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PETER

KEMPTHORNE:

Today's topic is portfolio theory. And it's really one of the most important topics in finance. We're going to go through the historical theory of portfolio optimization, beginning with Markowitz Mean-Variance Optimization, where we look at portfolios in terms of their performance characteristics as determined by the mean return and the volatility returns.

This analysis gets extended by looking at also investing with a risk-free asset. The initial theory on portfolio analysis didn't consider investing in cash, but just investing in risky assets. The problem changes quite dramatically when we add the risk-free asset.

Then the topic of utility theory, the Neumann Morgenstern utility theory, in statistical decision theory, we are trying to make decisions under uncertainty in a rational way. And Von Neumann and Morgenstern developed an expected utility hypothesis for rational decision making. And that's really very important motivator for decision analysis generally and portfolios selection in particular. So we'll go through that.

Then we'll turn to portfolio optimization constraints. There are realistic issues when you have-- well, there's how much money you have to invest, how much capital you have to invest, whether you could short securities or not, whether there are sort of limits in capacity of different assets you might want to trade. Those can come into play and affect the solutions.

All of these methods are driven by estimates of how well we expect assets to do, so estimating returns or estimating alphas, and also estimating the volatilities and correlations amongst the assets were trading in. Most of the theory we'll talk about today deals with estimates-- well, with the problem where we're assuming basically

certainty in terms of understanding those underlying return characteristics, the means, variances, correlations. But when we have to estimate those that raises problems. And finally, we'll finish up talking about some alternative risk measures, which extend the straightforward mean-- or simple mean, variance analysis.

Now the mean, variance analysis is a single period analysis. What we want to do is consider investing our capital in some assets. And we want to do so for just a single period and do so optimally.

So in terms of notation, we'll consider risky assets indexed by i , 1 through m . Single period returns will be a multivariate vector of those returns. The mean and covariance of these assets are represented in vector of returns. And covariance matrix of squared volatilities, the variances of the assets along the diagonals, and the covariances on the off diagonals.

A portfolio will be represented as basically a weighting of our investment in these m assets. So we'll have a vector w , consisting of w_1 up to w_m . I'm giving the relative weights and absolute weights of those investments. We'll assume that we have one unit of capital to invest. So the sum of these weights equals 1.

And the portfolio return has expected return given by the expectation of this linear combination of the returns, which is simply the same linear combination of the underlying expectations. And the variance of the portfolio is given by the variance of the weighted average of the individual returns, which is given by this quadratic form in the covariance matrix. So we've gone through this-- or you understand, I think, these calculations from our discussion in time series. And that's quite simple.

Now before going into the theory for this, let's look at the problem. Let's look at the portfolio analysis problem in a very simplified setting. So we're just going to consider two assets and talk about optimal portfolios investing in these two assets.

So m equals 2. The first asset has returned R_1 with mean 0.15. Now I'm going to think of this as annualized return, so a 15% annualized return and a volatility of 25%. The second asset has expected return of 20% and a volatility of 30%, σ_2 .

These assets are possibly correlated. We'll let ρ denote the correlation between the two. And a portfolio-- basically all the portfolios we could invest in so long as we are limited to a unit of capital and no shorting are given by portfolios indexed by w , basically the amount of money we invest in the second asset. So R_w is $R_2 w$ plus $1 - w$ times R_1 . That's the return on the portfolio that invests in those proportions in the two assets.

The expected return is this linear combination, which is given in matrix form in the previous page-- or in the lecture notes. And the variance R^2 volatility is-- what-- it's $(1 - w)^2$ times the variance of R_1 plus w^2 times the variance of R_2 . Then we have a covariance term that comes into play.

Now the mean-variance analysis that Markowitz addressed basically looks at the feasible portfolio set. We simplify things by saying let's focus just on the volatility or variance and the expected return of our portfolio. So we can define the feasible portfolio set as the collection of volatility return pairs for all the assets-- or for all the portfolios.

And well, what is π^* , that collection? Basically what's the universe of possible portfolios we could construct? In this two dimensional case it's going to be very simple. In multidimensional cases it's more complicated.

But what portfolios are optimal? Sub-optimal? How do we choose or identify a particular portfolio to invest in? How's that choice made? And is there any special structure to these portfolios?

What you'll see in the lecture today is that the Markowitz theory and extensions of that provide really elegant answers to these questions. But let's understand basically what's going on just with this two asset case. And what I'm going to do here is just simulate 500 weekly returns with different values of the correlation between the two assets.

And we want to examine the cumulative returns of each asset. The asset returns in terms of their means, volatilities, and correlations, a plot of π^* , and the

cumulative returns of each asset with a minimum variance portfolio. So let me just highlight here that what we want here is to be plotting sigma omega versus alpha omega.

And we have two assets. We have basically sigma 1 and alpha 1 corresponds to our first asset. Sigma 2 and alpha 2 corresponds to our second asset. So this point here corresponds to w equals 0. And the other corresponds to w equals 1, so R_1 and R_2 .

