i \in I_0}}{\downarrow} B \overset{trans}{\Rightarrow} a_{i \leq j} \underset{a_{i \in I_0}}{\downarrow} B$ ie $a_{i<j} a_j \underset{a_{i \in I_0}}{\downarrow} B$

$\overset{trans}{\Rightarrow} a_j \underset{a_{i<j}}{\downarrow} B \overset{trans}{\Rightarrow} a_j \underset{A a_{i<j}}{\downarrow} B$

But a_j in a Morley sequence $\Rightarrow a_j \underset{A}{\downarrow} a_{i<j}$

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So by transitivity $a_j \underset{A}{\perp\!\!\!\perp} B$

Now put $a' = a_j$ & note that $a \equiv^{Ls}_{A} a'$ by previous lemma.

<u>Lemma</u> (T simple) Let $p_1(x,a)$ & $p_2(x,b)$ be partial types over Aa & Ab respectively. Assume $b \equiv^{Ls}_{A} b'$ & $a \underset{A}{\perp\!\!\!\perp} bb'$

Also assume that $p_1(x,a) \cup p_2(x,b)$ dnd / A.

Then $p_1(x,a) \cup p_2(x,b')$ dnd / A.

<u>Proof</u> $\cancel{a \underset{A}{\perp\!\!\!\perp} b_0 b_1 \Rightarrow a \underset{Ab_0}{\perp\!\!\!\perp} b_1}$

~~By Since b~~

First assume that $d_A(b,b') \le 1$.

Then $\exists$ A-indiscernible sequence $(\underset{b}{\underset{\parallel}{b_0}}\ \underset{b'}{\underset{\parallel}{b_1}}\ b_2 \cdots)$

So $(b_1, b_2, \dots)$ is an Ab_0-indiscernible sequence.

By assumption, $a \underset{A}{\perp\!\!\!\perp} b_0 b_1$ so $a \underset{Ab_0}{\perp\!\!\!\perp} b_1$

So $\exists$ Ab_0-automorphic image of $(b_1, b_2, \dots)$ ~~st~~, say

$(b_1', b_2', \dots)$ s.t. $(b_1', b_2', \dots)$ is Aab_0-indiscernible in $tp(b_1/Ab_0)$

But since $tp(b_1'/Aab_0) = tp(b_1/Ab_0)$,

we may assume that $b_1' = b_1$. b_2 etc weren't important to begin with, so

we may assume $(b_1, b_2, \dots)$ is Aab_0-indiscernible.

So $\forall i > 0$, $tp(a b_0 b_i / A) = tp(a b b' / A)$.

let $f_n \in Aut_A \mathfrak{U}$ s.t. $f_n(b_i) = b_{i+n}$. $\forall i \geq 0$.

let $a_n = f_n(a)$.

So we have a sequence $(a_i b_i)$ s.t. $tp(a_i b_i b_{i+j} / A)$
$= tp(a_0 b_0 b_j / A) = tp(a b b' / A)$.

So we can find an A-indiscernible sequence $(a_i' b_i')$ s.t.

$\forall i \, \forall j > 0 \quad a_i' b_i' b_{i+j}' \equiv_A a b b'$.

So we may assume $a_0' = a$, $b_0' = b$ & $b_1' = b'$.

So we have $p_1(x, a_0') \cup p_2(x, b_0')$ dnd $/A$. by assumption.

So by a previous lemma (from last lecture), $/A$

$\bigcup \{ p_1(x, a_i') \cup p_2(x, b_i') \}$ dnd $/A$.

In particular $p_1(x, a_0') \cup p_2(x, b_1')$ dnd $/A$

ie $p_1(x, a) \cup p_2(x, b')$ dnd $/A$.

So now say $d_A(a, b) = n < \omega$.

So $\exists$ sequence $a = c_0, \dots, c_n = b$. s.t. (c_i, c_{i+1}) starts an A-indiscernible sequence.

