

Lecture 25: Can We Recover?

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Can We Recover?

- The question in the title of this talk is intended as a triple entendre.
- “Can We Recover?” could refer either to:
 - 1 the systemic risk arising from the credit crisis, or
 - 2 the main result in a recent paper by MIT professor Steve Ross, or
 - 3 academic and practitioner reaction to item #2 (especially mine!)
- The title of this talk actually refers to items #2 and #3.

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The Recovery Theorem

- Steve Ross has a forthcoming JF paper called “The Recovery Theorem, which is also the title of his Theorem 1. The theorem gives a sufficient set of conditions under which “natural” probabilities at a point in time are *uniquely* determined (i.e. recovered) from exact knowledge of Arrow-Debreu (AD) security prices on that date.
- There are many ways to impose additional and consistent assumptions which uniquely determine a snapshot of AD security prices from a snapshot of derivative security prices.
- When both sets of assumptions hold, a snapshot of derivative security prices yields the market’s contemporaneous forward-looking view on the underlying. In particular, one gets the likelihood of large rare moves and one gets the mean. This contrasts with the classical time series approach which assumes that the future behaves like the past, looks backward, and generally only obtains a highly noisy estimate of these desired quantities.

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- Let \mathbb{P} be physical probability measure, whatever that means to you
- Assume no arbitrage and the existence of a money market account (MMA). The first fundamental theorem of asset pricing says there exists a probability measure \mathbb{Q} such that MMA-deflated prices evolve as \mathbb{Q} local-martingales.
- Suppose for a moment that the market's beliefs differ from \mathbb{P} . If we attempt to recover \mathbb{P} from \mathbb{Q} in such a world, we obtain a 3rd probability measure that we can call \mathbb{R} (for recovered).
- If the market's beliefs reflect reality, then $\mathbb{R} = \mathbb{P}$. We allow the possibility that they do not, so it is only the probability measures \mathbb{Q} and \mathbb{R} that will necessarily be equivalent.
- In this talk, we refer to \mathbb{R} as representative beliefs and we will show how to recover \mathbb{R} from \mathbb{Q} . Believers in market efficiency can replace \mathbb{R} with \mathbb{P} whenever they see an \mathbb{R} .

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Natural Assumptions for Obtaining Natural Probabilities

- Ross shows how to recover the natural probability measure \mathbb{R} from market prices of derivative securities. In Ross' theorem 1, one starts with “a world with a representative agent”.
- While worlds of this type may be natural for financial economists, my subsequent discussions with industry practitioners and math finance academics suggested that worlds of this type were less than natural for them.
- The purpose of this talk is to show that \mathbb{R} can be determined under an alternative set of sufficient conditions, which hopefully this alternative audience will find more natural.
- In a nutshell, we will switch attention away from “worlds with a representative agent” and instead model the value of a portfolio which has been called “the natural numeraire” (cf. Flesaker & Hughston and Platen), the “growth optimal portfolio”, (cf. Kelly, 1956), and the “numeraire portfolio” (cf. Long, 1990).

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Overview of this Talk

- There are six parts to this talk:
 - ① Arrow-Debreu Security Prices & Market Beliefs
 - ② Ross Recovery for Finite State Markov Chains
 - ③ Change of Numeraire in a Univariate Diffusion Setting
 - ④ The Numeraire Portfolio in a Univariate Diffusion Setting
 - ⑤ Recovering \mathbb{R} for Time Homog. Diffusion over a Bounded State Space
 - ⑥ Failures and Successes for Unbounded State Space
- The operating assumptions will be different in different sections. Within a section, only one set of assumptions holds.

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Part I: Arrow Debreu (AD) Security Prices & Market Beliefs

- A binary option pays one unit of a specified currency eg. \$1, if an event comes true, eg $S_T > K$, and they pay zero otherwise.
- AD securities are defined as binary claims trading implicitly or explicitly on some underlying uncertainty X in a spot market. In a single-period discrete-state setting, $A_{j|i}$ is the dollar price paid at time 0 given $X_0 = i$ for an A-D security paying \$1 at time 1 if $X_1 = j$, and zero otherwise.
- From Breeden and Litzenberger (1978), $A_{j|i}$ is the market spot price given $X_0 = i$ of a single period butterfly spread centered at $X_1 = j$. Knowing the market prices of options of all strikes determines $A_{j|i}$ for one initial state i and for all terminal states j .
- By restricting dynamics sufficiently, eg. nearest neighbor transitions or spatial homogeneity, one can also determine how $A_{j|i}$ varies across initial states i , for each j .
- In this talk, we assume that the problem of determining all of the elements of the matrix A has been solved, even when we later go to continuous time and a continuum state space.

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Arrow Debreu Securities in a Multi-period Setting

- Now consider a multi-period discrete-time discrete-state setting.
- An Arrow-Debreu security pays \$1 if a particular path occurs and zero otherwise.
- For example, with 2 periods/3 dates, $A_{jk|i}$ is the dollar price paid at time 0 given $X_0 = i$ for an A-D security paying \$1 at time 2 if and only if $X_1 = j$ and $X_2 = k$.
- The payoff of this A-D security is path-dependent, whereas the payoff of the one period AD security priced at $A_{j|i}$ is path-independent.

