2/17 <u>Defn</u>: $\underline{\kappa^0(T)}$ = least cardinal (if one exists) s.t.

$\forall$ singleton a and tuple b there is a subtuple $b' \subseteq b$ with $|b'| < \kappa^0(T)$ and $a \underset{b'}{\downarrow} b$

$\kappa^0_r(T)$ = least such regular cardinal.

[$\kappa^0_r(T) = \begin{cases} \kappa^0(T) & \text{if } \kappa^0(T) \text{ is regular} \\ \kappa^0(T)^+ & \text{otherwise} \end{cases}$

<u>Defn</u>: T is <u>simple</u> if $\kappa^0(T)$ exists.

Assume that T is simple.

① Then $\forall$ finite a and infinite b $\exists\, b' \in [b]^{<\kappa^0(T)}$ (subtuple of b of card $< \kappa^0(T)$) s.t. $a \underset{b'}{\downarrow} b$

② Then $\forall a, b$ $\exists\, b' \in [b]^{<\kappa^0_r(T) + |a|^+}$ s.t. $a \underset{b'}{\downarrow} b$

<u>Proof</u> ① write: $a = (a_0, a_1, \ldots, a_{n-1})$

$\forall i < n$ $\exists\, b_i' \in [b]^{<\kappa^0(T)}$, $I_i \subseteq i$ s.t. $a_i \underset{b_i' a_{\in I_i}}{\downarrow} b a_0 \cdots a_{i-1}$

(b) $\downarrow$ (a) Let $b' = \bigcup b_i'$ so $b' \in [b]^{<\kappa_0}$

and (a) $a_i \underset{b' a_{<i}}{\downarrow} b a_{<i} \Rightarrow a_i \underset{b' a_{<i}}{\downarrow} b$ by trans.

Now by induction: $a_{<i} \underset{b'}{\downarrow} b$: 0 ✓

$i+1$: $a_{i+1} \underset{b'a_{\leq i}}{\downarrow} b$ and $a_{\leq i} \underset{b'}{\downarrow} b \overset{(b)}{\Rightarrow} a_{\leq i+1} \underset{b'}{\downarrow} b$

$\Rightarrow a_{<n} \underset{b'}{\downarrow} b.$ $\square$

(2) Exercise (as above & use finite character).

Assume T is Thick (without defining) (Blackbox for now...)

Thm: Let T be a thick simple theory$_{CAT}$.

Then $\downarrow$ satisfies:

1. automorphism invariance: If $f \in Aut(\mathfrak{U})$ then $a \underset{c}{\downarrow} b \iff f(a) \underset{f(c)}{\downarrow} f(b)$.
2. finite character: $a \underset{c}{\downarrow} b \iff a' \underset{c}{\downarrow} b'$ $\forall a' \subseteq a, b' \subseteq b$ finite.
3. symmetry: $a \underset{c}{\downarrow} b \iff b \underset{c}{\downarrow} a$
4. transitivity: $a \underset{c}{\downarrow} b, d \iff a \underset{c}{\downarrow} b$ & $a \underset{cb}{\downarrow} d$.
5. extension: $\forall a, b, c$ $\exists a' \equiv_c a$ st. $a' \underset{c}{\downarrow} b$
6. local character: $\forall$ finite a, any b $\exists b' \in [b]^{\leq |T|}$ $(|T| = |\Delta| = |L|)$.
   st. $a \underset{b'}{\downarrow} b$ (saying: $\kappa^0(T) \leq |T|^+$).
7. The Independence Theorem. (I think I have to do this...)

Moreover: If T is any theory and $\mathop{\smile\!\!\!\!\!|}'$ is a notion of independence satisfying ①-⑦, then T is simple and $\mathop{\smile\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!|}'$

Cameron proves this later.

We assume ⑤ (to be proved later)

①, ② ✓.

③

Defn: A sequence $(a_i : i < \alpha)$ is a Morley sequence over C

if
1. it is indiscernible over C.
2. for all $i < \alpha$, $a_i \mathop{\smile\!\!\!\!\!|}_C a_{<i}$

Lemma: Assume $(a_i : i < \omega)$ is a Morley sequence over C.

Then $\forall n < \omega$: $a_{\geq n} \mathop{\smile\!\!\!\!\!|}_C a_{<n}$

Proof By induction on m:

$\forall n \quad a_n, \dots, a_{n+m} \mathop{\smile\!\!\!\!\!|}_C a_{<n}$

$m = 0$: defn of Morley seq. ✓

$m+1$: $a_n, \dots, a_{n+m} \mathop{\smile\!\!\!\!\!|}_C a_{<n}$ by ind. hyp.

