18.S34 (FALL 2007)

PROBLEMS ON ROOTS OF POLYNOMIALS

NOTE. The terms "root" and "zero" of a polynomial are synonyms. Those problems which appeared on the Putnam Exam are stated as they appeared verbatim (except for one minor correction and one clarification).

1. (39P) Find the cubic equation whose roots are the cubes of the roots of
$$x^3 + ax^2 + bx + c = 0.$$

2. (a) (40P) Determine all rational values for which a, b, c are the roots of
$$x^3 + ax^2 + bx + c = 0.$$
   (b) (not on Putnam Exam) Show that the only real polynomials $\prod_{i=0}^{n-1}(x - a_i) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in addition to those given by (a) are $x^n, x^2 + x - 2$, and exactly two others, which are approximately equal to
$$x^3 + .56519772x^2 - 1.76929234x + .63889690$$
   and
$$x^4 + x^3 - 1.7548782x^2 - .5698401x + .3247183.$$

3. (51P) Assuming that all the roots of the cubic equation x^3+ax^2+bx+c are real, show that the difference between the greatest and the least roots is not less than $\sqrt{a^2 - 3b}$ nor greater than $2\sqrt{(a^2 - 3b)/3}$.

4. (56P) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z) + 1$ and $Q(z)+1$. Prove that $P(z) = Q(z)$. (On the original Exam, the assumption that $P(z)$ and $Q(z)$ are nonconstant was inadvertently omitted.)

5. (58P) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying
$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$
   show that the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ has at least one real root.

6. (68P) Determine all polynomials of the form

$$\sum_{0}^{n} a_i x^{n-i} \quad \text{with } a_i = \pm 1$$

$(0 \le i \le n, \ \ 1 \le n < \infty)$ such that each has only real zeros.

7. (81P) Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2+1)P(x)P'(x) + x(P(x)^2 + P'(x)^2).$$

Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n-1$ distinct real roots.

8. (89P) Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

9. (90P) Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for each $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

10. (91P) Find all real polynomials $p(x)$ of degree $n \ge 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

(i) $p(r_i) = 0, \quad i = 1, 2, \ldots, n,$

and

(ii) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \quad i = 1, 2, \ldots, n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

11. (a) (85P) (relatively easy) Let k be the smallest positive integer with the following property:

There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x-m_1)(x-m_2)(x-m_3)(x-m_4)(x-m_5)$ has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

(b) (considerably more difficult) Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

12. (99P) Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

13. (a) (05P) Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

    (b) (00P) Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k f}{dt^k}$ in the half-open interval $[0, 1)$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \quad \text{and} \quad \lim_{k\to\infty} N_k = 2N.$$

(On the original Exam, it was not stated that the zeros should be taken in $[0, 1)$.)

14. Let $ax^3 + bx^2 + cx + d$ be a polynomial with three distinct real roots. How many real roots are there of the equation

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2?$$

15. Does there exist a finite set M of nonzero real numbers, such that for any positive integer n, there exists a polynomial of degree at least n with all coefficients in M, all of whose roots are real and belong to M?

16. Suppose that the polynomial $ax^2+(c-b)x+(e-d)$ has two real roots, both greater than 1. Prove that $ax^4+bx^3+cx^2+dx+e$ has at least one real root.

17. Suppose that $a, b, c \in \mathbb{C}$ are such that the roots of the polynomial z^3+az^2+bz+c all satisfy $|z|=1$. Prove that the roots of $x^3+|a|x^2+|b|x+|c|$ all satisfy $|x|=1$.

18. Let $P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ be a monic polynomial of degree n with complex coefficients a_i. Suppose that the roots of $P(x)$ are $x_1, x_2, \cdots, x_n$, i.e., we have $P(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$. The *discriminant* $\Delta(P(x))$ is defined by

$$\Delta(P(x)) = \prod_{1\le i<j\le n}(x_i-x_j)^2.$$

Show that

$$\Delta(x^n+ax+b) = (-1)^{\binom{n}{2}}\left(n^n b^{n-1}+(-1)^{n-1}(n-1)^{n-1}a^n\right).$$

HINT. First note that

$$P^{'}(x) = P(x)\left(\frac{1}{x-x_1}+\cdots+\frac{1}{x-x_n}\right).$$

Use this formula to establish a connection between $\Delta(P(x))$ and the values $P^{'}(x_i)$, $1\le i\le n$.