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Replicating Arrow Debreu Securities

- Recall that with 2 periods/3 dates, $A_{j|i}$ is the initial \$-price in state i for a path-independent AD security paying \$1 at time 1 if and only if $X_1 = j$, while $A_{jk|i}$ is the initial \$-price in state i for a path-dependent AD security paying \$1 at time 2 if and only if $X_1 = j$ and $X_2 = k$.
- Let $A_{k|i}$ be the initial \$-price in state i for a path-independent AD security paying \$1 at time 2 if and only if $X_2 = k$.
- Suppose we buy all of the path-dependent securities for a total cost of $\sum_j A_{jk|i}$. Then we replicate the payoff of the later-dated path-independent security.
- No arbitrage implies $A_{k|i} = \sum_j A_{jk|i}$.

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Conditioning Arrow Debreu Security Prices

- Recall that with 2 periods/3 dates, $A_{j|i}$ is the initial \$-price in state i for a path-independent AD security paying \$1 at time 1 if and only if $X_1 = j$, while $A_{jk|i}$ is the initial \$-price in state i for a path-dependent AD security paying \$1 at time 2 if and only if $X_1 = j$ and $X_2 = k$.
- We have $A_{jk|i} \neq A_{j|i}$ because an additional condition is required for the path-dependent AD to pay off and we have $A_{jk|i} \leq A_{j|i}$ if the interest rate over $[1, 2]$ is non-negative given $X_1 = j$.
- Let $A_{k|ij} \equiv A_{jk|i}/A_{j|i}$ denote the proportion of the larger earlier value that ends up in the smaller later value. We refer to $A_{k|ij}$ as the price of the AD security conditioned on both i and j .
- The conditioned AD security prices $A_{j|i}$ and $A_{k|ij}$ are both positive measures, but they are not probability measures unless interest rates vanish.

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Replicating Path-Indep. AD using Path-Dependent AD

- Recall that with 2 periods/3 dates, $A_{j|i}$ is the initial \$-price in state i for a path-independent A-D security paying \$1 at time 1 if and only if $X_1 = j$, while $A_{jk|i}$ is the initial \$-price in state i for a path-dependent A-D security paying \$1 at time 2 if and only if $X_1 = j$ and $X_2 = k$.
- Recall $A_{k|ij} \equiv A_{jk|i}/A_{j|i}$ is the price of the AD security conditioned on both i and j .
- Finally recall $A_{k|i}$ is the initial \$-price in state i for a path-independent A-D security paying \$1 at time 2 if and only if $X_2 = k$ and that no arbitrage implies $A_{k|i} = \sum_j A_{jk|i}$.
- It follows that no arbitrage also implies $A_{k|i} = \sum_j A_{k|ji}A_{j|i}$.

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Markovian and Time-Homogeneous A-D Securities

- Recall that no arbitrage implies $A_{k|i} = \sum_j A_{k|j} A_{j|i}$.
- If the positive transition measure $A_{k|ij}$ does not depend on i , i.e. if $A_{k|ij} = A_{k|j}$ for all j and k , then we say X is a Markov process under \mathbb{A} .
- In this case, no arbitrage implies $A_{k|i} = \sum_j A_{k|j} A_{j|i}$ which is a matrix multiplication.
- In this talk, we will always assume that the A-D security prices extracted from market prices are consistent with X being a Markov process, even when we later go to continuous time and a continuum state space.
- If in addition, shifting all 3 dates by the same positive integer does not affect $A_{k|ij}$, then we can say X is a time-homogeneous Markov process under \mathbb{A} .

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Market Beliefs in a Multi-period Markovian Setting

- Suppose that derivatives trade on a single underlying uncertainty X .
- Suppose we would like to know what “the market” believes about the likelihood that X is in each possible state at each future date.
- It's tempting to try to infer these beliefs from presumed knowledge of AD security prices, but these prices are contaminated by effects from both “time-value-of-money” and from “risk-aversion”.
- What we would like to do is decontaminate these prices and thereby learn both market beliefs and the combined effect of time-value-of-money/risk-aversion.
- Mathematically, we want to find a 1-1 map between A-D security prices, quantified by a positive measure \mathbb{A} , and market beliefs, quantified by a probability measure \mathbb{R} .

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Part II: Ross' Recovery for Finite State Markov Chains

- Think of the probability measure \mathbb{R} as quantifying the market's beliefs about the frequencies of future states, to the extent that these frequencies end up in market prices. Suppose that \mathbb{R} is ex ante unknown by us, but we know market prices of derivative securities.
- From these prices and a sufficiently strong set of assumptions, we can learn the positive measure \mathbb{A} describing Arrow Debreu security prices. Having done so, we know \mathbb{A} ex ante, but not \mathbb{R} .
- In 2011, Professor Steve Ross of MIT began circulating a working paper called "The Recovery Theorem", whose first theorem gives sufficient conditions under which knowing \mathbb{A} implies knowing \mathbb{R} exactly.

