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PAIGE BRIGHT: Last time we started talking about metric spaces being compact on Euclidean space in particular, where we showed on Rn that sequentially compact was the same as closed and bounded, which was the same as topologically compact. And notice that in these two theorems, we use some very important propositions that we've learned from real analysis. To prove that closed and bounded implied sequentially compact, we have to use the Bolzano-Weierstrass theorem, which essentially states exactly the condition of sequentially compactness. Bolzano-Weierstrass.

And to use closed and bounded, to show topologically compact, we had to use the Heine-Borel theorem, where we trapped our compact set inside of a closed cube. And we know that that closed cube is going to be compact, but we don't always have these closed cubes on metric spaces, right? What does it mean for there to be a cube of functions?

And in fact, in general, these implications are not going to be precisely true. And small error from last time-- last time, I stated that sequentially compact was the same as closed and bounded on metric spaces. This is not the case. Sorry about that.

We are using Bolzano-Weierstrass to prove this theorem. And we need it to show the other implication. If it was always the same, then all three of these would always be true. And it's not.

Metric spaces where closed and bounded implies topologically compact have the Heine-Borel property. And not every metric space has this property. But what we're going to show today is that if we're not on Rn-- if we're on a general metric space, we still have the following implications.

We already know that sequentially compact and topologically compact implies closed and bounded, which is a very helpful thing, right? If you want to show that a metric space isn't compact, you just have to show that it's not closed or at least not bounded. But we're still going to have the following implication-- what we're going to show today.

We're going to show that sequentially compact is the same as topologically compact. In fact, we're going to show two more properties that emphasize the importance of compact metric spaces in general. It's going to be a very proof-intensive day, but I hope you stick with it with me.

So yeah, that's our goal for today. Before we show sequentially compact implies topologically compact, and furthermore, the opposite direction, first, we're going to start by proving some lemmas about sequential compactness to make our job just a little bit easier in the long term. And in fact, these lemmas are going to be used throughout the rest of the class, so please pay attention.

So first, we have what is known as the lemma-- the big number lemma-- where here I'm just not writing out number. But the idea is given our metric space x is sequentially compact, if I have an open cover of x-- let's call this our Ui's-- then there exists an R bigger than 0 such that for all x in x, the ball of radius r around x is contained in one of the Ui's for some i.

Now, let's draw a picture of what I'm actually describing here because this quantifier logic can be a little bit confusing the first time you see it. So here. Let's just draw a little blob for a metric space x. And what we're going to do is find an open cover of this by the Ui. So I'll draw these by circles.

So these are our Ui's. OK, there we go. And now, for every x in x, we claim that there exists a ball of radius R around x such that it's contained in only one of these open sets-- or sorry, not only one-- that it's fully contained in one of these sets. And this might ring out to you as being true if you want to prove topologically compact is the same as sequentially compact.

What we're going to do in the long term is assume that we're sequentially compact, take an open cover, and use this R to construct our conclusion. But first, we have to prove this fact. And the proof is going to be pretty usual for sequential compactness. So our proof is going to be by contradiction.

We're going to use contradiction, assume not true. What does it mean for this statement to be not true? Well, it means that for every R bigger than 0, there exists some x in x such that the ball of radius R around an x is not contained in any of the Ui's. It's contained in a cover of the Ui's. It's just not fully contained in one of them, at least.

And what we're going to do is use this to construct a sequence and take the convergent subsequence and reach contradiction. So for each R equal to 1 over n, choose x n to satisfy the above property. So we're going to choose the points in our sequence to be the xn, or at least one of the xn, such that the ball of radius R or the ball of radius 1 over n around x is not contained in any of the Ui's. And we know that this is going to be true because one of our n is going to be bigger than 0.

So what are we going to do with the sequence? Well, we're going to start by taking a convergent subsequence, right? We're going to use the fact that we're sequentially compact.

And this trend will carry on throughout the day. So we have a convergent subsequence-- let's call it xnk-- converging to x, of course. That's what we usually let it be.

