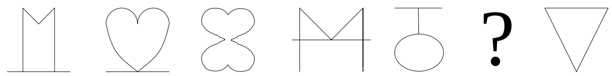
18.A34 PROBLEMS #6

64. [1] Find the missing term:



65. [1] (a) A drugstore received a shipment of ten bottles of a certain drug. Each bottle contains one thousand pills. The drugstore received a telegram from the drug company saying that the pills in one bottle each weigh 10 milligrams too much and should be returned immediately. How can the faulty bottle be found with only one weighing?

(b) The next time the druggist received a shipment of ten bottles of the same drug, he again received a telegram from the drug company, this time saying that any any number of the bottles might contain pills each of which was 10 milligrams too heavy. Can all the faulty bottles still be determined with only one weighing?

66. (a) [2] A person has twelve coins, one of which is counterfeit and is either lighter or heavier than the other eleven. How few weighings are necessary on a balance to determine the counterfeit coin and to decide whether it's heavy or light?

(b) [2.5] Fix a positive integer k. Suppose one is given n coins, one of which is counterfeit and is either lighter or heavier than the other $n-1$. What is the largest possible value of n such that in k weighings on a balance it is possible to determine the counterfeit coin and whether it is heavy or light?

67. (a) [1] Show that for any integer x, $x^2 - x$ is divisible by 2.

(b) [1] Show that for any integer x, $x^3 - x$ is divisible by 6.

(c) [2.5] Let $f(n)$ be the largest integer k such that $x^n - x$ is divisible by k for all integers x. For instance, (a) and (b) assert that $f(2) \geq 2$ and $f(3) \geq 6$. Find a "nice" number-theoretic description of $f(n)$ not directly involving $x^n - x$. Show for instance that $f(2) = 2$, $f(3) = 6$, $f(4) = 2$, and $f(5) = 30$.

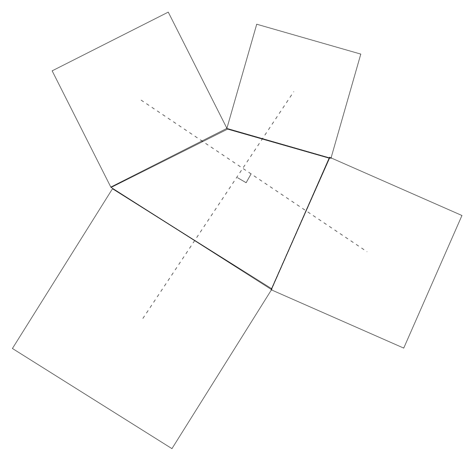
68. [1] (a) A person starts at the point $x = 1$ ft. at time $t = 0$ sec. and moves along the x-axis so that his velocity in feet per second is equal to his distance in feet from $x = 0$. (So in particular at time $t = 0$ his velocity is $1\,\frac{\text{ft}}{\text{sec}}$.) Where will he be in one second?

(b) What if his velocity is $x^2\,\frac{\text{ft}}{\text{sec}}$ when his/her distance from $x = 0$ is x ft?

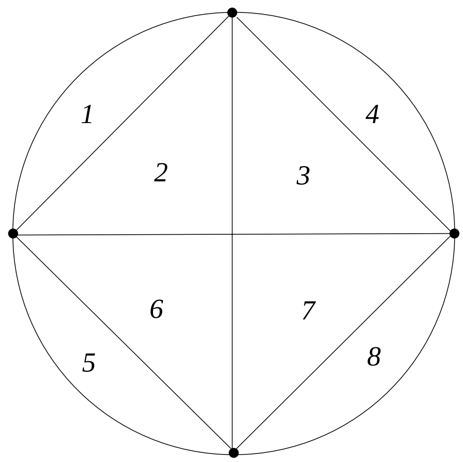
69. [2] Let $P(x)$ be a polynomial of degree n satisfying $P(i) = 2^i$ if $0 \leq i \leq n$. What is $P(n+1)$?

70. [2] Let n be an integer greater than one. Show that $n^4 + 4^n$ is not prime.

71. [2] Erect a square (facing outwards) on each side of a convex quadrilateral. Join the centers of each pair of opposite squares by a straight line. Show that these two straight lines intersect in right angles.



72. [2] Let $h(n)$ be the number of regions formed inside a circle by choosing n points on the circumference and drawing a straight line segment between every two of the points. Assume the points have been chosen so that no three of these chords intersect in a single point in the interior of the circle. For instance, $h(4) = 8$:



Find a simple formula for $h(n)$.

73. (a) [3] Let V be a finite set of vertices in the plane, no three colinear, and let E be a set of straight line segments (edges) joining two of the vertices in V. Suppose that any two edges in E either have a vertex in common or properly cross (i.e., intersect in their interiors). Show that $\#E \leq \#V$.

    (b) [5] Suppose that the edges in E joining two vertices need not be straight lines. We now assume that every pair of edges intersects exactly once, either at a common vertex or at a proper crossing. Show that the conclusion still holds.

74. [2] A man and a fly both start out at the point $x = 0$ at time $t = 0$. The man walks 4 mi/hr in the positive x-direction. The fly flies at a rate of 10 mi/hr. It continually flies between the man and the point $x = 0$. Where will the fly be after one hour? (Do not confuse this problem with a similar one where a fly flies between two trains moving toward each other.)

75. [5] Show that 462 is the largest integer that cannot be written in the form $ab + ac + bc$, where a, b, c are positive integers. (It is known that there is at most one such integer $n > 462$. If it exists, then it satisfies $n > 2 \cdot 10^{11}$.)

76. [3] Does there exist an infinite binary word $w = a_1 a_2 \cdots$ ($a_i = 0$ or 1) that is not eventually periodic, such that for all n sufficiently large,

the prefix $a_1a_2\cdots a_n$ ends in a (nonempty) square, i.e., we can write $a_1a_2\cdots a_n = xyy$ (concatenation of words) with y nonempty?