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Some Terms Used in Ross Theorem 1 (My Notation)

- pricing matrix $A(x, y)$
- natural probability transition matrix $p(x, y)$
- pricing kernel $\phi(x, y) \equiv \frac{A(x, y)}{p(x, y)}$
- “world with a representative investor”: In an intertemporal model with additively time separable preferences and a constant discount factor δ , the pricing kernel can be written as:

$$\phi(x, y) = \frac{\delta U'(c(y))}{U'(c(x))},$$

where c denotes consumption at time t as a function of the state.

- Intuitively, the pricing kernel captures the combined effect of time-value-of-money and risk-aversion.

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Ross Theorem 1 (My Notation)

- The theorem below is quoted verbatim from Ross, except for notation changes:

In a world with a representative agent, if the pricing matrix A is positive or irreducible, then there exists a unique (positive) solution of the problem of finding P , the discount rate δ , and the pricing kernel ϕ . That is for any given set of state prices, there is one and only one corresponding natural measure and therefore a unique pricing kernel ϕ .

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Remarks on Ross Theorem 1

- The input pricing matrix A must be unique (which is equivalent to complete markets under no arbitrage).
- The proof in Ross' paper assumes that under A , the single state variable X is a finite-state time-homogeneous Markov chain.
- Ross' assumptions imply that \mathbb{R} exists and that X is also a finite-state time-homogeneous Markov chain under \mathbb{R} .
- The assumptions also imply uniqueness for the transition probability matrix P of X , the discount factor δ (which can exceed one), and the pricing kernel $\phi(x, y) = \frac{\delta U'(c(y))}{U'(c(x))}$, which could be increasing in y .

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Are Ross' Assumptions Necessary?

- My co-author and I wondered whether it was necessary that the Markovian state variable X transition between a finite number of states. In industry, we often use diffusions which have a continuous state space. Supposing that X is a univariate time-homogeneous diffusion, could the infinitesimal generator of X under \mathbb{R} , $G_x^{\mathbb{R}} \equiv \frac{a^2(x)}{2} \frac{d^2}{dx^2} + b^{\mathbb{R}}(x) \frac{d}{dx}$, be determined by the infinitesimal generator of X under \mathbb{A} , viz $G_x^{\mathbb{A}} \equiv \frac{a^2(x)}{2} \frac{d^2}{dx^2} + b^{\mathbb{Q}}(x) \frac{d}{dx} - r(x)\mathcal{I}$?
- We also wondered whether it was necessary to consider “a world with a representative agent”. When X is a univariate diffusion rather than a finite state Markov chain, Ross' use of a representative investor forces the state variable X underlying AD securities to drive the price of every asset in the whole economy. While some asset prices may be driven by a single uncertainty, it's a stretch to assume all are. Could we bypass the notion of a representative investor and hence consider some strict subset of the economy? If so, then for a small enough subset of the assets, it would be reasonable that their prices are all driven by a single diffusing state variable X .

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Part III: Changing Numeraire w. a Univ. Diffusion Driver

- From now on, we work in a continuous time setting with a continuous state space.
- The purpose of this part of the talk is to develop the theory for changing the numeraire from the dividend-paying asset widely known as “one dollar” to some other non-dividend paying asset. In contrast, the standard change of numeraire theorem due to El Karoui, G eman and Rochet involves switching between 2 non-dividend paying assets.
- In the next part, we will discuss an important choice for the new numeraire, which we call John Long’s numeraire portfolio.
- The standard change of numeraire theorem and John Long’s numeraire portfolio are both well known to be very general results. We present both results from a very restricted perspective, simplifying their derivation.
- The perspective will in fact be specialized further in a later part to achieve our objective of recovering the probability measure \mathbb{R} from the positive measure \mathbb{A} .

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Assumptions

- Suppose that we have a money market account (MMA) whose growth rate at time t defines the stochastic short rate $r_t, t \geq 0$.
- Suppose we also have a set of $n \geq 1$ risky assets, which in general would be a strict subset of all of the assets in the economy. We impose two restrictions on the subset:
 - ① there is no arbitrage between the $n + 1$ assets.
 - ② The observed Arrow-Debreu security prices are consistent with a univariate diffusion X driving all $n + 1$ prices.
- The first restriction implies there is a probability measure \mathbb{Q} under which the cum-dividend prices of all $n + 1$ assets grow in expectation at rate r .
- The second restriction implies \mathbb{Q} is unique and its effects are explored in more detail on the next slide.

