18.S190 PSET 3

IAP 2023

Review / helpful information:

- Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$. Then, we define

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

- We define the ℓ^p norm on a sequence $a = \{a_n\}_n$ of real numbers as

$$\|a\|_p = \left(\sum_{j=1}^{\infty} |a_n|^p \right)^{1/p}$$

where $1 \leq p < \infty$. We define ℓ^p space as the space of (infinite) sequences $x = \{x_n\}_n$ in $\mathbb{R}$ such that $\|x\|_p < \infty$.

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1. Let A be a closed subset of $\mathbb{R}^n$ and let B be a compact subset of $\mathbb{R}^n$. Show that $A + B$ is closed.
Hint: Let $\{a_n\}_n$ be a sequence in $A + B$ such that $a_n \to z \in \mathbb{R}^n$. Show that $z \in A + B$.

Remark 1. *In fact, you can show that the sum of a closed set and a compact subset in a "topological vector space" is a closed set, but that goes beyond the scope of this class.*

2. Let (X, d) be a metric space and $S \subset X$. Then, a point $x \in S$ is an **isolated point** if there exists an $\epsilon > 0$ such that $B_\epsilon(x)$ contains no other points of S. Show that a point $x \in S$ is an isolated point if and only if give a sequence $\{a_n\}$ in S converges to x, it must be the case that there exists an N such that for all $n \geq N$, $a_n = x$.
3. Let X be a metric space. Show that a finite union of compact subsets in X is compact.
4. Let $1 \leq p < \infty$. Consider the set

$$S := \{a = \{a_n\} \in \ell^p \mid \|a\|_p \leq 1\}.$$

Explain why S is closed and bounded in ℓ^p (under the metric induced by the norm in PSET 2), and prove that S is not a compact subset of ℓ^p.
Hint: Let $e_n = \{\delta_{k,n}\}_k \in S$ where

$$\delta_{k,n} = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}.$$

Show that $\{e_n\}_n$ does not have a convergent subsequence in S. Make sure to explain why this shows S is not compact.
5. Consider the set

$$S := \{f \in C^0([0,1]) \mid \|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \leq 1\}.$$

Prove that S is not compact.
Hint: consider the sequence $f_n(x) = x^n$.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = kx + b$ for $0 < k < 1$ and $b \in \mathbb{R}$. Show that f is a contraction, find the fixed point of f, and directly show the fixed point is unique.

7. (Optional) Consider the space ℓ^2. Show that the set

$$A = \{a = \{a_k\}_k \in \ell^2 \mid |a_k| < k^{-3}\}$$

is a compact subset of ℓ^2.

8. (Optional) Consider the set of functions of the form

$$\sum_n a_n e^{inx}$$

with $|a_n| < (1 + |n|)^{-2}$.

(a) Show that every function in this set lies in $C^0([0, 2\pi])$.

(b) Show that this set is compact in $C^0([0, 2\pi])$.

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