Review / helpful information:

• The uniform distance on $C^0([a, b])$ is defined as
  \[ d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|. \]

• Given a vector space $V$ over the real numbers (i.e. a space where addition of vectors and multiplication by real numbers is well-defined), we define a norm to be a function $\| \cdot \| : V \rightarrow [0, \infty)$ satisfying the following properties:
  
  – Positive Definite: $\| v \| \geq 0$ and $\| v \| = 0 \iff v = 0$.
  
  – Homogeneity: $\| \lambda v \| = |\lambda| \| v \|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.
  
  – Triangle Inequality: $\| x + y \| \leq \| x \| + \| y \|$.

• We denote "$K$ is a subset of a metric space $X$" by $K \subset X$.

• Let $A$ be a set of real numbers and $a = \sup A < \infty$. Then, for all $n \in \mathbb{N}$ there exists an $x_n \in A$ such that
  \[ a - \frac{1}{n} < x_n \leq a. \]

Throughout this problem set, let $(X, d)$ be a metric space.

1. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in $X$. Show that $d(x_n, y_n)$ converges.
   Hint: show that in a metric space, $|d(a, b) + d(a', b')| \leq d(a, a') + d(b, b')$.

   **Remark 1.** You may not assume $x_n$ and $y_n$ converges. This is only true in a Cauchy complete space.

2. In class, we have defined a set $A \subset X$ to be closed if its complement is an open set in $X$. There is another useful definition of a closed set however. Show that $A \subset X$ is closed if and only if every convergent sequence in $A$ converges in $A$. In other words, if $\{x_n\}$ is a convergent sequence in $A$ such that $x_n \rightarrow x$, then $x \in A$.

3. Here, we will show that $C^0([0, 1])$ is Cauchy complete with respect to the uniform distance. Suppose that $f_n \in C^0([0, 1])$ is a Cauchy sequence.
   
   (a) Fix an arbitrary $x_0 \in [0, 1]$. Show that $\lim_{n \rightarrow \infty} f_n(x_0)$ exists.
      
      Hint: $\mathbb{R}$ is Cauchy complete.

   (b) Define $f : [0, 1] \rightarrow \mathbb{R}$ by
   \[ f(x) = \lim_{n \rightarrow \infty} f_n(x). \]
Show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
$$|f_n(x) - f(x)| \leq \epsilon$$
for all $x \in [0, 1]$ and for all $n \geq N$.

(c) Show that $f(x)$ is continuous on $[0, 1]$. I.e., $f \in C^0([0, 1]).$

Hint: To show $f(x)$ is continuous at $x_0$, consider
$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$  

(d) Using parts a-c, explain why $\lim_{n \to \infty} f_n = f$ as a sequence in $C^0([0, 1]).$

4. Let $||\cdot||$ be a norm on a vector space $V$, and let $d(x, y) = ||x - y||$ for all $x, y \in V$.
Show the following three properties:

(a) $d(\lambda x, \lambda y) = |\lambda|d(x, y)$ for all $\lambda \in \mathbb{R}$, and for all $x, y \in V$.

(b) Translation invariance: $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in V$.

(c) Prove $d$ is a metric on $V$. This metric is called the metric induced by the norm.

5. The following are important properties of compact sets in $\mathbb{R}$.

(a) Let $K \subset \mathbb{R}$. Show that there exists a maximum and a minimum value in $K$.

(b) (Optional) Generalize the Heine-Borel theorem to $\mathbb{R}^n$. (This proof is very similar to that in class.)

6. (Optional) Let $U$ be an open set in the metric space $(X, d)$. Show that $U$ can be written as a union of arbitrarily many open balls.

7. (Optional) Show that a function $f : X \to Y$ is continuous if and only if given $U \subset Y$ where $U$ is open in $Y$, $f^{-1}(U)$ is open in $X$.

8. (Optional) We call two norms $||\cdot||_1, ||\cdot||_2$ equivalent if there exists constants $C_1 > 0$ and $C_2$ such that
$$C_1||x||_1 \leq ||x||_2 \leq C_2||x||_1.$$  

One can similarly define equivalent metrics. On $\mathbb{R}^n$ we define the supremum norm and $\ell^p$ norms (for $1 \leq p < \infty$):
$$||x||_\infty = \max_i |x_i| \quad \text{and} \quad ||x||_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$  

(You can check that these are in fact norms, but do not have to.) Show that the supremum norm, $\ell^1$, and $\ell^2$ norms are equivalent on $\mathbb{R}^n$ by showing
$$||x||_\infty \leq ||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2 \leq n||x||_\infty.$$  

Briefly explain why this shows the norms are pairwise equivalent.