Have $b = c_0 \equiv_A^{LS} \cdots \equiv_A^{LS} c_n = b'$ since $d_A(c_i, c_{i+1}) \leq 1$.

X

$\cdot/2$

Remember $a \underset{A}{\downarrow} bb'$.

By extn, $\exists\, a' \underset{Abb'}{\equiv} a$ s.t. $a' \underset{Abb'}{\downarrow} c_1, \dots, c_{n-1}$

So We may assume $a \underset{Abb'}{\downarrow} c_1 \cdots c_{n-1}$

$\Rightarrow \quad a \underset{A}{\downarrow} c_0 \cdots c_n$.

Then $p_1(x,a) \cup p_2(x, c_i)$ dnd A $\forall i \le n$

So in particular, $p_1(x,a) \cup p_2(x,b')$ dnd $/A$. $\square$.

<u>Exercise</u> T simple & $(I, <)$ any linear order.

$\{a_i : i \in I\}$ satisfying $\forall i \ a_i \underset{C}{\downarrow} a_{<i}$

$\Rightarrow \ \forall J_1, J_2 \subseteq I$. s.t. $J_1 \cap J_2 = \emptyset$ we have $a_{\in J_1} \underset{C}{\downarrow} a_{\in J_2}$

Hint: 1st consider J_1, J_2 finite.

My lecture...

2/27 <u>Corollary (The Independence Theorem)</u>

Let A be a set. Let b_1 & b_2 be tuples s.t. $b_1 \underset{A}{\downarrow} b_2$.

Let a_1 & a_2 be tuples s.t. $a_1 \overset{\text{is}}{\underset{A}{\equiv}} a_2$ and $a_i \underset{A}{\downarrow} b_i$, $i=1,2$.

Then there exists a tuple a s.t. $a \overset{\text{is}}{\underset{A}{\equiv}} a_i$, $a \underset{A}{\downarrow} b_1 b_2$

and $a \underset{Ab_i}{\equiv} a_i$, $i = 1, 2$

Claim: Let a_1, a_2, b be tuples st. $a_1 \equiv^{Ls}_A a_2$.

Then there exists a b' st. $b \equiv^{Ls}_A b'$ and $\exists f \in \mathrm{Aut}_A \mathfrak{U}$ st.

$f(a_2 b) = a_1 b'$.

Proof First assume $d_A(a_1, a_2) \leq 1$.

So $\exists (a_1, a_2, a_3, \dots)$ A-indiscernible.

Let $B = \{b_j\}$ enumerate representatives of all possible

Lstp(b''/A) where $b'' \equiv_A b$.

By extension/extraction, we get a similar sequence

$(a_i' : i > 0)$ to $(a_i : i > 0)$ st. (a_i') is $A \cup B$-indiscernible.

So $\exists g \in \mathrm{Aut}_A \mathfrak{U}$ st. $g(a_i) = a_i'$ for $i > 0$.

Let $b_j' = g^{-1}(b_j)$.

Then (a_i) is $A \cup B'$-indiscernible and $\forall b'' \equiv_A b$ $\exists j$ st. $b'' \equiv^{Ls}_A b_j'$

$(E(g(b''), b_j) \Leftrightarrow E(b'', b_j)$ so $g(b_2) \equiv^{Ls}_A b_j \Rightarrow b_2 \equiv^{Ls}_A b_j')$

So say $b \equiv^{Ls}_A b_m'$, some $b_m \in B$.

Let b' be a tuple st. $a_2 b \equiv_{A \cup B'} a_1 b'$

So in particular, $b b_m' \equiv_A b' b_m'$.

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So since $b \equiv^{LS}_A b_m'$ & $bb_m' \equiv_A b'b_m'$, we have $b' \equiv^{LS}_A b_m'$

So $b \overset{LS}{\underset{A}{\equiv}} b'$ and $a_2 b \equiv_{A \cup B'} a_1 b'$ implies

$\exists f \in \mathrm{Aut}_A \mathcal{U}$ s.t. $f(a_2 b) = a_1 b'$.