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Risk-Neutral Infinitesimal Generator

- On the last slide, we assumed that the observed Arrow-Debreu security prices are consistent with a univariate diffusion X driving the money market account and the $n \geq 1$ risky asset prices.
- This implies *inter alia* that there exists a spot rate function $r(x, t) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ such that $r_t = r(X_t, t)$, $t \geq 0$.
- This also implies that for each risky asset $i = 1, 2, \dots, n$, there exists a spot value function $S_i(x, t) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$ such that $S_{it} = S_i(X_t, t)$, $t \geq 0$.
- Under the unique risk-neutral measure \mathbb{Q} , each spot value function solves:

$$G_x^{\mathbb{Q}} S_i(x, t) = r(x, t) S_i(x, t), \quad i = 1, \dots, n,$$

where $G_x^{\mathbb{Q}}$ denotes the infinitesimal generator of X under \mathbb{Q} , viz

$$G_x^{\mathbb{Q}} \equiv \frac{\partial}{\partial t} + \frac{a^2(x, t)}{2} \frac{\partial^2}{\partial x^2} + b^{\mathbb{Q}}(x, t) \frac{\partial}{\partial x}.$$

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From Dollars to a New Numeraire

- Let $N_t > 0$ denote the \$ value of some new numeraire at time t .
- Let's call the new numeraire a "newm".
- Since A is the \$ value of an AD security, the newm value of an AD security is just the stochastic process: $\frac{A_{tT}}{N_t}$, $t \in [0, T]$.
- In our univariate diffusion setting, there exists a spot value function $n(x, t) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}^{++}$ such that $N_t = n(X_t, t)$, $t \geq 0$.
- Since N is the spot value of a self-financing portfolio of the $n + 1$ assets, the newm's spot value function $n(x, t)$ solves:

$$G_{xt}^{\mathbb{Q}} n(x, t) = r(x, t)n(x, t), \quad x \in \mathbb{R}, t \in [0, T].$$

- We say that the numeraire's value function n is space-time harmonic.

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The Numeraire Transform

- For each AD security with “strike” y & expiry T , consider holding a static position in $n(y, T)$ AD securities. Each static position pays off either zero or one newm and for each $t \in [0, T]$ has a spot value given by the fraction $F_{tT} \equiv \frac{A_{tT}n(y, T)}{N_t}$ newms.
- In our Markovian setting, we may also define the corresponding positive density function:

$$f(x, t; y, T) \equiv \frac{a^d(x, t; y, T)n(y, T)}{n(x, t)}, \quad x \in \mathbb{R}, t \in [0, T].$$

- When $r(t, x) \neq 0$, $\int a^d(x, t; y, T)dy \neq 1$, so a^d is *not* a probability density function (PDF) (think of the superscript d as denoting defective probability).
- When $r(t, x) = 0$, then a^d is a PDF. When in addition, $a^d(x, t; y, T) = a^d(x, T - t, y)$, and $n(x, t) = h(x)$, probabilists refer to our middle equation as Doob’s h transform (h for harmonic). In our more general setting, we analogously call f the n transform of a^d (n for numeraire).

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Convexity/Covariation

- Recall that the fraction f is the n transform of a :

$$f(x, t; y, T) \equiv a^d(x, t; y, T)n(y, T)/n(x, t), \quad x \in \mathbb{R}, t \in [0, T].$$

- Equivalently, defining $F_{tT} \equiv f(X_t, t; y, T)$, $A_{tT} \equiv a(X_t, t; y, T)$, and $N_t \equiv n(X_t, t)$ as 3 positive continuous processes for $t \in [0, T]$:

$$F_{tT} \equiv A_{tT}n(y, T)/N_t, \quad t \in [0, T].$$

- Computing the percentage change on each side:

$$\frac{dF_{tT}}{F_{tT}} = \frac{dA_{tT}}{A_{tT}} - \frac{dN_t}{N_t} - \frac{dN_t}{N_t} \frac{dA_{tT}}{A_{tT}} + \left(\frac{dN_t}{N_t} \right)^2, \quad t \in [0, T],$$

since $n(y, T)$ is invariant to t .

- Factoring out $\frac{dN_t}{N_t}$:

$$\frac{dF_{tT}}{F_{tT}} - \frac{dA_{tT}}{A_{tT}} + \frac{dN_t}{N_t} = -\frac{dN_t}{N_t} \left[\frac{dA_{tT}}{A_{tT}} - \frac{dN_t}{N_t} \right] = -\frac{dN_t}{N_t} \frac{dF_{tT}}{F_{tT}}, \quad t \in [0, T].$$

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N Transform of AD Prices

- We defined the F process as the N transform of the A process, i.e.:

$$F_{tT} \equiv A_{tT} n(y, T) / N_t, \quad t \in [0, T),$$

and from the bottom of the last slide:

$$\frac{dF_{tT}}{F_{tT}} + \frac{dN_t}{N_t} \frac{dF_{tT}}{F_{tT}} = \frac{dA_{tT}}{A_{tT}} - \frac{dN_t}{N_t}, \quad t \in [0, T).$$

- If N has sample paths of bounded variation, then $\frac{dN_t}{N_t} \frac{dF_{tT}}{F_{tT}} = 0$, so the percentage change in the fraction F is just the percentage change of the numerator process $A_{tT} n(y, T)$ less the the percentage change in the denominator process N_t .
- If N has sample paths of unbounded variation, then the percentage change in F can deviate from this difference.
- Multiplying both sides by F :

$$dF_{tT} + d \ln N_t dF_{tT} = \frac{n(y, T)}{N_t} dA_{tT} - \frac{A_{tT} n(y, T)}{N_t^2} dN_t, \quad t \in [0, T).$$