Now, what's going to happen when we combine assets in a portfolio? Well, let's take a look at the simulation. OK, here is a graph of the simulated asset returns with basically a mean returns given by 15%, 25%, volatility is 20 to 30, and with an asset correlation of 0.

So here's just a scatter plot of the weekly returns. There's basically no apparent correlation there. There actually is a sample correlation, because a sample from these distributions won't have perfectly 0 correlation.

On top, we have graphs of the cumulative returns of the two individual assets. Rather, obviously, the higher graph corresponds to the asset with higher return, which is asset 2. And the green corresponds to asset 1.

The graph on the right top right is the graph of the feasible set as we allocate between asset 1 and asset 2. And so by the simulation, this curve corresponds to the feasible set of portfolios. And what's really remarkable is that we can get basically a reduction in the volatility of the portfolio without compromising, and in fact improving, the return the portfolio.

So if we invest fully in asset 1, we're at this point. As we increase-- as we start allocating towards asset 2, not only does the return of asset of the portfolio go up, but volatility goes down. And let's see, in this simulation, what I've done is also plotted the return of the portfolio corresponding to the minimum variance.

So let's just look the minimum variance portfolio for a minute. We have sigma squared w is equal to $1 - w$ squared plus w squared sigma 2 squared-- this is sigma 1 squared. And then plus 0 in the case there's no correlation between the two

assets.

If we want to minimize this portfolio volatility, we can take the derivative of that with respect to the weight and set that equal to 0. So what's that equal to? Well, it's-- let's see, did I do this right? Actually just with the previous notes-- sorry, R_1 is equal to 0.

OK, so we have $2w_1 \sigma_1^2 - 2w_2 \sigma_2^2 = 0$. And so solving this, we get w_1 is equal to-- well, let's see, you get $\frac{1}{\sigma_1^2} = \frac{w_2 \sigma_2^2}{w_1 \sigma_1^2}$ sorry, $\frac{1}{\sigma_1^2} = \frac{w_2 \sigma_2^2}{w_1 \sigma_1^2}$ divided by $\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$.

If you solve this out, you basically get a weighting on the different assets, which weights them inversely proportional to their squared volatility. And with this graph here, you can see that the blue graph is a bit closer to asset 1 than to asset 2's cumulative return. That corresponds to giving to a slightly higher weight to asset 1, because $\frac{1}{\sigma_1^2}$ is bigger than $\frac{1}{\sigma_2^2}$.

Well, let's look at what happens if we consider negative correlations between the two assets. Well, actually, OK, before we do that, if you were going to choose one of these portfolios for investing, are there any portfolios that you wouldn't invest in? And are any of these portfolios sub-optimal in terms of mean variance?

AUDIENCE: w_1 .

PETER KEMPTHORNE: Well, w_1 is certainly infeasible because we can increase its mean and we can decrease its volatility. So actually all of these points here are really sub-optimal portfolios. And from the minimum variance portfolio, which is basically getting us the vertical tangent, all the points from here up to the asset 2, fully investing in asset 2 are feasible. And none of these portfolios dominates the other. Basically it's a trade off between return and volatility.

OK, so that's a really important point that there really are portfolios that you can just disregard considering that there are definite-- well, when there's zero correlation, there is a benefit to diversifying across these two assets. So if you have an asset

and you are considering pooling that asset with another in your portfolio, if those are fully uncorrelated, then you should be able to improve your portfolio by adding some allocation to that second asset.

Let's look at the a more negative correlation between the two assets of minus 0.4. Here, you'll see that there's basically a tilt, negative tilt, to the scatter plot of returns. And with the portfolios of two assets, basically, the feasible set, that feasible set is actually stretching further to the left.

So with a negative correlation between the assets, we're actually able to reduce the volatility even more, if that's what is as our preference. If we go up here to minus 0.8, it gets even more exaggerated. Now what's going to happen if we have a correlation of negative 1 between these two assets?

AUDIENCE: The portfolio can have 0?

PETER KEMPTHORNE: Yes, then a portfolio of the two assets can have zero variability. So indeed with the hedging strategies, one often is considering investing in assets consisting of perhaps an underlying security and some derivative on that. And that derivative, if it's a future say, could have basically a negative 1 correlation. So you could essentially hedge out the volatility almost perfectly.

So that special case actually does exist quite frequently and is exploited. But it falls out just from the simple analysis of this simple simulation. OK, let's also look at the going to increase in the correlation. Yes.

AUDIENCE: So if the volatility of the whole portfolio is zero, does that mean it will always be constant? It will be--

PETER Does that mean what?

KEMPTHORNE:

AUDIENCE: So would that mean since it has zero variance it will actually be constant over time? The whole portfolio?

