In this lecture, we investigate the relationship between total dual integrality and integrality of polytopes. We then use a theorem on total dual integrality to provide a new proof of the Tutte-Berge formula.

1 Total Dual Integrality

Consider the linear program defined as

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq b \end{array} \tag{1}$$

where A and b are rational and the associate dual program

$$\begin{array}{ll} \min & y^\top b \\ \text{s.t.} & A^\top y = c \\ & y \geq 0 \end{array} \tag{2}$$

Definition 1 *The system of inequalities by $Ax \leq b$ is* Total Dual Integral *or TDI if for all integral vectors c the dual program has an integral solution whenever the optimal value is finite.*

The main result for today is

Theorem 1 *If $Ax \leq b$ is TDI and b is integral then $P = \{x : Ax \leq b\}$ is integral*

Proof: We proceed by contradiction. Consider a vertex x^* of P such that $x_j^* \notin \mathbb{Z}$. We can construct an integral c such that x^* is the optimal solution corresponding to c by picking a rational c in the optimal cone of x^* and scaling. Consider $\hat{c} = c + \frac{1}{q} e_j$ where q is an integer. Since the cone is full dimensional, $\hat{c}$ will still be in the optimality cone of x^* for q sufficiently large. Now it follows that $q\hat{c} = qc + e_j$ and thus $(q\hat{c})^\top x^* - (qc)^\top x^* = x_j^* \notin \mathbb{Z}$. This means that either $(q\hat{c})^\top x^*$ or $(qc)^\top x^*$ are not integral which contradicts the assumption of total dual integrality. $\square$

Note that the converse doesn't generally hold. We can have $Ax \leq b$ integral with b an integral vector, but the system is not TDI.

1.1 Total Unimodularity

As an aside, we can consider an alternate condition which guarantees integrality.

Definition 2 *A matrix A is* totally unimodular *(TUM) if for any square submatrix A', $\det A' \in \{-1, 0, 1\}$.*

The following propositions hold for TUM matrices.

Proposition 2 *If A is totally unimodular then for all integral vectors b, $Ax \leq b$ is integral.*

This differs from Total Dual Integrality where the integrality was dependent on both A and b.

Proposition 3 *If A is totally unimodular, $Ax \leq b$ is total dual integral for any integral vector b.*

For the case of non-bipartite matching, A is not totally unimodular. Indeed, for the three cycle with edges $e_1 = (1, 2)$, $e_2 = (2, 3)$, $e_3 = (1, 3)$, the matrix of $x(\delta(v)) \leq 1$ is given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{3}$$

and $\det A = 2$.

Often we can find subsystems of inequalities which define specific solutions of interest and are totally unimodular. Such a technique is employed in Lovász's proof of the Lucchesi-Younger theorem on dicuts.

1.2 An alternate proof of Theorem 1

We begin by proving a theorem of Kronecker.

Theorem 4 (Kronecker Approximation Theorem (1884)) *$Ax = b$ has an integral solution if and only if $y^\top b$ is an integer whenever $y^\top A$ is an integral vector.*

Proof: To prove the forward implication, take an integral solution x^*. Then $y^\top Ax^* = y^\top b$ and if $y^\top A$ is integral then $y^\top b$ must be an integer.

To prove the converse, first note that there must be some solution to the system of equations; otherwise there would be a solution to $y^\top A = 0$ with $y^\top b \neq 0$ and by scaling y, we can get $y^\top b \notin \mathbb{Z}$. For the remainder, we will consider only a full row rank part of A.

We proceed by introducing operations on the matrix A which preserve integrality. Let the jth column of A be denoted by a_j. First note that exchanging two columns of A preserves both the existence of an integral solution of $Ax = b$ and the property that $y^\top b \in \mathbb{Z}$ whenever $y^\top A \in \mathbb{Z}$. Second, note that we can add any integral multiple of one column to another column and still preserve the assumptions. Indeed, for $\lambda \in \mathbb{Z}$, if $Ax = b$, construct the matrix A' with columns identical to those of A but with $a'_i = a_i + \lambda a_j$. Let x' be a vector with $x'_k = x_k$ except for $x'_j = x_j - \lambda x_i$. Then it is clear that $A'x' = b$ and x' is integral (whenever x is). Conversely if $A'x' = b$, we can define x by $x_k = x'_k$ except for $x_j = x'_j + \lambda x'_i$ and x is integral and satisfies $Ax = b$. The preservation of the second assumption is proved similarly.

Using these elementary operations, we can transform A into the form

$$A' = \begin{bmatrix} B & 0 \end{bmatrix} \tag{4}$$

with B lower triangular as follows. For the first row, we can pair any two nonzero entries and compute their gcd using Euclid's algorithm

$$\gcd(x, y) = \begin{cases} \gcd(x - y, y) & \text{if } x \geq y \\ \gcd(y, x) & \text{if } x < y \end{cases} \tag{5}$$

since these operations are elementary, we can perform them on the columns and reduce the first row to one nonzero entry. We can then put this column as column 1 and proceed to the next row leaving column 1 fixed. Proceeding in this manner results in the desired form for A'.

Now observe that B is nonsingular because we have assumed that A has full row rank. $B^{-1}A' = \begin{bmatrix} I & 0 \end{bmatrix}$ and hence $B^{-1}b$ must be integral (since every row of B^{-1} is a possible candidate for $y^\top$). Since

$$A' \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = b \tag{6}$$

we have found an integral solution to the system $A'x = b$ and this completes the proof.