OK. So what are we going to do? We want to reach the contradiction that there is, in fact, a ball of radius R around one of these x's contained in the Ui. How do we do that?

Well first, we notice that x has to be in one of the Ui's. x in Ui0 for some i0. And this is true given that it's an open cover of our space.

And therefore, because the Ui0 is open, there exists an R0 bigger than 0 such that the ball of radius R0 is contained in Ui0. Right? Now, we want to find a subset of this such that we reach for contradiction.

So choose n sufficiently large so that 1 over n is less than R0 over 2 and do the distance from x to xn is less than R0 over 2. And we're going to use this to create a subset of b R0 of x. So how do we do that?

Well, we consider the ball of radius 1 over n around x-- or sorry, around xn. So for all y in this set, we notice that the distance from x to y is less than or equal to the distance from x to xn plus the distance from y to xn. This is by the triangle inequality.

This term is going to be less than R0 over 2. It's less than 1 over n, which is less than R0 over 2 plus 1 over something that's less than 1 over n, which is less than R0 over 2. So this is R0, which tells us that the ball of radius 1 over n around x is fully contained. And the ball of radius R0 around x, which is fully contained, and Ui0, which is our contradiction, right? Because we claimed at the very beginning that none of these balls are going to be fully contained in any of the Ui's.

So this is the big number lemma. As a small note, the largest such R such that this property holds is known as the Lebesgue number. And this comes up in a number of other applications. We're just not going to be talking about them today.

So now, we have a way of refining our open cover to refine our open cover to the set of open balls of all the same radius, but what we need to do is get finitely many of these open balls for our open cover. And as such, we're going to define a new term. So definition-- the term is known as totally bounded.

A-- or I should say metric space-- let's call it x-- is totally bounded if, for all epsilon bigger than 0, there exists y1 through yk such that the union of balls of radius epsilon around the finitely many yi's contains x. So it's a very similar picture to this, only we have all the same radius R instead of our regular open cover. And what we're going to do is show that sequential compactness implies totally bounded.

This is another lemma of ours. OK. Lemma-- sequentially compact implies totally bounded as metric spaces. So if your metric space is sequentially compact, then it's going to be totally bounded. Let's prove this.

Proof-- again, our proof is going to be by contradiction. Assume that it's not totally bounded. Then for all epsilon bigger than 0, there exists-- or sorry, I should say there does not exist finitely many epsilon balls that cover our space.

What does it mean to be not existent, right? What does it mean for there to not exist finitely many epsilon balls? Well, this means that there are infinitely many of them.

So it takes epsilon balls to cover x. Now, what are we going to do with this? We're once again going to construct a sequence and a convergent subsequence. But we need to somehow use the fact that it takes infinitely many epsilon balls to cover x.

Well, let's just start by taking any x-- x1-- be an x. And now we want to construct an x2 such that there is no Cauchy subsequence of the xn's. So let's take x2 to be an x not including the ball of radius epsilon around x1.

Let's continue this process. Let xn be in x not including the union of balls of radius epsilon around xi. Now, why are we doing this? We want to create a sequence of points such that the distance between every point is bigger than epsilon, which we have immediately from this.

The distance from xi to xj is always bigger than epsilon by construction because if it wasn't less than epsilon, then it would be contained in the ball of radius epsilon around one of the other xi's. So what does this tell us? This tells us that our sequence can never be Cauchy, right?

If our sequence was Cauchy, or a subsequence was Cauchy, then the distance between subsequent points has to be able to get less than epsilon, which is not the case. So what we notice is that there are no Cauchy subsequences of the sequence xi. And this is going to be our contradiction.

Why? Because this is a sequence in a sequentially compact metric space. So we know that there has to exist a convergent subsequence. But being convergent implies that it's going to be Cauchy. We've shown this on the second day in our general theory of metric spaces.

So what we've shown is that there should be a Cauchy subsequence, but there is no Cauchy subsequence. So this is our contradiction. And that concludes the proof.