PETER OK, I mean, if it has zero variance, it should be constant. Zero variance means that
KEMPTHORNE: over-- that its value is a constant.

AUDIENCE: Is it a good idea to do that?

PETER Well, in terms of the markets nothing has zero-- almost nothing has zero volatility.

KEMPTHORNE: And so indeed-- but in the pricing theory that [? Zhun ?] [? Bung ?] will be discussing in subsequent lectures, if we have basically a portfolio which has no volatility, then its return should be equal to whatever a risk-free rate ought to be. And so this particular portfolio out to be structured so that it achieves a return equal to a risk-free rate, barring transaction costs and frictions and all that kind of thing. But yeah.

OK, so if we increase the correlation from 0 to 0.4, well, we still get a benefit of diversification but less. We're basically not able to lower the variance as much. And it's even more exaggerated with a correlation a 0.8.

Now, in looking at the simulation, one thing I want to highlight to you now, which will come up later, is here I've simulated returns according to Gaussian distributions with these means, volatilities, and correlations. And in the lower left panel, I give the sample statistics. Basically, the maximum likely estimate for all the parameters.

And what you'll see is that these sample estimates differ quite a bit from-- well, they differ from the theoretical parameters. In this case, the sample volatility is almost exactly on point for the first asset. It's a bit below 0.287 for the second. The sample mean is 0.21, which is a bit high for the first. And it's 0.321, which is also a bit high for the second.

Let's just go back and look at a couple of the others, because it can get-- OK, well here's one where the sample means are 0.144 and 0.343. And so those are actually quite different from the population parameters. And at least, the second asset has a much higher sample mean.

So it's important to note that sample estimates can have a certain amount of variability. It turns out that there's less variability in estimating covariances and

correlations, if the assumptions hold, and greater variability in the sample means. So at the end of the day we really need to be very sensitive to what estimates we use and how much uncertainty there is those.

OK, let's go back to the lecture notes now. View, screen, all right. All right, so we've just gone through how we are evaluating different portfolios in terms of the pair of the-- basically the mean of the portfolio and the squared volatility of the portfolio. Higher expected returns are obviously desirable. Low volatility is desirable.

And so what Markowitz did was to pose this is a quadratic programming problem, where what we want to do is minimize the squared volatility of the portfolio subject to a constraint on the mean of the portfolio and considering that we're fully invested. So this mathematical problem is a standard convex optimization problem, a quadratic programming problem, which is very simple.

And we solve it by defining a Lagrangian. Basically, we take our objective function of the volatility. And we want to minimize that. We're going to use a half factor just to simplify the computations. And then we add Lagrangians for different constraints of the problem. So we have λ_1 times α_{naught} minus $w' \alpha$.

We want the mean return, $w' \alpha$, to be constrained to equal α_{naught} . And we also want the sum of the weights, $w' \mathbf{1}$ to equal 1. And the first order conditions of the Lagrangian basically give us those two constraints for differentiate with respect to λ_1 and λ_2 . The initial first order condition with respect to the portfolio weights basically allows us to solve for those weights.

Now, this first order condition is going to solve our problem, because if we take the second order derivative of this Lagrangian, it's going to be σ . So let me just point out if we take $d^2 L$ by $dw dw'$, that's equal to our covariance matrix σ . And that is a positive definite or positive semi-definite matrix. So we indeed are minimizing the problem.

So this is just a generalization of basically a parabola in multi dimensions. And we're trying to minimize that. Well, what's the solution to this?

Well, first we can solve for w , the weights, in terms of λ_1 and λ_2 . So we take the first equation of the first order conditions and basically premultiply by Σ^{-1} . And we get w_{naught} is equal to $\lambda_1 \Sigma^{-1} \alpha + \lambda_2 \Sigma^{-1} \mathbf{1}$, the unit vector or vector of units.

And then we can just solve for λ_1 and λ_2 by plugging this w_{naught} solution into the second two equations. And these two equations for the second and third first order conditions is just a very simple set of linear equations. We basically have $\alpha_{naught} = A \lambda_1 + B \lambda_2$.

And that matrix A , B , and C is given by $\mathbf{1}^T \Sigma^{-1} \alpha$ for the A element. And the B and C elements are corresponding elements. So this solves the problem.

And the variance of the optimal portfolio with a given return can also be solved by just substituting in these solutions. So we get that for a given α_{naught} , our target return, the squared volatility of that optimal portfolio is this has that form.

Now, what's that form? It's essentially a parabola in α_{naught} . OK, so when we're looking at-- is there an eraser? here it is-- when we're looking at this graph here in the two dimensional case, then there's basically a parabola. I can't draw parabolas very well. But there's a parabola between those two points that characterizes the thing. And in multi-dimensional-- multiple assets, it's just a multivariate extension of that.

This particular problem can also be looked at in two other ways. Before we're looking at minimizing the variance, subject to a constraint on the expected return. We can also say, well, let's maximize the return, subject to a constraint on the volatility. And that problem basically has the same Lagrangian.