Now let $d_A(a_1, a_2) = n < \omega$.

Then $\exists$ a sequence $a_1 = c_1, \ldots, c_n = a_2$ s.t. $c_i \overset{LS}{\underset{A}{\equiv}} c_{i+1}$ & $d_A(c_i, c_{i+1}) = 1$.

So given c_i, c_{i+1} & b_{i+1} s.t. $c_i \overset{LS}{\underset{A}{\equiv}} c_{i+1}$, $\exists b_i$ s.t. $b_i \overset{LS}{\underset{A}{\equiv}} b_{i+1}$

and $\exists f_{i+1} \in \mathrm{Aut}_A \mathcal{U}$ s.t. $f_{i+1}(c_{i+1} b_{i+1}) = c_i b_i$

So $c_n = a_2$, $b_n = b$, $c_1 = a_1$ and $b_1 := b'$

and $f_2 \circ \cdots \circ f_{n-1} \circ f_n (c_n b_n) = c_1 b_1$ and $b_1 \overset{LS}{\underset{A}{\equiv}} \cdots \overset{LS}{\underset{A}{\equiv}} b_n$ claim $\square$

<u>Proof of Independence Theorem</u>

By claim, $\exists f \in \mathrm{Aut}_A \mathcal{U}$ st. $f(a_2 b_2) = a_1 b_2'$ where $b_2' \overset{LS}{\underset{A}{\equiv}} b_2$.

By strong type extn, we can find $b_2'' \overset{LS}{\underset{Aa_1}{\equiv}} b_2'$ s.t. $b_2'' \underset{Aa_1}{\downarrow} b_1 b_2$.

So $b_2 \overset{LS}{\underset{A}{\equiv}} b_2''$ and $\exists g \in \mathrm{Aut}_{Aa_1} \mathcal{U}$ st. $g(b_2') = b_2''$ so $g \circ f(a_2 b_2) = a_1 b_2''$.

So we may assume $b_2' \underset{Aa_1}{\downarrow} b_1 b_2$

By assumption $a_2 \underset{A}{\downarrow} b_2$ and $a_1 b_2' \equiv_A a_2 b_2 \overset{\text{invariance}}{\Longrightarrow} b_2' \underset{A}{\downarrow} a_1$

So by transitivity $b_2' \underset{A}{\downarrow} a_1 b_1 b_2$.

Claim: (1) $a_1 \mathop{\downarrow}_A b_1 b_2'$ (2) $b_1 \mathop{\downarrow}_A b_2 b_2'$

(1): $a_1 \mathop{\downarrow}_A b_1 b_2' \iff a_1 \mathop{\downarrow}_A b_1$ & $a_1 \mathop{\downarrow}_{Ab_1} b_2'$

But $b_2' \mathop{\downarrow}_A a_1, b_1, b_2 \Rightarrow b_2' \mathop{\downarrow}_A a_1 b_1 \Rightarrow b_2' \mathop{\downarrow}_{Ab_1} a_1$

(2): $b_2' \mathop{\downarrow}_A a_1 b_1 b_2 \Rightarrow b_2' \mathop{\downarrow}_A b_1 b_2$ claim □

Now let $p(x, y_1) := tp(a_1 b_1/A)$ & $q(x, y_2) := tp(a_2 b_2/A)$.

Then since $f(a_2 b_2) = a_1 b_2'$, we have $a_1 \models q(x, b_2')$.

Thus $a_1 \models p(x, b_1) \cup q(x, b_2')$.

By (i) $p(x, b_1) \cup q(x, b_2')$ dnd $/A$.

By (ii) & previous lemma from last lecture, $p(x, b_1) \cup q(x, b_2)$ dnd $/A$.

So by improved extn, it is realised by some a st $a \mathop{\downarrow}_A b_1 b_2$

(and so $a \equiv_{Ab_i} a_i$ & $a \equiv^{Ls}_A a_i$). □

Cor We could have required $a \overset{Ls}{\equiv}_{Ab_i} a_i \quad \forall i \in \{1,2\}$.