Given: $a_{n+m+1} \mathop{\smile\!\!\!\!\!|}_C a_{\leq n+m}$

$\Rightarrow \quad a_{n+m+1} \mathop{\smile\!\!\!\!\!|}_{C, a_n, \dots, a_{n+m}} a_{<n}$.

$\Rightarrow \quad a_n \dots a_{n+m+1} \mathop{\smile\!\!\!\!\!|}_C a_{<n}$

Then by finite character: $a_{\geq n} \underset{c}{\perp} a_{<n}$

<u>Proposition</u> ~~(Universality of Morley~~

Assume $(b_i : i<\omega)$ is a Morley seq (M.S.) over c and that it is indiscernible ac. Then $a \underset{c}{\perp} b_{<\omega}$

<u>Proof</u> By finite character, we may assume $|a| < \omega$.

Set $\kappa := \kappa_r^o(T)$.

Let $(b'_i : i \in \kappa^*)$ be a similar sequence over ac, where κ^* is κ with inverse order. (use compactness)

By simplicity, $a \underset{cb'_{\in I}}{\perp} b'_{\in \kappa^*}$ where $I \in [\kappa^*]^{<\kappa}$

$\Rightarrow$ By reg of κ $\exists\, i < \kappa$ st. $I \subseteq \{j \in \kappa : j < i\} = \{j \in \kappa^* : j >^* i\}$

$\Rightarrow a \underset{cb'_{>^*i}}{\perp} b'_{\in \kappa^*} \Rightarrow a \underset{cb'_{>^*i}}{\perp} b'_{\leq^* i}$

We know $\forall n, m$: $b_n, \dots, b_{n+m-1} \underset{c}{\perp} b_{<n}$

By invariance: $\forall I, J \subseteq \kappa^*$ finite, if $I >^* J \Rightarrow b'_{\in I} \underset{c}{\perp} b'_{\in J}$

By finite character $b'_{>^*i} \underset{c}{\perp} b'_{\leq^* i}$

$\Rightarrow a, b'_{>^*i} \underset{c}{\perp} b'_{\leq i^*} \Rightarrow a \underset{c}{\perp} b'_{\leq i^*}$.

Again by invariance: $a \underset{c}{\perp} b_{<n}\ \forall n \overset{\text{fin char}}{\Rightarrow} a \underset{c}{\perp} b_{<\omega}$ $\square$.

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Lemma For all a, c $\exists$ a Morley sequence for a over c

ie a Morley sequence $(a_i : i<\omega)$ over c which is in the $tp(a/c)$.

Proof Define a sequence $(a_i : i<\lambda)$ (λ big enough) as follows:

Given $(a_j : j<i)$ ~~the~~ let a_i be st. $a_i \equiv_c a$ and $a_i \downarrow_c a_{<i}$ (by extension)

Extract an c-indiscernible sequence $(a_i' : i<\omega)$

st. $\forall n$ $\exists$ $i_0<\dots<i_{n-1}$ st. $a'_{\le n} \equiv_c a_{i_0},\dots,a_{i_n}$

Know: $a_{i_n} \downarrow_c a_{<i_n} \Rightarrow a_{i_n} \downarrow_c a_{i_0}\dots a_{i_{n-1}}$

By invariance, $a_n' \downarrow_c a'_{<n}$ $\square$.

Corollary from previous Proposition (Universality of Morley Sequences)

~~Let $tp(a/b,c) = p(x;b,c)$.~~

~~Then~~ Let $p(x,b)$ be a partial type over b.

① Then $p(x,b)$ divides over c iff ② for all Morley sequences (b_i) for b/c, we have

$\bigwedge p(x, b_i)$ is inconsistent

iff (3) for some MS (b_i) for b/c - $\bigwedge p(x, b_i)$ is inconsistent

<u>Proof</u> (2) $\Rightarrow$ (3) since Morley sequences exist.

(3) $\Rightarrow$ (1) by defn.

(1) $\Rightarrow$ (2). by contrapositive.