And we can also consider just maximizing a weighted average of the return and have a negative multiple on the variance and consider maximizing that expression. This turns out to be the risk aversion optimization, where we basically are penalizing portfolios w for how much variance they have. And the λ factor tells us how

much penalty to associate per variance unit.

And these are all equivalent problems being solved by the same Lagrangian. And from these problems we can define the Efficient Frontier, which is the collection of all possible solutions, where we range the target return amongst values that are feasible and the volatility amongst values that are feasible as well. So the Efficient Frontier will just trace these.

And in our two variable case, our two asset case-- we have sigma and alpha. We have two assets. It's basically a parabola like that.

If we have another two assets that we're investing with, then if we consider the two asset portfolios are just these two assets, there's basically another parabola there. And as we consider different other assets in the mix, we basically get all these parabolas of two asset portfolios. And then combinations of those two asset portfolio gives us our feasible set.

And so at the end of the day, we basically have a convex set of all feasible assets, which define the Efficient Frontier. And the Efficient Frontier is always going to be basically the top side of that curve.

Well, let's see, the next topic considers adding basically a risk-free asset to invest in. The problem as it's been stated so far says we want to be fully invested across our m assets. And what are the optimal portfolios?

Well, what if we want to invest some of our money, our capital, just in cash? Or not invest our full capital and invest the rest in the portfolio? So if we consider adding a risk-free asset, then this is an asset. We'll call it the 0th asset.

It has, say, some expected return or not. It's risk free. So that's a constant. It has zero variance.

And if we're investing as well in that possible risky asset, then we can basically consider investing w weights in the risky asset and $1 - w$ times the unit vector in the risk-free asset.

And let's see, I want to draw this graph here. Let's see. OK, suppose we have two assets where this is the efficient set. And we consider now allowing for a risk-free asset to be invested as well. Well, that risk-free asset is basically this point here. And it has mean r_{naught} and variance 0.

Now, if we combine this asset with any of these portfolios, what's the feasible set going to be? Well, we can basically invest in some money in the risk-free asset and some in asset 2. So we can get any point along this line.

Basically, if we invest in the risk-free asset-- if we invest some fraction in asset 2 and the remainder in the risk-free asset, then our expected return is the linear line between r_{naught} and α_2 . And the volatility of a weighted average-- of a fraction of asset 2 has a volatility given by whatever that weight is times the volatility.

So this point here corresponds to, say, w times σ_2 for its volatility and r_{naught} plus w times α_2 minus r_{naught} . Now the dramatic thing is that we actually are able to achieve an improvement over points on the efficient frontier, such as, say, the minimum variance portfolio.

And if we consider the fraction here, we have a higher return and a lower variance than the minimum variance portfolio that we considered before. And so investing in the risk-free asset does sort of enlarge our opportunity space quite dramatically. And there are some very special results that come from this.

So let's go through the mathematics for solving this problem. And in the lecture notes, I'm going to go over this pretty quickly, as really when you go through it slowly you'll say, oh, this is very straightforward and logical.

So we're basically going to minimize the volatility subject to the constraint that the return is equal to α_{naught} . We define the Lagrangian and do the first order conditions, solve those. And if we solve those, we get-- basically λ_1 has this nice form here. And this is obtained by just very simple equation solving of those two first order conditions.

And if we have m assets available when we look at the solution, then we get an optimal portfolio that has a w naught vector given by this solution and a λ_1 given by the solution. Now w naught is the proportion or is the allocation to the risky assets. What varies depending on our target return, α naught, is simply this λ_1 Lagrangian multiplier.

Basically, α naught, our target return, only affects the value of λ_1 . And our weights across the risky asset is a simple multiple of a fixed vector of relative weights. And this fixed vector of relative weights is the inverse of a covariance matrix across the assets times α minus $1 - m R_0$.

And so what we have is a portfolio that basically invests in the risky assets in the same way. The only thing that differs is how much weight we give to that particular portfolio. As we increase λ_1 , then we give more allocation to this fixed vector of weight. So we invest proportionally in the assets. And we just scale how much that overall factor is to achieve different levels of return.

OK, we can get nice, closed form expressions for the portfolio variance just as an extension of the two asset case. Basically the portfolio variance is a parabola in the target return. And so as we increase our target return, we have to increase our portfolio variability. There is that trade off when we use optimal portfolios.

And if we consider the fully invested optimal portfolio, well, the fully invested portfolio-- and that's nothing in cash, invest everything in the risky assets-- we actually are going to call that the market portfolio. And the expressions here give us the weights for that market portfolio. And this may look a little complicated in terms of the expressions. But it's actually quite simple closed form expressions for the expected return and the variance of the market portfolio.