Proof Find $a_1' \overset{Ls}{\equiv}_{Ab_1} a_1$ st. $a_1' \mathop{\downarrow}_{Ab_1} a_1$. WMA $a_1' \mathop{\downarrow}_{Aa_1b_2} b_2$. Since $a_1 \mathop{\downarrow}_A b_1$, we get $\boxed{a_1 \mathop{\downarrow}_A b_1 a_1'}$. Since $a' \mathop{\downarrow}_{Ab_1} a_1$ we get $a_1' \mathop{\downarrow}_{Ab_1} a_1 b_2 \Rightarrow$

$\boxed{a_1' b_1 \mathop{\downarrow}_A b_2}$ (Since $b_1 \mathop{\downarrow}_A b_2$). We may replace b_1 with $b_1 a_1'$.

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Similarly, we can replace b_2 with $b_2 a_2'$ s.t. $a_2 \overset{\mathrm{LS}}{\underset{Ab_2}{\equiv}} a_2'$.

Apply the independence thm to find a s.t.

① $a \underset{A}{\downarrow} b_1 b_2 a_1' a_2'$ ② $a \underset{Ab_i a_i'}{\equiv} a_i$

Then $a_i \overset{\mathrm{LS}}{\underset{Ab_i}{\equiv}} a_i' \Rightarrow a \overset{\mathrm{LS}}{\underset{Ab_i}{\equiv}} a_i' \Rightarrow a \overset{\mathrm{LS}}{\underset{Ab_i}{\equiv}} a_i$ □

Cor Let A be a set, a_0, a_1 tuples st. $a_0 \overset{\mathrm{LS}}{\underset{A}{\equiv}} a_1$ and $a_0 \underset{A}{\downarrow} a_1$. Then a_0, a_1 start a morley sequence over A. (The converse is obvious.)

Proof We construct a sequence $\{a_i : i < \lambda\}$ starting with a_0, a_1 st. $\forall i < j < \lambda$, $a_i a_j \underset{A}{\equiv} a_0 a_1$ and $a_i \underset{A a_{<i}}{\equiv} a_j$

Let $q(x,y) = tp(a_0 a_1 / A)$.

Assume we have $(a_i : i < \alpha)$.

Case 1: α limit, For $i < \alpha$, let $p_i(x) := tp(a_i / A a_{<i})$.

Then $\{p_i\}$ is an increasing sequence. Let $p_\alpha = \bigcup_{i<\alpha} p_i$

Let $a_\alpha \models p_\alpha$. Since $a_i \underset{A}{\downarrow} a_{<i}$ $\forall i < \alpha$, by finite character

$a_\alpha \underset{A}{\downarrow} a_{<i}$ $\forall i < \alpha \Rightarrow a_\alpha \underset{A}{\downarrow} a_{<\alpha}$.

$p_\alpha \supseteq \bigcup\{\underset{\substack{\| \\ p_1(a_0,x)}}{q(a_i,x)} : i<\alpha\} \Rightarrow \forall i<\alpha \;\; a_i a_\alpha \equiv_A a_0 a_1$

$\Rightarrow a_\alpha \overset{LS}{\underset{A}{\equiv}} a_0.$

Case 2: $\alpha = \beta+1$. We ~~ha~~ have $\overset{b_0}{a_{<\beta}} \underset{A}{\downarrow} \overset{b_1}{a_\beta}$ & $\overset{b_0}{a_{<\beta}} \underset{A}{\downarrow} \overset{c_0}{a_\beta}$ &

we can find a' st. $a' \models q(a_\beta, x)$

so $\underset{b_1}{a_\beta} \underset{A}{\downarrow} \underset{c_1}{a'}$ and $\underset{c_1}{a'} \overset{LS}{\underset{A}{\equiv}} \underset{c_0}{a_\beta}$ (because $q(a_\beta, a')$ says so).