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The Newm Numeraire Generates a New Generator

- Recall: $dF_{tT} + d \ln N_t dF_{tT} = \frac{n(y, T)}{N_t} dA_{tT} - \frac{A_{tT} n(y, T)}{N_t^2} dN_t$, $t \in [0, T)$.
- Taking \mathbb{Q} expectations and switching to the infinitesimal generator view:

$$\mathcal{G}_{xt}^{\mathbb{Q}} f(x, t; y, T) + a^2(x, t) \sigma_n(x, t) \frac{\partial f(x, t; y, T)}{\partial x} =$$

$$\frac{n(y, T)}{n(x, t)} \mathcal{G}_{xt}^{\mathbb{Q}} a^d(x, t; y, T) - \frac{a^d(x, t; y, T) n(y, T)}{n^2(x, t)} \mathcal{G}_{xt}^{\mathbb{Q}} n(x, t), \quad x \in \mathbb{R}, t \in [0, T),$$

where $\sigma_n(x, t) \equiv \frac{\partial \ln n(x, t)}{\partial x}$ and recall $\mathcal{G}_{xt}^{\mathbb{Q}} \equiv \frac{\partial}{\partial t} + \frac{a^2(x, t)}{2} \frac{\partial^2}{\partial x^2} + b^{\mathbb{Q}}(x, t) \frac{\partial}{\partial x}$.

- So far, we have used the positivity of A and N , but not their harmonicity, viz:

$$\mathcal{G}_{xt}^{\mathbb{Q}} n(x, t) = r(x, t) n(x, t), \quad \mathcal{G}_{xt}^{\mathbb{Q}} a^d(x, t; y, T) = r(x, t) a^d(x, t; y, T),$$

for $x \in \mathbb{R}, t \in [0, T]$. Using this harmonicity implies that:

$$\mathcal{G}_{xt}^{\mathbb{Q}} f(x, t; y, T) + a^2(x, t) \sigma_n(x, t) \frac{\partial f(x, t; y, T)}{\partial x} = 0,$$

so f is in the null space of a new generator $\mathcal{G}_{xt}^{\mathbb{F}} \equiv \mathcal{G}_{xt}^{\mathbb{Q}} + a^2(x, t) \sigma_n(x, t) \frac{\partial}{\partial x}$.

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Probability from Prices

- Recall that f is in the null space of the new generator $\mathcal{G}_{xt}^{\mathbb{F}}$:

$$\mathcal{G}_{xt}^{\mathbb{F}} f(x, t; y, T) \equiv \left[\mathcal{G}_{xt}^{\mathbb{Q}} + a^2(x, t) \sigma_n(x, t) \frac{\partial}{\partial x} \right] f(x, t; y, T) = 0.$$

- Also, the solution to the PDE $\mathcal{G}_{xt}^{\mathbb{Q}} n(x, t) = r(x, t) n(x, t)$ is just:
 $n(x, t) = \int_{-\infty}^{\infty} a^d(x, t; y, T) n(y, T) dy$, since a^d is the fundamental solution.
- Recalling that the fraction $f(x, t; y, T) = a^d(x, t; y, T) n(y, T) / n(x, t)$, it follows that f is a transition probability density function:

$$\int_{-\infty}^{\infty} f(x, t; y, T) dy = \int_{-\infty}^{\infty} a^d(x, t; y, T) \frac{n(y, T)}{n(x, t)} dy = 1, \text{ for all } x, y \in \mathbb{R}, T \geq t.$$

- If we use the frequency function f to define a probability measure \mathbb{F} , then under \mathbb{F} , the process X is a diffusion with drift coeff.
 $b^{\mathbb{F}}(x, t) = b^{\mathbb{Q}}(x, t) + a^2(x, t) \frac{\partial \ln n(x, t)}{\partial x}$ and the same diffusion coeff. $a(x, t)$ as under the risk-neutral probability measure \mathbb{Q} .
- Furthermore it follows from the top equation that the process $F_t \equiv f(X_t, t; y, T)$, $t \in [0, T]$, is a local \mathbb{F} martingale.

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Could \mathbb{R} be a Martingale Measure?

- To summarize, we n -transformed the positive transition density $a^d(x, t; y, T)$ into a transition probability density function $f(x, t; y, T)$.
- One can show that if the new numeraire is the money market account, then the resulting transition probability density function is just the risk-neutral one $q(x, t; y, T)$ associated to \mathbb{Q} .
- Other numeraires create other transition probabilities.
- We usually think of these equivalent martingale measures as fictitious, i.e. different from the probability measure \mathbb{R} capturing market beliefs.
- Might it be the case that there is some numeraire which numeraire transforms $a^d(x, t; y, T)$ into the transition probability density function $p(x, t; y, T)$ associated to \mathbb{R} ?
- The next part shows that under no arbitrage, the answer is yes.