$\square$

Corollary 5 *$P = \{x : Ax \le b\}$ is integral if and only if each supporting hyperplane contains and integral vector.*

Proof: The forward implication is immediate because every supporting hyperplane contains a vertex of P. For the converse, suppose x^* is a non-integral vertex of P. x^* is a unique solution of a subsystem $\hat{A}x = \hat{b}$ and by the Kronecker approximation theorem, there exists a vector y such that $y^\top b$ is non-integral and $y^\top \hat{A}$ is integral. By adding an integral constant to the components of y, we can assume that y is nonnegative. Let $c = \hat{A}^\top y$ and $\alpha = y^\top b$. Then $c^\top x = \alpha$ is a supporting hyperplane (teh fact that $c^\top x \le \alpha$ is valie follows from the nonnegativity of y) and $c^\top x$ is non-integral for all integral x which is a contradiction. □

This results in a new proof for Theorem 1.

Proof of Theorem 1: If $Ax \le b$ is TDI and b is integral, pick an integral c such that c_i and c_j are relatively prime for $i \neq j$. By linear programming duality, $\max c^\top x$ such that $Ax \le b$ will be an integer α and $c^\top x = \alpha$ will be a supporting hyperplane.

Since the entries of c are relatively prime, we can find an integral vector x contained in the supporting hyperplane. (Indeed, it can be shown easily by induction on n that if the gcd of the entries of c is g then there is an integral solution to $c^\top x = g$.) Therefore, we conclude that $Ax \le b$ is integral. □

2 Back to matchings

Given a graph G and a matching M define a vector $\chi^M \in \mathbb{R}^{|E|}$ as

$$\chi_e^M = \begin{cases} 1 & e \in M \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The *Matching Polytope* is the convex hull of all such incidence vectors.

Consider the polytope P defined by the inequalities.

$$\begin{aligned} x(\delta(v)) &\le 1 \quad \forall v \in V \\ x(E(U)) &\le \left\lfloor \frac{|U|}{2} \right\rfloor \quad \forall U \in \mathcal{P}_{odd} \\ x &\ge 0 \end{aligned} \tag{8}$$

where $\mathcal{P}_{odd}$ denotes the odd cardinality subsets.

Edmonds proved in 1965 that P was indeed the matching polytope. Cunningham and Marsh in 1978 proved Edmonds result by showing that P was TDI. Indeed this immediately implies that P is the matching polytope because all vertices of P would be integers, and any valid integer solution of P is a matching.

Explicitly we have

Theorem 6 (Cunningham-Marsh) *For all $w \in \mathbb{Z}^{|E|}$, there exist integral vectors y and z such that the maximum weight of any matching is equal to*

$$\begin{aligned} \min \sum_{v \in V} y_v &+ \sum_{S \in \mathcal{P}_{odd}} \left\lfloor \frac{S}{2} \right\rfloor z_s \\ \sum_{v \in V} y_v \chi^{\delta(v)} &+ \sum_{S \in \mathcal{P}_{odd}} z_s \chi^{E(S)} \ge w \\ y \ge 0,\ \ & z \ge 0 \end{aligned}$$

The proof of this theorem is in Schrijver's book. There are actually two proofs in the book, one that assumes the knowledge of the matching polytope, the other that's self-contained.

We will now show that the Cunningham-Marsh theorem implies the Tutte-Berge formula in the cardinality case ($w_e = 1$ for all e).

Theorem 7 (Tutte-Berge)

$$\nu(G) = \min_{U \subseteq V} \frac{1}{2} \left(|U| + |V| + o(G - U)\right)$$

Proof: Recall from lecture 1 that "$\leq$" was immediate.

For any solution of the Cunningham-Marsh dual problem, it is clear that y_v and z_S are at most 1 as $w_e = 1$ for all edges. Furthermore, for all $v \in V$ either $y_v = 1$ or $z_S = 1$ for some odd set S containing v.

Suppose z_S and z_T are such that $S \cap T \neq \emptyset$. If $S \cup T$ is an odd set, we can set $z_S = z_T = 0$ and $z_{S \cup T} = 1$ because

$$\left\lfloor \frac{|S|}{2} \right\rfloor + \left\lfloor \frac{|T|}{2} \right\rfloor = \frac{|S| + |T|}{2} - 1 \geq \frac{|S \cup T| + 1}{2} - 1 = \left\lfloor \frac{|S \cup T|}{2} \right\rfloor \tag{9}$$

and this assignment would reduce the cost function. If $S \cup T$ is an even set, take $j \in S \cup T$ and set $z_{S \cup T - \{j\}} = 1$ and $y_j = 1$. This will also never increase the cost function. Therefore, we conclude that for an optimal solution, the sets $\{S \in \mathcal{P}_{odd} : z_S = 1\}$ are not overlapping.

Let $U = \{v \in V : y_v = 1\}$ and $W = \{S \in \mathcal{P}_{odd} : z_S = 1\}$. If $v \in U$ and $v \in S$ with $S \in W$, then we can remove v and an additional vertex u from S and let $y_u = 1$ and this gives another feasible solution without increasing the cost function. Thus we can assume that U and all the sets S in W are disjoint. This implies that there cannot be any edges between the sets with $z_S = 1$, which means that $|W| = o(G - U)$. Therefore we have shown

$$\begin{aligned} \nu(G) &= \sum_{v \in V} y_v + \sum_{S \in \mathcal{P}_{odd}} \left\lfloor \frac{S}{2} \right\rfloor z_S \\ &= |U| + \frac{|V| - |U|}{2} - \frac{1}{2}|W| \\ &= \frac{1}{2} \left(|V| + |U| - o(G - U)\right) \end{aligned} \tag{10}$$

$\square$