By independence theorem $\exists\, a_\alpha \underset{A}{\downarrow} a_{<\alpha}$ st.

① $a_\alpha \equiv_{A a_{<\beta}} a_\beta$ & ② $a_\alpha \equiv_{A a_\beta} a'$.

① $\Rightarrow \forall i<\beta \;\; a_i a_\alpha \equiv_A a_i a_\beta \equiv_A a_0 a_1.$

$\forall i \leq \beta \;\; a_\alpha \equiv_{A a_{<i}} a_\beta \equiv_{A a_{<i}} a_i$

② $\Rightarrow q(a_\beta, a_\alpha) \Rightarrow a_\beta a_\alpha \equiv_A a_0 a_1$. & the construction is complete.

Extract an A-indiscernible sequence (a_i').

Then $\exists\, i<j$ st $a_0' a_1' \equiv_A a_i a_j \equiv_A a_0 a_1$.

So we may assume $a_0 a_1 \equiv_{A} a_0' a_1'$.

Also $a_i' \underset{A}{\downarrow} a'_{<i}$ (since $\forall i<\lambda \;\; a_i \underset{A}{\downarrow} a_{<i}$) $\square$

Now use this corollary ...

<u>Cor</u> $a \overset{LS}{\underset{A}{\equiv}} b \iff d_A(a,b) \leq 2.$

Assume $a \overset{LS}{\underset{A}{\equiv}} b$. Find c st. $c \underset{A}{\downarrow} ab$, $c \overset{LS}{\underset{A}{\equiv}} a \Rightarrow \underbrace{d_A(a,c)}_{\leq 1}$

$c \overset{LS}{\underset{A}{\equiv}} b \Rightarrow \underbrace{d_A(b,c)}_{\leq 1}.$

$\Rightarrow d_A(a,b) \leq 2.$

<u>Cor</u> Equality of Lascar strong types is type-definable,

ie $\exists\ E(x,y)$ over A st. $\models E(a,b) \iff a \overset{LS}{\underset{A}{\equiv}} b$.

Take $p(x) = E(x,a)$ then $p(x)$ is logically equivalent to $Lstp(a/A)$.

IOU From Previous lecture

If $a \underset{c}{\downarrow} b$ then $D(a/c, \Xi) = D(a/bc, \Xi)$.

Proof

$$D(a/c, \Xi) \geq D(a/bc, \Xi) \checkmark.$$

We prove by induction on α:

if $\xi \in D(a/c, \Xi) \cap \Xi^{\alpha} \Rightarrow \xi \in D(a/bc, \Xi)$.

$\alpha = 0$: $\checkmark$

$\alpha = \text{limit}$: $\checkmark$

$\alpha = \beta+1$: $\xi = \theta \wedge (\varphi, \psi)$.

Then $\exists d$ s.t. $\varphi(x,d)$ divides$_{/c}$ wrt ψ.

Write $p = tp(a/c)$.

Then $\theta \in D(p \wedge \varphi(x,d), \Xi)$.

This means that $div_{cd,\theta}(x) \wedge p(x) \wedge \varphi(x,d)$ is consistent.

Let a' realise it.

Then $a' \underset{c}{\equiv} a$, so find d' s.t. $a'd \underset{c}{\equiv} ad'$.

Furthermore, find $d'' \underset{ac}{\equiv} d'$ s.t. $d'' \underset{ac}{\downarrow} b$

then: (1) $\theta \in D(a'/cd, \Xi)$ so $\theta \in D(a/cd', \Xi) \Rightarrow \theta \in D(a/cd'', \Xi)$

& (2) $\varphi(x,d) \in tp(a'/cd) \Rightarrow \varphi(x,d'') \in tp(a/cd'')$;

since $d'' \underset{c}{\equiv} d$, $\varphi(x,d'')$ divides$/c$.