Assume not (2), ie there is a Morley sequence (b_i) for b/c

and a st. $\models \bigwedge p(a, b_i)$. Let $q_n = tp(b_0 \ldots b_{n-1}/c)$. $q_{<\lambda} = \left[\bigcup_{n<\omega} \bigcup_{i_0 \ldots i_{n-1} < \lambda} q_n(x_{i_0} \ldots x_{i_n}))\right] \cup \bigcup_{i<\lambda} p(a, x_i)$ So $q_{<\lambda}$ is finitely consistent & so consistent. Let $(b_i : i < \lambda)$ satisfy it.

Extract an ac-indiscernible sequence. $\forall n \exists i_0 \ldots i_n$ st. $b'_{\leq n} \equiv_{ac} b_{i_0} \ldots b_{i_n}$. $b_{i_n} \underset{c}{\downarrow} b_{i_0} \ldots b_{i_n} \Rightarrow b'_n \underset{c}{\downarrow} b'_{<n}$

By extension/extraction, we may assume that (b_i) is and $\models p(a, b_i)$ $\forall i<\omega$.

ac-indiscernible. $\overset{\text{by prop}}{\Rightarrow} a \underset{c}{\downarrow} b_0$

$\Rightarrow tp(a/b_0 c)$ does not divide over $c \Rightarrow p(x, b_0)$ dnd over c.

But $b \equiv_c b_0 \Rightarrow p(x, b)$ dnd over c. $\square$

<u>Improved Extension</u>

Assume $a \underset{c}{\downarrow} b$ and d is given. Then $\exists a' \equiv_{bc} a$ st.

$a' \underset{c}{\downarrow} bd$.

<u>Proof</u> Let $(b_i d_i)$ be a Morley sequence for bd/c.

Then (b_i) is a c-indiscernible sequence in $tp(b/c)$

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and as $a \underset{c}{\downarrow} b$ $\exists f \in Aut_c(\mathcal{U})$ st. $f(b_i)$ is ac-indiscernible.

write $b_i' = f(b_i)$ and $d_i' = f(d_i)$.

So $(b_i'd_i')$ is c-indiscernible and (b_i') is ac-indiscernible

By extn / extraction: $\exists\ (b_i''d_i'')$ st.

1. ac-indiscernible
2. similar to $(b_i'd_i')$ over c ($\Rightarrow$ to $(b_id_i)/c$).
3. (b_i'') is similar $/ac$ to (b_i').

$\Downarrow$ a MS$/c$.

prev prop. & ① & ②.

$\Rightarrow a \underset{c}{\downarrow} b_0''d_0''$

③ $\Rightarrow$ $b_0'' \equiv_{ac} b$ ie $ab_0'' \equiv_c ab$.

① $\Rightarrow$ $b_0''d_0'' \equiv_c bd$. $b_0''d_0'' \equiv_c b_0'd_0' \equiv_c b_0d_0 \equiv_c bd$.

send $b_0''d_0''$ to bd by a c-automorphism.

Let a' be the image of a under it.

invariance

$\Rightarrow$ $a' \underset{c}{\downarrow} bd$. & $a'b \equiv_c ab_0'' \equiv_c ab \Rightarrow a' \equiv_{bi} a$ $\square$.

<u>Corollary</u> Assume $a \underset{c}{\downarrow} b$. Then there exists a Morley sequence for a/c which is bc-indiscernible.

Proof: For $i<\lambda$ (λ big enough).

~~Find~~ ~~[illegible]~~ Find a_i st. $a_i \equiv_{bc} a$ $a_i \mathop{\smile\!\!\!\!\!\;|}_c b a_{<i}$

Extract a bc-indiscern seq $(a_i' : i<\omega)$.

By same argument: ~~[illegible]~~ $a_i' \mathop{\smile\!\!\!\!\!\;|}_c a'_{<i} b \Rightarrow a_i' \mathop{\smile\!\!\!\!\!\;|}_c a'_{<i}$ $\square$.

Cor $\mathop{\smile\!\!\!\!\!\;|}$ is symmetric.

Proof Assume $a \mathop{\smile\!\!\!\!\!\;|}_c b$.

Let (a_i) be a Morley sequence for a/c which is indiscernible over bc

$\Rightarrow b \mathop{\smile\!\!\!\!\!\;|}_c a_{<\omega} \Rightarrow b \mathop{\smile\!\!\!\!\!\;|}_c a_0 \overset{a_0 \equiv_{bc} a}{\Longrightarrow} b \mathop{\smile\!\!\!\!\!\;|}_c a$.

Cor $\mathop{\smile\!\!\!\!\!\;|}$ is transitive.

Proof right downward + left upward + symmetry $\square$.