And these two lemmas-- the Lebesgue number lemma and this one right here-- are going to be all we need to show that sequentially compact implies topologically compact. So let's start that proof. Theorem-- sequentially compact is the same as topologically compact as metric spaces.

So let's start the proof. Proof-- assume sequentially compact, and let Ui be an open cover of the metric space x. What we then know by the Lebesgue number lemma is that there exists an R bigger than 0 such that balls of radius R are contained in one of the Ui's. So therefore, by the Lebesgue number lemma, there exists an R bigger than 0 such that balls of radius all around x are contained in Ui for some i. I.e. for every x, there exists some i such that this is true.

And now, how do we go from this to finitely many of these Ui's? Well, here we use the following lemma. We know that it's totally bounded. So given this R bigger than 0, let R be equal to epsilon and the definition of totally bounded.

And then we know that there exists finitely many y1 through yk such that x is contained in balls of radius in the union of balls of radius R around yi from i equals 1 to k. But we want to go from this to a finite subcover. But what we notice is that each of these balls of radius R is contained in one of the Ui's.

So right now, this is going to be contained in the union of Uij, where ij is the one such that the balls of radius R around yi is contained in Uij. And then we're done with the forward direction, right? Because we've gone from an open cover of our sequentially compact metric space and reached a finite subcover. So that shows that sequentially compact implies topologically compact.

The other direction is going to be mildly harder, but it's still going to be essentially what we've been doing this entire time. We're going to assume that it's not sequentially compact, construct a sequence, and take the convergent subsequence, and reach a contradiction somehow. OK.

So suppose topologically compact-- oh, I'm holding it backwards. Let me rewrite this. Compact.

And let's suppose for the sake of contradiction-- assume for the sake of contradiction-- we're not sequentially compact. What we're going to do is-- or sorry, let me first state what does it mean to be not sequentially compact? Then there exists some sequence-- let's call it xn, of course-- with no convergent subsequence.

Our goal is to go from this to an open cover of our space and show that there is no finite subcover to reach our contradiction. Well, how do we do that? Well first, we need to note some facts about convergent subsequences. Firstly, none of the xi's-- actually, I guess I shouldn't say firstly. Just note that none of the xi can appear infinitely many times.

Actually, yeah. I'm going to call this firstly. Why is this true? Well, if any of the xi's appeared infinitely many times, then that would mean that we could just take a very trivial subsequence of xi's, and that would converge to xi.

So we know that none of them appears infinitely many times. So we can just assume that each one appears finitely many times. But furthermore, we know that for all epsilon bigger-- nope, that's not what I meant to say. We know that there exists an epsilon n bigger than 0 such that the ball of radius epsilon n around xn is simply the set xn, where here I mean the single point xn.

So I guess to be extra clear, I'll write this as xj and xj. Why is this true? Well, suppose for the sake of contradiction that for every epsilon bigger than 0, there exists some xi in a ball of radius epsilon around xj.

Well then what that would tell us is that we could take a convergent subsequence, which will converge to xj. We just choose closer and closer points to xj. So we know that there must exist some epsilon n such that this is the case.

So we're going to use this to create our open cover. We're going to let these be the open cover of uj. So let's call them ui, or uj's. Yeah. From j equals 1 to infinity.

But this isn't quite an open cover. We don't know that we've fully covered x yet. So what we're going to do is let U0 be equal to x not including any of the terms in our sequence.

Now, why does this make sense? Or sorry, why is this an open set? Well, we know that the complement of this set contains all of its limit points, which means that it's going to be closed, as you're shown on your second problem set.

So what this tells us is that the complement of the complement is therefore going to be open. So U0 is open. Therefore, we've reached an open cover of our metric space x.

We have that x is contained-- oh, wrong direction-- contained in U0 union the infinite union of ui's or uj's. j equals 1 to infinity.

Now, from here we want to reach a finite subcover, but notice that every finite subcover is going to be emitting infinitely many points, right? Because every finite subcover is going to have finitely many of these uj's. But what that tells you is that infinitely many of them-- sorry, let me double-check what I want to be saying here.