19. Let $P_n(x)=(x+n)(x+n-1)\cdots(x+1)-(x-1)(x-2)\cdots(x-n)$. Show that all the zeros of $P_n(x)$ are purely imaginary, i.e., have real part 0.

20. Let $P(x)$ be a polynomial with complex coefficients such that every root has real part a. Let $z\in\mathbb{C}$ with $|z|=1$. Show that every root of the polynomial $R(x)=P(x-1)-zP(x)$ has real part $a+\frac{1}{2}$.

21. Let $d\ge 1$. It is not hard to see that there exists a polynomial $A_d(x)$ of degree d such that

$$F_d(x) := \sum_{n\ge 0} n^d x^n = \frac{A_d(x)}{(1-x)^{d+1}}. \qquad (1)$$

For instance, $A_1(x) = x$, $A_2(x) = x + x^2$, $A_3(x) = x + 4x^2 + x^3$. Show that every root of $A_d(x)$ is real. HINT. First obtain a recurrence for $A_d(x)$ by differentiating (1).

22. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a monic polynomial with complex coefficients. Choose $j \in \{0, \ldots, n\}$ so that the roots of P can be labeled $\alpha_1, \ldots, \alpha_n$ with

$$|\alpha_1|, \ldots, |\alpha_j| > 1, \qquad |\alpha_{j+1}|, \ldots, |\alpha_n| \leq 1.$$

Prove that

$$\prod_{i=1}^{j} |\alpha_i| \leq \sqrt{|a_0|^2 + \cdots + |a_{n-1}|^2 + 1}.$$

HINT. One approach is to deduce this from an identity involving the polynomials $(z - \alpha_1) \cdots (z - \alpha_j)$ and $(\alpha_{j+1}z - 1) \cdots (\alpha_n z - 1)$.

23. Let $Q(x)$ be any monic polynomial of degree n with real coefficients. Prove that

$$\sup_{x \in [-2,2]} |Q(x)| \geq 2.$$

HINT. Let $P_n(x)$ be the monic polynomial satisfying

$$P_n(2\cos\theta) = 2\cos(n\theta) \qquad (\theta \in \mathbb{R}),$$

and examine the values of $P_n(x) - Q(x)$ at points where $|P_n(x)| = 2$.

OPTIONAL. Prove that equality only holds for $Q = P_n$.

24. Let $P(x), Q(x)$ be two polynomials with all real roots $r_1 \leq r_2 \leq \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_{n-1}$, respectively. We say that $P(x)$ and $Q(x)$ are *interlaced* if

$$r_1 \leq s_1 \leq r_2 \leq s_2 \leq \cdots \leq s_{n-1} \leq r_n.$$

Prove that $P(x)$ and $Q(x)$ are interlaced if and only if the polynomial $P + tQ$ has all real roots for all $t \in \mathbb{R}$.

25. Let $P(x)$ be a polynomial with real coefficients. For $t \in \mathbb{R}$, let $V(P, t)$ denote the number of sign changes in the sequence

$$P(t), P'(t), P''(t), \ldots.$$

(A *sign change* in a sequence is a pair of terms, one positive and one negative, with only zeros in between.) Prove that for any $a, b \in \mathbb{R}$, the number of roots of P in the half-open interval $(a, b]$, counted with multiplicities, is equal to $V(P, a) - V(P, b)$ minus a nonnegative even integer. Then deduce Descartes's rule of signs as a corollary.

26. Let $P(x)$ be a squarefree polynomial with real coefficients. Define the sequence of polynomials $P_0, P_1, \ldots$ by setting $P_0 = P$, $P_1 = P'$, and

$$P_{i+2} = -\mathrm{rem}(P_i, P_{i+1}),$$

where $\mathrm{rem}(A, B)$ means the remainder upon Euclidean division of A by B; upon arriving at a nonzero constant polynomial P_r, stop. Prove that for any $a, b \in \mathbb{R}$, the number of zeros of P in $(a, b]$ is $\sigma(a) - \sigma(b)$, where $\sigma(t)$ is the number of sign changes in the sequence

$$P_0(t), P_1(t), \ldots, P_r(t).$$