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Part IV: John Long's Numeraire Portfolio

- In 1990, Long introduced a notion which he called the *numeraire portfolio*.
- Long showed that if any set of assets includes the MMA and is arbitrage-free, then there always exists a self-financing portfolio of them whose value is always positive, i.e. a numeraire.
- Furthermore, if the spot price S_i of each asset is expressed relative to the value $L > 0$ of Long's numeraire portfolio, then the relative price S_i/L is a \mathbb{R} martingale.

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Existence of the Numeraire Portfolio

- Let S_{0t} be the spot price of the MMA and suppose that we have n risky assets with spot prices S_i for $i = 1, \dots, n$.
- Assuming no arbitrage between these $n + 1$ assets, Long (1990) proved that there exists a portfolio with value $L > 0$ such that for all times u and t with $u \geq t \geq 0$:

$$E^{\mathbb{R}} \frac{S_{iu}}{L_u} \Big|_{\mathcal{F}_t} = \frac{S_{it}}{L_t}, \quad i = 0, 1, \dots, n.$$

- In words, assuming no arbitrage between a set of assets, implies that one can always construct a portfolio of them with value $L > 0$ such that each asset's relative price S_i/L is a \mathbb{R} martingale. Hence, when P&L is measured in units of the numeraire portfolio, all assets have the same mean $P\&L$.

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Numeraire Transforming the \$ Value of an AD Security

- Let P_{tT} denote the probability measure capturing the market's beliefs regarding X_T at time t , A_{tT} denote the \$ price of an AD security at time t maturing at T , and let L_t denote the \$ value of Long's NP at time t .
- From Long:

$$E^{\mathbb{R}} \frac{S_T}{L_T} \Big|_{\mathcal{F}_t} = \frac{S_{it}}{L_t}, \quad i = 0, 1, \dots, n.$$

- When a market is complete, we can replicate every AD security and hence:

$$E^{\mathbb{R}} \frac{A_T}{L_T} \Big|_{\mathcal{F}_t} = \frac{A_t}{L_t}, \quad i = 0, 1, \dots, n.$$

- Equivalently, in our univariate diffusion setting:

$$p(x, t; y, T) = \frac{a^d(x, t; y, T)L(y, T)}{L(x, t)}, \quad t \in [0, T].$$

- In words, the transition probability density function $p(x, t; y, T)$ capturing market beliefs \mathbb{R} is just the L numeraire transform of $a^d(x, t; y, T)$.

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Drift Effect of Numeraire Change

- Recall that the PDF $f(x, t; y, T)$ obtained by $n(x, t)$ transforming $a^d(x, t; y, T)$ solves:

$$\left\{ \frac{\partial}{\partial t} + \frac{a^2(x, t)}{2} \frac{\partial^2}{\partial x^2} + [b^{\mathbb{Q}}(x, t) + a^2(x, t)\sigma_n(x, t)] \frac{\partial}{\partial x} \right\} f(x, t; y, T) = 0.$$

- When $n(x, t) = L(x, t)$, then $\sigma_n(x, t) = \sigma_L(x, t)$ and $f(x, t; y, T) = p(x, t; y, T)$.
- It follows that:

$$\left\{ \frac{\partial}{\partial t} + \frac{a^2(x, t)}{2} \frac{\partial^2}{\partial x^2} + [b^{\mathbb{Q}}(x, t) + a^2(x, t)\sigma_L(x, t)] \frac{\partial}{\partial x} \right\} p(x, t; y, T) = 0.$$

- Hence, changing measure from \mathbb{Q} to \mathbb{R} raises X 's drift by $a^2(X_t, t)\sigma_L(X_t, t)$.
- If the short rate $r_t = r(X_t, t)$ for some known function $r(x, t)$, then one thinks of $a^2(X_t, t)$ as directly observed through the volatility of the short rate. In the next part, we will also restrict the form and dynamics of L_t , so as to achieve both uniqueness and identification of σ_{L_t} .

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Part V: Recovering \mathbb{R} for Time Homog. Diffusion over a Bounded State Space

- We add a few more assumptions in order to identify the lognormal volatility function of Long's NP.
- In particular, we assume that X and L are both time-homogeneous under \mathbb{A} . As a result, $a(x, t) = a(x) > 0$, $b^{\mathbb{Q}}(x, t) = b^{\mathbb{Q}}(x)$, $r(x, t) = r(x)$, and $\sigma_L(x, t) = \sigma_L(x)$.
- In this section, we also assume that the state space (ℓ, h) of X is bounded. In the next section, we explore some examples with unbounded state space.
- We show that these additional assumptions determine the \mathbb{R} dynamics of X and all of the spot prices of the assets in the given set.

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Our Assumptions In Equations

- We assume no arbitrage for some finite set of assets which includes a money market account (MMA).
- As a result, there exists a risk-neutral measure \mathbb{Q} under which spot prices deflated by the MMA balance evolve as local martingales.
- We assume that under \mathbb{Q} , the driver X is a time-homogeneous diffusion:

$$dX_t = b^{\mathbb{Q}}(X_t)dt + a(X_t)dW_t, \quad t \in [0, T],$$

with bounded state space (ℓ, u) , $t \geq 0$, $a(x) > 0$, and where W is \mathbb{Q} -SBM.