Yeah. Because it's going to emit infinitely many points, this tells us that x-- or sorry, there is no finite subcover, which is our contradiction because we're assuming topologically compact. So we've shown that topologically compact implies sequentially compact. And that concludes the other direction of our proof. And thus, we're done.

So this is essentially what we've set off to do from the very beginning, which is very motivated by metric spaces. But as you may guess from the fact that sequentially compact implies totally bounded, there's quite a bit more that we can actually be assuming. So what we're going to show for the rest of today is two more equivalent definitions of compactness on a metric space.

And I'll again reiterate these are for a metric space, not inherently a topological space, as you might see in 18901. But this is the definition of compactness or equivalent definitions of compactness for a metric space. But before I do that, I want to first state some lemmas or some corollaries of these facts that we've shown so far because they're deeply important.

Compactness has some very deep results. And I know that this is going to be a very proof-intensive class. So I want to first start by stating some facts that are a little bit easier to prove, but also mildly intuitive.

So first, recall a function f from x to y, where x and y are metric spaces, is continuous if and only if, for every u open in y, f inverse of u is open in x. Sorry, I know I didn't fully write it out, but I at least said it verbally. f inverse of you is going to be open in x. This is the definition of continuous.

And what we're going to do is therefore show lemma. If k is a compact subset of x, where here the double subsets implies compact-- so recall that notation-- we're going to note that f of k is compact in y. Over here f is a continuous map from x to y.

So this is our first statement that we're going to prove today. Or sorry, the first statement that we're going to prove right now. Proof-- well, we want to somehow reach an open cover of k. So let Ui be an open cover of f of k in y.

Then what do we know? Well, what this will imply is that f inverse of Ui will be a finite-- sorry, not finite-- will be an open cover of k. The fact that the inverse images cover k is by the very definition of f of k, but how do we know that they're open?

Well, this is from the fact that it's continuous. These are the inverse images of open sets. And so what we know is that we can reach a finite subcover-- a finite subcover given by f inverse Ui from, let's say, i equals 1 to k. And what we're going to do is map this back to f of k.

Notice that then f of f inverse of Ui equals Ui, which is therefore going to give us our open-- sorry. How should I say this? How should I say this? This is not necessarily directly true.

What I mean to say is that then Ui from i equals 1 to k is open cover of f of k. How do we know this? Well, we know that the Ui's are such that-- or sorry, the f inverse of Ui's cover k. So therefore, the images of them must cover f of k.

And so we've gone from an open cover of f of k to a finite subcover, which shows that it's compact in y. So that is our proof. That shows that the image of compact sets is going to be compact under continuity.

And using this and the fact that we have sequential compactness is going to allow us to state some very helpful lemmas. So lemma-- or sorry, not lemma-- corollary. On that f from x to, let's say, the real numbers, continuous has a max and a min achieved on compact sets, by which I mean for every compact subset of x, the image of it to R is going to have a maximum and a minimum achieved.

And the proof is very short at this point. We know that f of k is going to be compact in R, which implies that it's going to be closed and bounded. Closed and bounded, which you can think of as sequentially compact in the same way, implies that we can find a maximum and a minimum of f of k.

This is essentially just the extreme value theorem. But it's much more general than the extreme value theorem. Before, the extreme value theorem was a map from the real numbers, or at least a compact subset, like a to b, to the real numbers. Now, all of a sudden, we have the extreme value theorem for metric spaces.

This fact is extremely helpful. I cannot iterate how important it's going to be in the long term if you keep taking analysis courses, but the fact that images of compact sets are going to be compact is very, very helpful, and helps us with finding a maximum and a minimum. In fact, one other way to think of this theorem is just to take a sequence that converges to the maximum and converges to the minimum. And there has to be a convergent subsequence of that sequence because it's sequentially compact.

Besides this, I also want to note-- corollary-- maps f given x is compact, f from x to R continuous is going to be bounded. This is the second corollary, right? Before, we were using closedness.

