4 January 13, 2022

Compact Metric Spaces

Last time, we showed that a set in $\mathbb{R}^n$ is sequentially compact if and only if it is topologically compact, by showing

$$\text{sequentially compact} \iff \text{closed and bounded} \iff \text{Heine-Borel} \iff \text{topologically compact}.$$ 

However, by the previous remark, we don’t have Heine-Borel for arbitrary metric spaces. Which begs the question: is sequentially compact the same as topologically compact in metric spaces? The answer is yes. To prove this, we first show a handful of preliminary results.

**Lemma 1 (Lebesgue Number Lemma)**

Let $(X, d)$ be a sequentially compact metric space and $\{U_i\}_{i \in I}$ be an open cover of $X$. Then, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subseteq U_i$ for some $i \in I$.

**Proof:** Before proving this, try to visualize the result!

We prove this lemma through contradiction. Assume that for some $r > 0$ there exists an $x \in X$ (possibly depending on $r$) such that for each $i \in I$, $B_r(x) \nsubseteq U_i$. Consider the sequence $\{x_n\}_n$ in $X$ such that $B_{1/n}(x_n) \nsubseteq U_i$ for all $i \in I$.

Given that $X$ is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to x \in X$. Given that $\{U_i\}$ is an open cover of $X$, there exists a $U_0$ such that $x \in U_0$. Given $U_0$ is open, it also follows that there exists an $r_0$ such that $B_{r_0}(x) \subseteq U_0$. Hence, choose $N$ large enough such that $d(x, x_N) < \frac{r_0}{2}$ and $\frac{1}{N} < \frac{r_0}{2}$. Then, if $y \in B_{1/N}(x_N)$, then

$$d(x, y) \leq d(x, x_N) + d(x_N, y) < r_0.$$ 

Therefore, $y \in B_{r_0}(x) \subseteq U_0$. Hence,

$$B_{1/N}(x_N) \subseteq B_{r_0}(x) \subseteq U_0$$

which is a contradiction. 

We call this $r$ the **Lebesgue number** of the open cover of $X$, which is useful in other applications.

**Definition 2**

A metric space $X$ is **totally bounded** if, for every $\epsilon > 0$, there exists $x_1, x_2, \ldots, x_k \in X$ with $k$ finite such that $\{B_\epsilon(x_i) \mid 1 \leq i \leq k\}$ is an open cover of $X$. 

Lemma 3
A metric space $X$ is sequentially compact implies that $X$ is totally bounded.

**Proof:** Assume that $X$ is sequentially compact and not totally bounded. Therefore, there exists an $\epsilon > 0$ such that $X$ cannot be covered by a collection of open sets of only finitely many $\epsilon$-balls. Hence, let $x_1 \in X$, $x_2 \in X \setminus B_\epsilon(x_1)$, then $x_3 \in X \setminus B_\epsilon(x_1) \setminus B_\epsilon(x_2)$ and so on. We know that there exists such $x_i$ by the previous statement. Hence, for all $i \neq j$, $d(x_i, x_j) \geq \epsilon$. Therefore, $\{x_n\}_n$ has no convergent subsequence as if there was a convergent subsequence it would be Cauchy, and the previous line shows that no subsequence of $\{x_n\}$ will be Cauchy. This is a contradiction to $X$ being sequentially compact.

Theorem 4
A metric space $X$ is (topologically) compact if and only if $X$ is sequentially compact.

**Proof:** We first show that topologically compact implies sequentially compact. Assume for the sake of contradiction there there exists a sequence $\{x_n\}_n$ in $X$ with no convergent subsequence. Notice that no term in the sequence can appear infinitely many times, as otherwise there would be a trivial subsequence of $\{x_n\}$. Hence, we assume without loss of generality that $x_i \neq x_j$ if $i \neq j$. Furthermore, notice then that for every $n$ there exists an $\epsilon_n > 0$ such that $B_{\epsilon_n}(x_n)$ contains no other terms in the sequence. If this wasn’t the case, then there would again be a convergent subsequence of $\{x_n\}_n$. Therefore, for each $i$, there exists an open ball $U_i$ centered at $x_i$ such that $x_j \notin U_i$ for all $i \neq j$.

Additionally, consider $U_0 = X \setminus \{x_n \mid n \in \mathbb{N}\}$. $U_0$ is open, as $U_0^c = \{x_n \mid n \in \mathbb{N}\}$ is closed (it contains all of it’s limit points). Hence,

$$U_0 \cup \{U_n \mid n \in \mathbb{N}\}$$

is an open cover of $X$. However, this open cover has no finite subcover as any finite collection of the cover will fail to include infinitely many terms from the sequence $\{x_n\}_n$. This is a contradiction, and thus topologically compact implies sequentially compact.

