18.S190 PSET 1

IAP 2023

1. Consider the following map: $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ where

$$d(x, y) = \begin{cases} \|x - y\|_{\mathbb{R}^2} & x, y, 0 \text{ collinear} \\ \|x\|_{\mathbb{R}^2} + \|y\|_{\mathbb{R}^2} & \text{otherwise.} \end{cases}$$

Here, I use $\|\cdot\|_{\mathbb{R}^2}$ to denote the Euclidean norm/magnitude of a vector in $\mathbb{R}^2$. Show that this map is a metric on $\mathbb{R}^2$. This is called the British Railway metric. (Try to figure out why!)
Hint: Try drawing a picture, and use the fact that $\|\cdot\|_{\mathbb{R}^2}$ is a metric.

2. Is $d : C^1([0, 1]) \times C^1([0, 1]) \to [0, \infty)$ defined by

$$d(f, g) = \sup_{x \in [0,1]} |f'(x) - g'(x)|$$

a metric on $C^1([0, 1])$? If so, prove it. If not, show what properties of a metric d satisfies, and explain which properties of a metric d fails.

3. Show that $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ where

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

is a metric on $\mathbb{R}$.

4. Define a *semi-metric* on X as a metric that satisfies symmetry, the triangle inequality, and $d(x, y) \geq 0$ for all $x, y \in X$, but doesn't necessarily satisfy $d(x, y) = 0 \iff x = y$. Specifically, $x = y \implies d(x, y) = 0$ but the opposite implication need not be true. Show that the sum of a metric and a semi-metric on X is a metric on X. In other words, if d is a metric on X, and d' is a semi-metric on X, then $d + d'$ is a metric on X.

5. Show that $I_t : C^0([a, b]) \to C^1([a, b])$ is a continuous map where

$$I_t(f) = \int_a^t f(x)\,\mathrm{d}x$$

for some $t \in [a, b]$.
Hint: This proof is semi-similar to an example done in class, though you will need to mess with ϵs and δs.

6. (Optional) In this problem, you will show that the ℓ^p-metric is in fact a metric.

(a) (Hölder's Inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}$, $1 \leq k \leq n$. Prove that if $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q}.$$

Hint: Prove that if $A, B > 0$ and $t \in (0,1)$, then $A^t B^{1-t} \leq tA + (1-t)B$ by showing the function

$$f(x) = tx + (1-t)B - x^t B^{1-t}, \quad x > 0,$$

has a minimum at $x = B$.

(b) (Minkowski's inequality) Suppose that $n \in \mathbb{N}$ and let $a_k, b_k \in \mathbb{R}$, $1 \leq k \leq n$. Prove that if $1 \leq p < \infty$, then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}.$$

Hint: by the triangle inequality,

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1}.$$

Now apply Hölder's inequality.

7. (Optional) We denote the space of infinitely differentiable functions on an interval $[a, b]$ as $C^\infty([a, b])$. Denote

$$\sup_{x \in [a,b]} |f^{(n)}(x) - g^{(n)}(x)| = d_n(f, g).$$

Problem 2 shows that d_n is a semi-metric on $C^\infty([a, b])$ for all $n \in \mathbb{N}$, and d_0 is a metric as we showed in class. Show that

$$d(f, g) := \sum_{n=0}^{\infty} 2^{-n} \frac{d_n(f, g)}{1 + d_n(f, g)}$$

is a metric on $C^\infty([a, b])$.

Remark 1. *This concept is related to what is called a Fréchet space, named after Maurice Fréchet who first wrote about metric spaces!*