Right now, we're using boundedness. So this is very helpful just in order to be able to conceptualize what's happening. So the proof is already complete by before.

So I will stop there for the moment being. Now, we have one more theorem that I want to show that will have some implications that will be conceptually helpful. But before I do that, I'm going to drink some water.

The next theorem is known as Cantor's intersection theorem. It's one that tells us things about nested sequences of compact sets. And it's helpful conceptually. It's used all the time.

In fact, if you were to take a complex analysis class, it's one of the first things that's used to show convergence of line integrals for holomorphic functions. So what's the statement? Theorem-- this is known as Cantor's intersection theorem.

So given K1 containing K2 containing K3, so on and so forth, each compact, each nonempty and compact-- by compact, I mean topologically or sequentially, which we know. That's a simple fact that-- let me reiterate. Buy nonempty and compact, I mean sequentially or topologically because they're the equivalent on a metric space.

And here I'm assuming that the compact subsets are in a metric space. Then the intersection of the ki's from i equals 1 to infinity is nonempty. How do we prove this?

Well, we prove this by taking a sequence of terms in ki's. So proof-- let wi be in ki for all i. And we know that there exists such an xi because they're nonempty. So right off the bat, we're using that hypothesis.

And what we want to note is then that there exists-- or sorry, let's focus on K1 for now. Notice xi is in K1 for all i. This follows from the fact that it's a nested sequence of compact sets.

So therefore, there exists a convergent subsequence xnk converging to sum a in K1. And what you can continue to show is that it's going to continue to convergence into a in every ki. So secondly, notice-- xi from i equal to 2 to infinity as opposed to including 1 is going to be a subset of K2.

And therefore, there exists a convergent subsequence. And the convergent subsequence has a unique limit point, right? This is what we said on the second day.

Convergent sequences have equivalent or have the same-- how do I want to say this? Have the same limit points. So therefore, a will be in K2.

And we can continue this process over and over again, ending up with a single point at least in all of the ki's. So this concludes our proof. And this is used in complex analysis, as I stated, to go from the sequence of triangles, let's say.

So let's say we have the sequence of triangles, and we know that there exists a point in every one of them. We can keep reiterating this pattern and end up with a point in the limit. So we know that the limits of the intersection is going to be nonempty. So that's how it's used in complex analysis.

Of course, we can't discuss this more fully now because it requires a discussion that takes numerous days of complex analysis to prove. But if you take complex analysis, this will be one of the first things you see. OK. So what are we going to do with this new theorem?

Well, we're going to define a new property. Definition-- a collection of closed sets-- notice here that I'm assuming closed and not compact-- is or has the finite intersection property if every finite subsection-- i.e. if I take finitely many of the closed sets-- we're going to assume that it has nonempty n intersection. We know that this is already going to be true for compact sets because the infinite intersection of them is nonempty.

But this statement is a little bit more general in that we're only assuming that it's going to be closed. And so we're assuming that every finite subcollection has a nonempty intersection. And so what we're going to show is that in fact, there's a relationship between the finite intersection property and topologically and sequentially compact.

That's going to be our huge theorem for today. It's going to take a little bit to prove, but will be deeply powerful once we're done. Our theorem is that the following are equivalent. The following are equivalent, which I'll write out more fully in case you haven't seen this abbreviation before on a metric space x.

One, x is topologically compact. Topologically. Two, x is sequentially compact, which makes sense. We've already shown this implication, but we're going to show two more properties right now.

Three, x is totally bounded. We've already shown sequentially compact implies totally bounded, but there's one more thing we need to include. We need to include that it's Cauchy complete.

And finally, our fourth property is going to be the following, which I'll state out clearly because it's one of the more confusing ones. Every collection of closed sets of closed subsets of x with the finite intersection property has nonempty intersection. Here the difference is that we know that every finite subcollection has nonempty intersection, but what this theorem is telling you is that the intersection of all of the closed subsets, if they have the finite intersection property, is also going to be nonempty.

Now, what we're going to do is show that 1 is true if only if 4 is true. Let me write this out more fully. We've already shown that 1 is true if an only if 2 is true.