- We also assume that under \mathbb{Q} , the value L_t of the numeraire portfolio just depends on X_t and t and solves:

$$\frac{dL_t}{L_t} = r(X_t)dt + \sigma_L(X_t)dW_t, \quad t \in [0, T].$$

- We know the functions $b^{\mathbb{Q}}(x)$, $a(x)$, and $r(x)$, but not $\sigma_L(x)$. How can we find it?

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Value Function of the Numeraire Portfolio

- Recalling that X is our driver, we have assumed that:

$$L_t \equiv L(X_t, t), \quad t \in [0, T],$$

where $L(x, t)$ is a positive function of $x \in \mathbb{R}$ and time $t \in [0, T]$.

- Applying Itô's formula, the volatility of L is:

$$\sigma_L(x) \equiv \frac{1}{L(x, t)} \frac{\partial}{\partial x} L(x, t) a(x) = a(x) \frac{\partial}{\partial x} \ln L(x, t).$$

- Dividing by $a(x) > 0$ and integrating w.r.t. x :

$$\ln L(x, t) = \int^x \frac{\sigma_L(y)}{a(y)} dy + f(t), \text{ where } f(t) \text{ is the constant of integration.}$$

- Exponentiating implies that the value of the numeraire portfolio separates multiplicatively into a positive function $\pi(x)$ of the level x of the driver X and a positive function $p(t)$ of time t :

$$L(x, t) = \pi(x)p(t),$$

$$\text{where } \pi(x) = e^{\int^x \frac{\sigma_L(y)}{a(y)} dy} \text{ and } p(t) = e^{f(t)}.$$

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Separation of Variables

- The numeraire portfolio value function $L(x, t)$ must solve the following linear parabolic PDE to be self-financing:

$$\frac{\partial}{\partial t} L(x, t) + \frac{a^2(x)}{2} \frac{\partial^2}{\partial x^2} L(x, t) + b^{\mathbb{Q}}(x) \frac{\partial}{\partial x} L(x, t) = r(x)L(x, t), \quad x \in (\ell, u).$$

- On the other hand, the last slide shows that this value separates as:

$$L(x, t) = \pi(x)p(t), \quad x \in (\ell, u), t \in [0, T].$$

- Using Bernoulli's classical separation of variables argument, we know that:

$$p(t) = p(0)e^{\lambda t}, \quad t \in [0, T],$$

for each separating constant λ and that:

$$\frac{a^2(x)}{2} \pi''(x) + b^{\mathbb{Q}}(x) \pi'(x) - r(x) \pi(x) = -\lambda \pi(x), \quad x \in (\ell, u).$$

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Regular Sturm Liouville Problem

- Recall the problem at the bottom of the last slide:

$$\frac{a^2(x)}{2}\pi''(x) + b^{\mathbb{Q}}(x)\pi'(x) - r(x)\pi(x) = -\lambda\pi(x), \quad x \in (\ell, u),$$

where $\pi(x)$ and λ are unknown.

- Whichever boundary conditions we are allowed to impose, they will be separated. As a result, we have a *regular* Sturm Liouville problem.
- From Sturm Liouville theory, we know that there exists an eigenvalue $\lambda_0 > -\infty$, smaller than all of the other eigenvalues, and an associated positive eigenfunction, $\pi_0(x)$, which is unique up to positive scaling.
- All of the eigenfunctions associated to the other eigenvalues switch signs at least once.
- One can numerically solve for both the smallest eigenvalue λ_0 and its associated positive eigenfunction, $\pi_0(x)$. The positive eigenfunction $\pi_0(x)$ is unique up to positive scaling.

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Value Function of the Numeraire Portfolio

- Recall that λ_0 is the known lowest eigenvalue and $\pi_0(x)$ is the associated eigenfunction, positive and known up to a positive scale factor.
- Knowing $\pi_0(x)$ up to positive scaling and knowing λ_0 implies that we also know the value function of the numeraire portfolio up to positive scaling, since:

$$L(x, t) = \pi_0(x)e^{\lambda_0 t}, \quad x \in [\ell, u], t \in [0, T].$$

- As a result, the volatility of the numeraire portfolio is *uniquely* determined:

$$\sigma_L(x) = a(x) \frac{\partial}{\partial x} \ln \pi_0(x), \quad x \in [\ell, u].$$

- Mission accomplished! Let's see what the market believes.

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\mathbb{R} Dynamics of the Driver X

- Recall that the \mathbb{Q} dynamics of X were assumed to be:

$$dX_t = b^{\mathbb{Q}}(X_t)dt + a(X_t)dW_t, \quad t \geq 0,$$

where recall W is a standard Brownian motion under \mathbb{Q} .

- By our previous analysis, the dynamics of the driver X under the probability measure \mathbb{R} are:

$$dX_t = [b^{\mathbb{Q}}(X_t) + \sigma_L(X_t)a(X_t)]dt + a(X_t)dB_t, \quad t \geq 0.$$

- Hence, we now know the \mathbb{R} dynamics of the driver X .
- We still have to determine the \mathbb{R} transition density of the driver X .