And what's happening with this is we have-- basically, for the risky assets we have some efficient or set of feasible portfolios, which is all in this range. And then with our risky asset, we could be investing here. And so the optimal portfolio, in the case where we can invest in the risk-free asset, will basically correspond to sort of maximizing the mean return for across all portfolio. So it's actually going to

correspond to the simple tangent line, which basically crosses the market portfolio, which is the fully invested portfolio.

Now, this structure of the problem is actually incredibly powerful. And there's an important paper by Tobin, which states that basically every portfolio is going to-- under these assumptions for the investment problem-- every optimal portfolio investment combination of the risk-free asset and the market portfolio. So regardless how much risk you want to take, the optimal portfolio is essentially the same. It just depends on how much capital you're going to put into that optimal portfolio. And so all of the optimal portfolios will invest in the same risky assets as the market portfolio in same proportions. And the only difference is their total weight.

Now, plugging in the expressions for the different Lagrange multipliers for a given portfolio, p with return-- and we get that expression. So let me just summarize that. So suppose we want the portfolio p such that the expectation of our p is equal to α_{naught} , then this portfolio R_p is going to equal-- basically, it's going to invest in the risk-free asset with $1 - w_m$ plus w_m times the return of the market. And the actual weights are given by this expression here, which gives us the expression that I was just writing up on the board. We have the expected return is $r_{naught} + w_m$ times market return minus r_{naught} . And the volatility, σ_p^2 , is simply the square of the weight in the market times the market volatility.

This leads to the definition of basically the capital market line, which is essentially this line here. This is our capital market line for the portfolio optimization. And the structure of this line is such that this point for the market portfolio that has volatility given by the market portfolio's volatility σ_m and the return on the market, α_m , or expected value of R_m .

And the slope of this line is given by the $\alpha_m - r_{naught}$ over σ_m . And points along the line are given by $r_{naught} + \sigma_p$ times this factor. So this is the expected value of R_p is any optimal portfolio has return is equal to the risk-free

rate plus a multiple of the return per risk of the market portfolio.

This term here is called the price of market risk. Yes?

AUDIENCE: Is that the same as the Sharpe ratio?

PETER
KEMPTHORNE: It's close to the Sharpe ratio. The Sharpe ratio is-- yes, this is the Sharpe ratio for the market portfolio.

And so, if we want to invest in the market, our decision is reduced to how much risk are we willing to take. And the compensation for taking extra risk is extra return. And we achieve that by essentially investing in the same portfolio, only we change the scale at which we're investing in that portfolio.

So far we've been considering sort of points between being fully invested in the market or fully invested in cash. If we are able to borrow money at the risk-free rate, then we can basically allocate additional weights to the market portfolio and achieve points that are beyond-- have higher return, higher volatility than the market portfolio. We can basically lever the strategy by borrowing money and investing that in this market portfolio. So Efficient Frontier, if we can borrow risk freely would just be the capital market line extended.

Here's a listing of the papers that go through sort of the classical foundations for this. These are all accessible on the web. And I encourage you to actually look at these papers, because the arguments are very straightforward. The motivation and background is interesting to read.

Virtually everyone on this page actually has a Nobel Prize, except Lintner, I think he died before they gave the Nobel Prize to Markowitz and Sharpe, but he certainly would've been included in this case.

AUDIENCE: And the latest one.

PETER Pardon me?

KEMPTHORNE:

AUDIENCE: The latest.

PETER And the latest one, yes, yes, Fama was just awarded, what, two weeks ago. All

KEMPTHORNE: right, let's move on to Von Neumann-Morgenstern Utility Theory.

Now, in the Markowitz mean-variance analysis we sort of reduced all portfolios to the properties of what's their expected return and what's their variance or volatility of the returns. Under what circumstances would that be a really good decision to be making for how to do portfolio optimization, portfolio allocations? Well, the Von Neumann Morgenstern theory. is a theory which motivates making decisions under uncertainty, where you should specify a utility function for your wealth, and you should engage in decisions that maximize the expected utility of your wealth.

And the theory is really very powerful in that when you're making decisions under uncertainty, there are sort of rational things you should do. If you like higher return, the decision should be consistent with preferring outcomes with higher returns. If you don't like variability, then you should be preferring returns that have the same expected return, that have lower volatility. But depending on how your utility function is defined, you may get different outcomes.

And so to set up this problem, let's just consider the problem that Markowitz addressed. You have a one-period investment. You start with the initial wealth, w_0 . You're going to choose a portfolio p . And the wealth after one period is simply going to be 1 plus the return on that portfolio given terminal wealth w . And our utility function is going to be some quantitative measure of the outcome, what's the value to the investor. And we'd simply want to compute the expected volatility of different strategies and find the strategy that maximizes the expected utility.

What are the basic properties of utility functions? Well, if we graph over wealth, our utility of wealth, starting at w_0 , our initial wealth, if we have greater wealth, presumably we'll have greater utility. So the slope of the utility function should be increasing. Perhaps as we get more and more wealthy, the marginal benefit of additional wealth isn't quite as much as it was when we were poor. And so perhaps the curve for the utility function should taper off a bit.