We now prove the other direction. Let $X$ be sequentially compact and let $\{U_i\}_{i \in I}$ be an open cover of $X$. By the Lebesgue number lemma, there exists an $r > 0$ such that for each $x \in X$, $B_r(x) \subset U_i$ for some $i \in I$. Furthermore, by Lemma 5, $X$ is totally bounded. Hence, there exists $y_1, \ldots, y_k \in X$ such that

$$X \subset B_r(y_1) \cup \cdots \cup B_r(y_k).$$

However, for each $i \in I$, we have $B_r(y_i) \subset U_{j(i)}$ for some $j(i) \in I$. (This notation just means for each $i$, there exists a $j \in I$ which depends on $i$ such that $B_r(y_i) \subset U_j$). Thus, $\{U_{j(1)}, \ldots, U_{j(k)}\}$ is a finite subcover for $X$. Therefore, every open cover of $X$ has a finite subcover, and thus sequentially compact implies topologically compact. \qed

Remark 5. Notice that we technically could’ve used this proof in the previous lecture, but the Heine-Borel Theorem is so vastly important that I decided to do that proof before today’s lecture.

We will now start to look at some illuminating applications of compact sets to reach an even more powerful theorem.

Recall 6
Let $X, Y$ be metric spaces and $f : X \to Y$ be a continuous function. Then, for all $U$ open in $Y$, $f^{-1}(U)$ is open in $X$. 

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**Theorem 7**

Let $X, Y$ be metric spaces and $f : X \to Y$ be continuous. Given $K \subseteq X$, $f(K) \subseteq Y$ is compact.

**Proof:** Let $\{U_i\}_{i \in I}$ be an open cover of $f(K)$. Then, define $V_i = \{f^{-1}(U_i)\}_{i \in I}$, which is open as $f$ is continuous. Therefore, $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of $K$. Hence, there exists a finite subcover $\{V_{i_1}, \ldots, V_{i_n}\}$ of $K$ as $K$ is compact. Thus, $\{U_{i_1}, \ldots, U_{i_n}\} = \{f(V_{i_1}), \ldots, f(V_{i_n})\}$ is a finite subcover of $f(K)$. Therefore, $f(K)$ is compact. \qed

**Corollary 8**

Let $X$ be a metric space and $K \subseteq X$. Then, given a continuous function $f : X \to \mathbb{R}$, $f$ obtains a maximum and minimum finite value on $K$.

**Proof:** The proof follows from the previous theorem, and Problem 5.(a) on PSET 2. \qed

**Corollary 9**

Sometimes in particular we want to study bounded continuous functions, and the previous corollary gives us a nice property. Given a compact metric space $X$, every continuous function on $X$ is bounded.

**Proof:** Follows immediately. \qed

**Theorem 10** (Cantor’s Intersection Theorem)

If $K_1 \supset K_2 \supset K_3 \supset \ldots$ is a decreasing sequence of nonempty sequentially compact subsets of $\mathbb{R}^n$, then $\cap_{i \geq 1} K_i$ is non-empty.

**Proof:** Choose a sequence $\{a_n\}_n$ such that $a_n \in K_n$ for each $n$. We know that there exists such an $a_n$ as each $K_n$ is nonempty. Then, $\{a_n\}_n$ is a sequence in $K_1$, and thus there exists a convergent subsequence $\{a_{n_k}\}_k$ such that $a_{n_k} \to a \in K_1$. Furthermore, $\{a_n\}_{n \geq 1}$ is a sequence in $K_2$, and thus contains a convergent subsequence. Therefore, $a \in K_2$. Continuing this process, we get that $a \in K_i$ for all $i$. Thus, $a \in \cap_{i \geq 1} K_i$. \qed

**Definition 11** (Finite Intersection Property)

A collection of closed sets $\{C_i\}_i$ has the **finite intersection property** if every finite subcollection has a nonempty intersection.

Given Lemma 5 and the Cantor Intersection Theorem, it is clear that there are some relations between compact sets, nonempty intersections of sets, and totally bounded sets. We hence show the following theorem.

**Theorem 12**

Given a metric space $(X, d)$, the following are equivalent.

1. $X$ is compact.
2. $X$ is sequentially compact.
3. $X$ is Cauchy complete and totally bounded.
4. Every collection of closed subsets of $X$ with the finite intersection property has a non-empty intersection.
We have shown (1) $\iff$ (2), and thus we show (1) $\iff$ (4) and (2) $\iff$ (3) to finish the proof.