(3) $d'' \underset{ac}{\downarrow} b$ & $a \underset{c}{\downarrow} b \Rightarrow d''a \underset{c}{\downarrow} b$

3a: $a \underset{d''c}{\downarrow} b \ \wedge\ \theta \in D(a/cd'', \Xi)$ & ind hyp

$\Rightarrow \theta \in D(a/bcd'', \Xi)$.

3b: $d'' \underset{c}{\downarrow} b \Rightarrow \varphi(x, d'')$ div /bc wrt ψ

(ie $\exists$ c-indiscernible sequence (d_i) in $tp(d''/c)$
st. $\models \psi(d_0 \cdots d_{k-1})$)
Since $\underbrace{d'' \downarrow b}_{c}$, this sequence has an automorphic image
in $tp(d''/bc)$. $\Rightarrow \varphi(x, d'')$ divides / bc wrt ψ.

All together: $\xi = \theta \wedge (\varphi, \psi) \in D(a/bc, \Xi)$ $\square$.

<u>Corollary</u> $a \underset{c}{\downarrow} b$ iff For some $\xi \in D(a/c, \Xi)$ maximal,
$\xi \in D(a/bc, \Xi)$
iff $D(a/bc, \Xi) = D(a/c, \Xi)$. $\square$

<u>Important Corollary</u>: Given any a, c (tuple of elts), and y (tuple of vars),
there is a partial type $p(y, a, c)$ such that for b of the appropriate length, $b \underset{c}{\downarrow} a \Leftrightarrow b \models p(y, a, c)$.

<u>Proof</u> Let $\xi \in D(a/c, \Xi)$ be maximal.

Let $p(y, a, c) := \operatorname{div}_{(c,y),\xi}(a)$.

Then $a \underset{c}{\downarrow} b \Leftrightarrow \xi \in tp(a/bc) \Leftrightarrow \models \operatorname{div}_{cb,\xi}(a)$

$\Leftrightarrow b \models p(y, a, c)$. $\square$

Definition Let $p(x)$ be a partial type over c.

Then we say that p has definable independence over c

if for all tuples y $\exists q(x,y)$ over c such that

$\forall a,b$ (of the right lengths), $\models q(a,b) \Leftrightarrow \models p(a)$ & $a \underset{c}{\downarrow} b$.

Then we proved: complete types have definable independence $(/c)$

Ex: Assume that $p(x)$ & $q(y)$ have definable independence.

Then $\exists r(x,y) := p(x) \otimes_c q(y)$ ie. $a,b \models r \Leftrightarrow a \models p$, $b \models q$ & $a \underset{c}{\downarrow} b$

and it has definable independence$/c$.

Proof First: $r(x,y)$ exists if $p(x)$ (or q) has definable independence.

Now let z be any tuple. Let $s(x,y,z)$ be:

$p(x) \wedge q(y) \wedge x \underset{c}{\downarrow} yz \wedge y \underset{c}{\downarrow} z$.

Then: $abd \models s \Rightarrow a \underset{c}{\downarrow} bd \wedge b \underset{c}{\downarrow} d \Rightarrow a \underset{bc}{\downarrow} d$

$\Rightarrow ab \underset{c}{\downarrow} d$ & $ab \models r$.

Conversely: Assume $ab \models r$ and $d \underset{c}{\downarrow} ab \Rightarrow a \underset{c}{\downarrow} b$ and $d \underset{bc}{\downarrow} a$

$\Rightarrow a \underset{c}{\downarrow} bd$ & $d \underset{c}{\downarrow} ab \Rightarrow b \underset{c}{\downarrow} d$ & $p(a)$ & $q(b)$

$\Rightarrow abd \models s$. $\square$

1.

Another IOU: $a \underset{c}{\downarrow} b \iff$ for all ϕ-indiscernible sequences (b_i) in $tp(b/c)$, there exists a ϕ-automorphic image in $tp(b/ac)$.