OK, these conditions would correspond to the first derivative being increasing always and the second derivative being less than 0 perhaps. There are definitions in the literature of risk aversion, absolute risk aversion, and relative risk aversion. And these are simple functions of the first and second derivatives of the utility function.

To see where these come into play, let's assume the utility function is a smooth function and consider Taylor's theories of approximation of that. So if we consider expanding the utility function about some base wealth, w^* , then it's simply equal to that value of w^* plus first derivative times $w - w^*$ plus $1/2$ the second derivative times the squared deviation of the wealth from w^* .

And if we take expectations of this, and if w^* is the actual expected wealth random variable, then this expected utility is actually proportional to the expected, w^* minus $1/2$ λ times the variance of the wealth. So sort of to a second order of approximation, this expected utility is a function which is looking at expected return minus a multiple of the volatility of that return, or the wealth if we're considering it on that scale.

So there are various different utility functions that economists have worked with. And basically the kinds of functions they work with are all the simple functions we as mathematicians know about, linear functions, quadratic functions, exponential functions, power functions, and log functions. So these are just the first ones to come to mind for economists perhaps. But there's actually some rich theory in terms of sort of investment choice with different utility functions of these types. And there is some interesting work there in the economics literature.

One thing that is to be pointed out is that with quadratic utility, then if we consider the expected utility under quadratic utility, that is the expected utility function that only depends on expected wealth and the variance of the wealth, which depends only on the expected return and the variance of the return of the portfolio. So if we are working with a quadratic utility function, then this means, variance analysis is the right thing to be doing under the Von Neumann Morgenstern expected utility theory. So doing that is a good decision under that if that's the utility function.

Now when is that solution not to be preferred? Well, really it's not be preferred if you have a different utility function possibly. It may be the utility function should be adding some penalties for skewness or kurtosis. And that's being ignored here. So extensions can be looked at.

An interesting mathematical fact is that under the assumption, or should I say the imagination that returns our Gaussian distributed. Then the sort of means and variances of portfolios completely characterize all the distributions of portfolios of assets, if the underlying assets are Gaussian distributions. So a mean-variance analysis is actually optimal under non-quadratic utility functions, if the underlying assets are Gaussian distributions, because only the means and variances are going to characterize the optimal portfolios and properties of those. And I guess, the stochastic dominance of functions of these variables will generally apply when there's that corresponding stochastic dominance in terms of their means and variances.

Anyway, this kind of theory can get involved. It allows us to extend the basic model that we consider here. Let's turn then to the topic of the portfolio constraints.

So far, in looking at this problem, we haven't made any constraints on the problem. We just want to maximize return, minimize variance, and consider trade offs between those. With practical portfolio optimization problems, there are different constraints that come into play. Portfolios that are long only constrain the weights to only be positive. There can be holding constraints, which it may be that we don't want the amount of a given asset to be too big.

There's the equity strategies, you don't want to be holding-- if you have a sort of medium frequency trading strategy in equities, you don't want to be holding much more than the trading volume of a few hours or maybe a day in the asset, because if you happen to have to sell it, there really won't be liquidity to trade that. But we can add--

There's a simple linear constraints on the holdings. There could be turnover

constraints. If we're considering basically adjusting our portfolio from one period to the next, there are limits in terms of how much we can actually trade of the different assets. There can be a benchmark exposure constraints.

Suppose we want to invest in a portfolio that's very much like there's a market index, say, the S&P 500. But we want to try and do better than the S&P 500. But we want to protect ourselves from basically not being very close to the benchmark if our bets, basically, an allocation, are wrong.

Then one basically can control how different from the benchmark allocation we are limit, then limit that departure so we can basically limit how far from the benchmark weights we want our portfolio to be. And this is useful basically for considering strategies that do as well or better than particular benchmark indices.

Related to this is our tracking error constraints. In addition to basically not having allocation weights that depart much from the benchmark weights, we can consider how much variation is there in our portfolio compared with an underlying benchmark. And we can consider tracking error between our portfolio p and the underlying benchmark and measure that tracking error in terms of variability, that difference, and want to control that.

There are also risk factor constraints. In equity markets there are many different factors that affect returns. And these factors can be identified empirically and controlled for by limiting the exposure to different factors in the portfolio. We'll see this in next lecture. But basically if there are underlying factors f , which basically a return on the asset i can be explained by sort of an idiosyncratic alpha for that asset, plus certain correlations with market factors f , and then a residual innovation ϵ .

And so the returns on these assets can be explained perhaps 50% by underlying market forces, and given by the underlying factors, that jT , which are constant across i . So these are affecting all the stocks with different coefficients β_{ik} . And in our portfolio, we may want to limit the exposure to a given factor. And with many strategies, you actually want to neutralize the portfolio to those market risk factors.