What we're now going to do is show that 1 is true if and only if 4 is true, and then show that 2 is true if and only if 3 is true because we've already shown sequentially compact implies totally bounded. So we would just have to show Cauchy completeness. And then that's going to be the end of today. We're going to show these four properties of compact sets, and then we're going to end today's discussion.

So let's start with showing that 1 is true implies 4 is true. So the proof is going to start with supposing, for the sake of contradiction, there exists a collection of closed sets with the finite intersection property, but with empty intersection. So suppose there exists ci-- i equals 1 to infinity-- closed with the finite intersection property, i.e. intersection of finite subcollections is going to be nonempty, and the intersection of all the ci's is empty.

That's going to be our assumption for contradiction. Well, knowing that the intersection of these closed sets is going to be empty, what do we know about the complements? Well, then we know that x, which is the complement of the empty set, is going to be the complement of the intersection of ci's.

But then we know by De Morgan's laws, which we introduced on the second day in general theory, it's going to be the union of the complements of ci's from i equals 1 to infinity. What do we know then? Well, the complement of a closed set is going to be open. So what we've gone from is this collection of closed sets to an open cover of x.

Therefore, there exists a finite subcover given by, let's say, the union from i equals 1 to k of ci complement. And what we're going to show is then that it fails to have the finite intersection property. So notice then that the intersection of ci from i equals 1 to k is going to be the same as the union of the complements from i equals 1 to k, which we know is x. So what this tells us is that the complement of this-- sorry, that's not what I meant to say.

Let me reiterate. What we know then is that x is equal to the union of ci complement from i equals 1 to k, which is, of course, equal to the intersection from n equals 1-- oh, OK, I see-- from i equals 1 to k ci complement. So we've gone from the union of the complements to the complement of the intersection.

And then this implies that the intersection from i equals 1 to k of ci must be the empty set, which is a contradiction because we're assuming that it has the finite intersection property, which again means that every finite subcollection has a nonempty intersection. So this is our contradiction. And we've shown that topologically compact implies that every collection of closed sets has with the finite intersection property has nonempty intersection.

The proof of the other direction is going to be very similar. We're going to show that it's topologically compact given property 4. So we're going to show that 4 implies 1.

Suppose z Ui's are an open cover of x. What we want to go from, then, is we want to use this to find information about closed sets so that we can apply property 4. Well, what we'll do is let Ui complement these ci, which are going to be closed. How do you know that they're closed?

Because the complement is going to be the Ui's, which are open, so that implies that it's closed. And so what we're going to show is that the ci's have finite intersection. So of the final intersection property-- assume for the sake of contradiction that they don't have the finite intersection property-- for the sake of contradiction-- that the intersection of ci1 cik is empty, i.e. it doesn't have the finite intersection property.

Well then we'll have is that the union from i equals 1 to k of the Ui's is equal to the intersection of the ci k, ci n from n equals 1 to k complement complement, which is, of course, then the complement of the intersection of the cosets, which is x. Do I want to read you this proof?

I will read you this proof. I want to rewrite this out more carefully because we have two nested contradictions. So firstly, assume for the sake of contradiction that there is no finite subcover of the Ui's. Finite subcover of the Ui's-- i.e. we're assuming for the sake of contradiction that x is not topologically compact.

And furthermore, we're going to assume for the sake of contradiction that the ci's don't have the finite intersection property. Therefore, because we want to show that they do have the finite intersection property, so the intersection from n equals 1 to k of ci n is empty. That's what we're going to assume.

Well, then what do we know? We know then that the union of the Ui's-- Ui n from n equals 1 to k-- is the intersection of the complements-- complement from n equals 1 to k. But then this is going to be all of x, right? Because it's a complement of the empty set.

And that's a contradiction because we're assuming that there's no finite subcover of the Ui's. So what we know then is that the ci's have the finite intersection property, i.e. the intersection of all of the ci's from i equals 1 to infinity is nonempty. So what do we know?