**Proof:** (1) $\implies$ (4): Assume for the sake of contradiction that there exists a collection of closed subsets $\{C_i\}_{i \in I}$ with the finite intersection property such that $\bigcap_{i \in I} C_i = \emptyset$. Given $C_i$ is closed in $X$ for all $i$, $U_i = C_i^c$ is open in $X$ for each $i$. Then,

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} C_i^c = \left(\bigcap_{i \in I} C_i\right)^c = \emptyset^c = X.$$

Hence, the $U_i$ cover $X$. Given $X$ is compact, there exists a finite subcover $\{U_n, \ldots, U_k\}$ of $X$. Thus,

$$X = \bigcup_{n=1}^k U_n = \left(\bigcap_{n=1}^k U_n^c\right)^c = \left(\bigcap_{n=1}^k C_n\right)^c.$$

Therefore, $\bigcap_{n=1}^k C_n = \emptyset$ which is a contradiction with the finite intersection property.

(4) $\implies$ (1): Suppose that $\{U_i\}_{i \in I}$ is an open cover of $X$, and let $C_i = U_i^c$ for each $i \in I$. Assume for the sake of contradiction that no finite subset of the $U_i$ covers $X$. We show that $C_i$ has the finite intersection property. Assume for the sake of contradiction that $\{C_n, \ldots, C_m\}$ satisfies $C_n \cap \cdots \cap C_m = \emptyset$. Then,

$$\bigcup_{i=1}^k U_n = \left(\bigcap_{n=1}^k U_n^c\right)^c = \left(\bigcap_{n=1}^k C_n\right)^c = \emptyset^c = X.$$

This is a contradiction with the assumption that no subset of the $U_i$ covers $X$. Hence, $\{C_i\}_{i \in I}$ satisfies the finite intersection property. Therefore, $\{C_i\}_{i \in I}$ has non-empty intersection; i.e. $\bigcap_{i \in I} C_i \neq \emptyset$. Then, $\bigcup_{i \in I} U_i \neq X$, which is a contradiction to the $U_i$ being an open cover for $X$. Thus, every open cover of $X$ has a finite open subcover.

(2) $\implies$ (3): We have already shown that $X$ being sequentially compact implies totally bounded, and hence we only need show that sequentially compact implies Cauchy complete. Let $\{x_n\}$ be a Cauchy sequence in $X$. Given $\{x_n\}$ is a sequence in $X$, there exists a convergent subsequence $\{x_{n_k}\}$ in $X$ such that $x_{n_k} \to x \in X$. Let $\epsilon > 0$, and choose $N$ such that $d(x_i, x_j) < \epsilon/2$ whenever $i, j \geq N$. Next, choose $n_k > N$ such that $d(x_{n_k}, x) < \epsilon/2$. Then,

$$d(x, x_N) \leq d(x, x_{n_k}) + d(x_{n_k}, x_N) < \epsilon.$$

Thus, $x_n \to x \in X$ as $n \to \infty$. Therefore, every Cauchy sequence in $X$ converges to a point in $X$. Hence, $X$ is Cauchy complete.

(3) $\implies$ (2): This part of the proof is quite difficult. Consider a sequence $\{x_n\}_n$ in $X$. Given $X$ is totally bounded, for every $n \in \mathbb{N}$, there exists a finite set of points $\{y_1^{(n)}, \ldots, y_r^{(n)}\}$ such that $X \subset B_{\frac{1}{2}}(y_1^{(n)}) \cup \cdots \cup B_{\frac{1}{2}}(y_r^{(n)})$. Define

$$S_n = \{y_1^{(n)}, \ldots, y_r^{(n)}\}.$$

We want to find a convergent subsequence of $\{x_n\}$. We do so by construction. Given $S_1$ is finite, there exists a $y_1^{(1)} \in S_1$ such that $B_1(y_1^{(1)})$ contains infinitely many points from $\{x_n\}$. Choose $z_1$ from this ball. Then, given $S_2$ is finite, there is a $y_2^{(1)}$ such that infinitely many points from $\{x_n\}$ are in $B_1(y_1^{(1)}) \cap B_{1/2}(y_2^{(1)})$. Choose $z_2$ from this set. Continue this procedure for each $k > 1$, selecting a $z_k$ from $\bigcap_{i=1}^k B_{1/2}(y_{i(k)})$. Then, we show $\{z_n\}$ is Cauchy. Let $\epsilon > 0$. Then, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Hence, for all $n, m \geq N$,

$$d(z_n, z_m) < \frac{1}{N} < \epsilon.$$

Therefore, by the Cauchy completeness of $X$, $\{z_n\}$ converges to a point in $X$. \(\square\)

**Remark 13.** Where do we use the fact that each ball has infinitely many points? We do in fact use this property in
the proof. Try to figure out how!

5 January 18, 2022