So we can actually constrain our weights to have zero exposure across these different market factors.

There are other constraints, minimum transaction size-- generally it's the case that trades in equities are in 100 share units, although that's changing. There can be minimum holding sizes. And there's also integer constraints that can be applied.

If you're trading assets that have sort of large values, then these integer constraints actually come into play. If you're trading Google stock, and how much is Google stock worth now?

AUDIENCE: \$800.

PETER KEMPTHORNE: Something like that. So compare that with Ford, which is like \$50, or I don't know, but there's orders of magnitude difference. So integer constraints can come into play. If you're dealing in a very large size, then these things don't really have much impact. But they can with smaller portfolios.

Now, all of these different constraints can be expressed as linear and quadratic constraints on the weights. So the set up for the portfolio optimization problem can be the same as before, except we add in these additional constraints. So we basically add in additional Lagrange multipliers times these particular constraints with their linear quadratic. And we can implement the portfolio optimization problem that way.

Let me turn to an example. Let's go-- OK, I want to go through an example. of-- OK, I want to consider US sector exchange traded funds between 2009 and last week. Basically, exchange traded funds allow you to invest in equity markets with sort of single assets that represent different sectors. The ones that are considered here are basically spider traded funds, which invest in the different major industrial sectors of the US market, ranging from materials, health care, consumer staples, down to technology and utilities.

Here is a graph of the cumulative returns of these nine different exchange traded funds and over the period from 2009 up through last week. And what one can see is

basically they perform differently. And what I'd like to do is just examines what would have been an optimal allocation across these exchange traded funds over this period. So busy looking back at the data and seeing how our portfolio analysis tools would result in particular application allocations.

So we can look at the risk versus return annualized of these. Let's see-- this is-- I guess that good enough.

Anyway, what you can see is a volatility range between 0 and 30. And you want your trim between 0 and 25. We have the different sectors in this plot. So this is our feasible set-- this is the plot return versus volatility for these nine exchange traded funds.

This happens to be utilities. There's financials over there. Here's consumer staples. Very top is a consumer discretionary.

Let's see, I applied a mean-variance optimization to this problem, assuming that there's a constraint of 30% of the capital per asset. So we don't want to invest more than 30% of the capital in any single exchange traded fund. And this graph here shows how, as we vary our target return from sort of the minimum value up to the maximum value for being fully invested, how much of the capital we're going to invest in the different sectors.

So what you can see is this yellow one, which is consumer staples, is coming in with a really high weight. And the green is energy-- let's see, I think it's energy-- that is the next one. And oranges is health.

And what's important to see is that when we're looking at sort of the lower range of the returns, basically expecting returns above the risk-free rate that's very low, then the relative proportion investment in the different assets is the same. Basically, these allocations are slowly scaling up, just linearly. And that corresponds to investing in the optimal portfolio without constraints with a fixed value.

But once we hit the 30% then that constraint starts to be active. And we can't invest anymore in that. So we have to add more to other securities. And so basically more

is given to the consumers discretionary and the other.

Now actually here's another graph of the same data, just stacking the allocation. So we consider ranging from how are we investing our capital. This dark red is how much money we invest in cash. And then these colored lines indicate by their vertical length how much we're investing in the different exchange traded funds.

So basically as the constraints get hit, the portfolio sort of change in terms of their overall structure. And what can happen is-- actually as we're trying to achieve more and more return, well, there may not be assets that provide that additional return, or there may be a very few assets that provide that return.

And so in this case, technology stocks are coming in as we're trying to get really high levels of return in our optimal portfolios. And so if we really want that higher return, that's going to come at higher risk. And we actually are de-allocating some from the consumer staples and consumer discretionary at that point.

OK, here's a graph of the Efficient Frontier that has estimated with the data. And what you can see is that these portfolios optimal portfolios yield improvements over each of the individual exchange traded funds, in terms of having higher return with lower risk. And if we consider just a target return of 10%-- OK, there's the portfolio.

I've graphed here sort of in solid blue what the optimal portfolio with a 10% volatility. That's the solid blue line there. So these are the results with this 30% capital constrained.

Now how is this problem going to change if we reduce the capital constraint from 30% per asset to 15%? Who can comment on what will change?

AUDIENCE: I'm a little confused. Could you put a constraint on the risk-free asset. Could you do that as well?

PETER
KEMPTHORNE: Well, the risk-free asset has no risk. And so putting a constraint-- well, you could put a constraint on it in terms of it not going negative and it not being greater than one. So in fact that is a constraint. Those are constraints being imposed here. But, I

guess, in some-- well, there are, like I guess, endowments, say, that have investment policies that say that they want a certain fraction assets invested in risky assets and not in cash. So that is a realistic constraint in certain circumstances. And this is just highlighting how the allocations vary across the risky asset if we only have very simple constraints on the cash investment.