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\mathbb{R} Transition PDF of the Driver X and r

- From the change of numeraire theorem, the Radon Nikodym derivative $\frac{d\mathbb{R}}{d\mathbb{A}}$ is:

$$\frac{d\mathbb{R}}{d\mathbb{A}} = \frac{L(X_T, T)}{L(X_0, 0)} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{\lambda_0 T},$$

since $L(x, t) = \pi_0(x) e^{\lambda_0 t}$.

- Solving for the PDF $d\mathbb{R}$ gives:

$$d\mathbb{R} = \frac{\pi_0(X_T)}{\pi_0(X_0)} e^{\lambda_0 T} d\mathbb{A}.$$

- As we know the positive function $\frac{\pi_0(y)}{\pi_0(x)}$, the positive function $e^{\lambda_0 T}$, and the Arrow Debreu state pricing density $d\mathbb{A}$, we know $d\mathbb{R}$, the transition PDF under \mathbb{R} of X .
- Since the short rate $r_t = r(X_t)$ for known function $r(X)$, we also know the transition PDF under \mathbb{R} of r .

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- Also from Girsanov's theorem, the dynamics of the i -th spot price S_{it} under \mathbb{R} are uniquely determined as:

$$dS_{it} = [r(X_t)S_i(X_t, t) + \sigma_L(X_t) \frac{\partial}{\partial x} S_i(X_t, t) a^2(X_t)] dt + \frac{\partial}{\partial x} S_i(X_t, t) a(X_t) dB_t,$$

where for $x \in (\ell, u)$, $t \in [0, T]$, $S_i(x, t)$ solves the following linear PDE:

$$\frac{\partial}{\partial t} S_i(x, t) + \frac{a^2(x)}{2} \frac{\partial^2}{\partial x^2} S_i(x, t) + b^{\mathbb{Q}}(x) \frac{\partial}{\partial x} S_i(x, t) = r(x) S_i(x, t),$$

subject to appropriate boundary and terminal conditions.

- The instantaneous (arithmetic) risk premium of S_i , $\sigma_L(X_t) \frac{\partial}{\partial x} S_i(X_t, t) a^2(X_t) dt$, is just $d\langle S_i, \ln L \rangle_t$, i.e. the increment of the quadratic covariation of S_i and $\ln L$.

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Part VI: Failures & Successes for Unbounded State Space

- Most diffusions used in derivatives pricing have an unbounded state space.
- For example, the standard model for a stock price is geometric Brownian motion, whose state space is the unbounded interval $(0, \infty)$.
- The analysis presented thus far has required that the state variable X diffuses over a bounded state space.
- To let X diffuse over an unbounded interval, we have been using a Hilbert space approach, which requires that functions in the domain of the infinitesimal generator be square-integrable.
- So far, these theoretical results don't apply if X follows geometric Brownian motion.
- Hence, when we focus on the Black Scholes model, our technical assumptions prevent us from learning the \mathbb{R} dynamics of a stock price from stock option prices and our assumptions.

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Short Rate Models

- A classical approach for valuing interest rate derivatives assumes that the short interest rate r is a so-called mean-reverting square-root process under the risk-neutral measure \mathbb{Q} .
- In this so-called Cox, Ingersoll, and Ross (CIR) model, the state space is again an unbounded interval, now $[0, \infty)$.
- When we focus on CIR model, we find that one can learn the \mathbb{R} generator of the time homogeneous diffusion r from the risk-neutral generator $\mathcal{G}_r^{\mathbb{A}} = \frac{\sigma^2 r}{2} \frac{d^2}{dr^2} + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - r) \frac{d}{dr} - r\mathcal{I}$. The \mathbb{R} generator is also in the CIR class, but with a higher speed of mean reversion $\kappa^{\mathbb{R}} \geq \kappa^{\mathbb{Q}}$.
- More generally, Vadim Linetsky and a student have determined that the infinitesimal generator under \mathbb{R} can be recovered from its \mathbb{A} counterpart whenever the state variable X is a continuous multi-variate diffusion with affine coefficients.
- Knowing in general when you can recover on unbounded state space is at present an open problem.

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Summary

- We highlighted Ross' Theorem 1 and proposed an alternative preference-free way to derive the same financial conclusion.
- Our approach is based on imposing time homogeneity on the \mathbb{Q} dynamics of the value L of Long's numeraire portfolio, when it is driven by a time-homogeneous diffusion process X with bounded state space.
- Under these assumptions, separation of variables allows us to separate beliefs from preferences. We learn both the market's beliefs and the risk premium.
- Lately, we have been exploring diffusions with unbounded state space. Sometimes our technical restrictions prevent recovery (eg. Black Scholes) and in other examples (eg. CIR model for the short interest rate), we were able to recover the \mathbb{R} dynamics of the short rate.
- At present, we do not have a general theory giving sufficient conditions for when Ross recovery succeeds for a diffusion over an unbounded state space. Since these diffusions are widely used, this is a good open problem for future research.

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