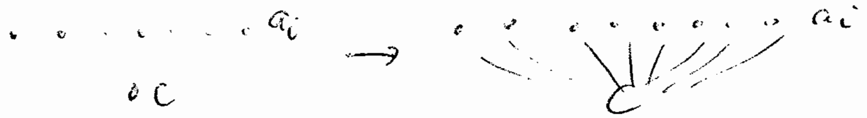
Recall: - We know this when replacing ϕ with c.
- If $(a_i : i \leq \omega)$ is c-indiscernible then $a_\omega \underset{a_{<\omega}}{\downarrow} c$

Lemma: Assume that (a_i) is a Morley sequence over c in $i<\omega$ (but doesn't matter). $p := tp(a_0/c)$.

Let (a_i') be any ϕ-automorphic image of (a_i), also c-indiscernible in p.

Then (a_i') is also a Morley sequence over c.

(Eg random graph



Proof of lemma Let a_ω, a_ω' extend these sequences to c-indiscernible sequences of length $\omega+1$ etc.

Still: $a_{\leq\omega} \equiv_\phi a'_{\leq\omega}$.

Then $D(p, \Xi) = D(a_\omega/c, \Xi) = D(a_\omega/c a_{<\omega}, \Xi)$
(since $a_\omega \underset{c}{\downarrow} a_{<\omega}$ since (a_i) is Morley seq $/c$).

$= D(a_\omega/a_{<\omega}, \Xi)$ (since $a_\omega \underset{a_{<\omega}}{\downarrow} c$ by c-indiscern seq)

$= D(a'_\omega / a'_{<\omega}, \equiv) = D(a'_\omega / c a'_{<\omega})$ (since $a'_\omega \underset{a'_{<\omega}}{\downarrow} c$).

$\subseteq D(a'_\omega / c, \equiv) = D(p, \equiv)$.

$\Rightarrow$ equality holds on the way

In particular: $D(a'_\omega / c)^{\equiv} = D(a'_\omega / c a'_{<\omega})^{\equiv}$

$\Rightarrow a'_\omega \underset{c}{\downarrow} a'_{<\omega} \Rightarrow \forall i < \omega \ a'_\omega \downarrow a'_{<i} \Rightarrow a'_i \underset{c}{\downarrow} a'_{<i}$ $\square$

Proof of 100 #2

If $a \underset{c}{\downarrow} b$ and (b_i) is $\emptyset$-indiscernible in $tp(b/c)$.

Then it has an automorphic image which is c-indiscernible in $tp(b/c)$ (extension/extraction), which in turn has a c-automorphic image in $tp(b/ac)$.

Conversely assume right-hand statement.

Let (b_i) be a Morley sequence for b/c.

Then it has an automorphic image (b_i') in $tp(b/ac)$, which we may assume is ac-indiscernible.

By the lemma: (b_i') is a Morley sequence$/c$.

(we don't have $b'_{<\omega} \underset{c}{\equiv} b_{<\omega}$ only $b'_{<\omega} \underset{\emptyset}{\equiv} b_{<\omega}$).

Since it is ac-indiscernible: $a \underset{C}{\downarrow} b'_{<\omega} \Rightarrow a \underset{C}{\downarrow} b'_0 \Rightarrow a \underset{C}{\downarrow} b$

<u>Random Graph</u> $\mathcal{L} = \{R\}$ binary predicate.

$T_0 = R$ is symmetric & antireflexive.

$T_1 = T_0 \cup \{ \forall x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1} \exists z \bigwedge_{i<n} R(x_i, z) \wedge \bigwedge_{j<m} \neg R(y_j, z) \; m, n < \omega \}$.

I. T_1 is complete & has QE & is ω-categorical.

II. T_1 is the model completion of T_0.

III. T_1 is simple and $A \underset{C}{\downarrow} B \iff A \cap C = A \cap (B \cup C)$
$\iff B \cap C = B \cap (A \cup C)$.
$\iff B \cap A \subseteq C$.

Josh's Example lecture ...