Well, the question I want to focus on is what happens if we make the maximum location constraint more severe. So it's 50% instead of 30%. Then what's going to happen is these capital constraints are going to start hitting sooner.

So we have to allocate to the other exchange traded funds. So let me show you how that works out with the same graph. Let's see-- OK, let me just close that and then we go here.

OK, this shows with a 50% allocation, how the allocations vary. And I'll be posting this on the course website. But it turns out the Efficient Frontier is actually lower. Basically, for higher returns, we actually can't get those higher returns by getting the biggest bang for the buck. We don't want to allocate too much to those higher return, higher risk securities. And so the optimal, the Efficient Frontier has estimated basically slopes down.

Now, with this particular example of these exchange traded funds and how you allocate across those, this provides some insight into portfolio optimization. It's not really a realistic setting, because we're looking over the past and using the actual sample performance to define these portfolios.

Let me just highlight an example of applying these kinds of methods with a-- is it here? Yep, OK. Is that the right graph?

OK, suppose we consider not investing in exchange traded funds. But we are a hedge fund. And we have sector pricing models across all these different sectors.

And what we can do is consider going long, short in these sectors. And in fact, we consider sort of a market neutral strategy, sector by sector, and consider investing sector by sector in these different sector-based models.

OK, here's a graph of different multifactor pricing models for trading market neutral programs within each of the nine industrial sectors, the same ones corresponding to the exchange traded funds. And because these strategies are market neutral, the sort of total returns over-- this is a five-year period-- are rather modest, namely sort of 60% for some of the models, 20% for other models

What's particularly relevant with these models though is that they tend to be less correlated. And the diversification benefits can be rather dramatic. And so here is a graph of the optimal allocations across these different sector market neutral models.

And we can see that this red model is getting a lot of weight. That actually is the utility sector. And then these other models are industrials and energy. And so if we consider investing in these, we can actually achieve a target volatility of 10%-- we're going to achieve this solid blue line-- as the portfolio strategy.

And one can see how by combining these assets together-- these assets are different market neutral trading models-- we have to get quite a bit of benefit from the portfolio optimization. And the greater benefit in the portfolio optimization is because these strategies tend to have much lower correlations with each other.

Let's go back now, finish up with-- how do we go back here? Where's my red dot?

OK, just to finish up the discussion, with these methods it's important to highlight that what we've been using were estimated returns, estimated volatilities, and correlations. And these estimates can have a huge impact on the results.

There's basically choices of estimation period. There's estimation error. Different techniques can modulate these issues. Exponential moving average are often applied. Having dynamic factor models is used. Basically, rather than using sample estimates of the variance, covariance matrix using factor models to estimate the variance covariance matrix results in more precise inputs to the optimization.

And finally, just with different risk measures-- we've been focusing on portfolio volatility and minimizing that-- the methodologies can be extended to have different

measures of risk-- mean absolute deviation, for example, or semi-variance. In terms of the motivation for squared volatility and minimizing that, you'll recall that in likelihood analysis of Gaussian distributions, the sort of squared deviations from the means are characterizing variability.

Well, if we focus on mean absolute deviation, then probability distributions that where the distributions relate to the absolute deviation from the mean characterization distribution might be appropriate. And indeed, distributions with heavier tails, like exponential distributions, are appropriate when we consider mean absolute deviation. So some alternate risk measures are appropriate when the volatility measure itself isn't necessarily appropriate because of the distributional assumptions.

Semi-variance was introduced actually by Markowitz, where you're interested in controlling the downside risk as opposed to the upside risk. There's value risk measures which are now completely standard in portfolio management and management of risky assets, where one is actually-- well, this is introduced by Ken Abbott in one of the first few lectures. Value at Risk is a very simple idea of characterizing what outcome is sort of the threshold of extreme, at say the 5% level or the 1% level, and to basically keep and monitor the Value at Risk.

So if we had a risk that was basically in the worst 5% or 1% of the time. What is that level? That's the Value at Risk.

That as a nice risk measure is simple to define and reasonably simple to estimate. But it doesn't characterize sort of what the potential exposure is if you exceed that. And there are extensions of that.

The conditional value at risk which is looking at the expected loss, given that you exceed the value at risk threshold. And this method is now-- I think it's been going to be incorporated into regulatory requirements for banks in terms of how they measure risk. Right now, I believe it's almost all value at risk. Extensions like this conditional value at risk is definitely going to be applied.

And in the discussion of different risk measures, there's literature talking about what are appropriate risk measures and how do we define those. Well, it really depends on what your assets are and if your assets are simply simple investments in stocks or bonds, sort of cash assets or cash instruments, then value at risk is quite reasonable. If you're investing in derivatives, which have non-linear payoffs, then things get complicated. And so things need to be handled on a case by case basis there.

But if you're interested in risk analysis, there's a whole discussion on coherent risk measures that you can look into. That's quite interesting. OK, let's stop there. Thank you.