We know then that the complement of these is going to be not all of x. So we know that x, which is the complement of the empty set, is not going to be equal to the complement of the intersection of the ci's. i equals 1 to infinity, which is, of course, the union of the Ui's.

But this is a contradiction because we've assumed from the very beginning that the Ui's are an open cover of x. So what we've shown is that our open cover is not, in fact, an open cover, which is our contradiction. So we've shown so far that 1 implies 4, which implies 1, and 1 implies 2, which implies 1.

So now, what we need to show is that 2 is the same as 3, i.e. sequentially compact is the same as totally bounded and Cauchy complete. Now, one of these directions has already been done for us. We've already shown that totally bounded is a consequence of sequentially compact.

What we now need to show is that it's Cauchy complete. So we're going to first show that 2 implies 3. OK.

So let xi be a Cauchy sequence of x, i.e. for all epsilon bigger than 0, there exists an n in the natural numbers such that for all n and m bigger than or equal to n, the distance between xn and xm is less than epsilon. We're going to use this to our advantage. So furthermore, what we know is that by sequential compactness, there's going to exist a convergent subsequence of the xi's.

And let's say it converges to x. And what we're going to show then is that in fact, xi will converge to x. We're going to show that the limit point of the Cauchy sequence is, in fact, going to be in our metric space. To do that, let me move to-- actually, I can fit this in here.

Notice then for all nk and n bigger than or equal to n, look at the distance from x to xn. And this will be less than or equal to the distance from x to xnk plus the distance from xnk to xn. And we know that this term is going to be less than epsilon over 2. Or sorry, we should assume epsilon over 2 from the beginning.

But furthermore, we know the xnk converges to x. So we can assume that this one is less than epsilon over 2, which is epsilon. So we've shown that xn is-- so what we've shown, therefore, is that xi will converge to x. And so we've shown that every Cauchy sequence is convergent.

So we've shown Cauchy completeness, which is what we wanted to show. We wanted to show that sequential compactness was the same as totally bounded and Cauchy complete, which concludes the first part of our proof. The second part, the second direction, is going to be quite a bit harder because we want to show that every sequence has a convergent subsequence, assuming totally bounded and Cauchy complete, which is going to require quite a bit of work, or at least quite a bit of mental work, even if on the chalkboard it's not so much.

It's a little bit confusing, how we go from totally bounded and Cauchy complete to a convergent subsequence. OK. We now want to show that property 4 implies property 1. And this will conclude our proof.

OK. How are we going to do this? Well, going to assume that x is totally bounded and Cauchy complete. And we want to show sequential compactness.

How do we do that? Well, we're going to start with an arbitrary sequence in x. So let xi be an arbitrary sequence in x. And we want to show that there exists a convergent subsequence, which is going to be a little bit difficult.

OK. To do so, what we're going to do is start with totally bounded. We're going to note that for all n to the natural numbers, we have that there exists finitely many y1, or in fact, I'll notate this yn1 through yr of nn such that the balls of radius 1 over n around yi of n cover x. This is our assumption from totally boundedness.

Here r of n is just a function of r. It's a function r that depends on n because the number of finitely many terms can depend on n. So in other words, we don't know exactly how many finitely many terms we have, but the more important thing to note is that we have finitely many of these for every single n, which is our superscript.

OK. What are we going to do with this? Well, we're going to go from this to-- OK, we want to go from this to a Cauchy subsequence of the xi's. And once we know that it's Cauchy, we're going to know that it's convergent.

So we know that there must exist 1-- how do I want to say this? Yeah, let's let S1 be the set of these points-- y1 n through yr of nn. What we know is that given that xi is an infinitely long sequence in x, one of the balls of radius 1 over n around these points-- sorry, I should call this Sn.

We know that around the balls of radius 1 over n, around 1 of these points, has to contain infinitely many of the terms. This is by the pigeonhole principle. So therefore, there exists in xi-- or sorry, I should say there exists a ball of radius 1 around some yi of 1 containing infinitely many xi.

And what we're going to do is choose our first point in our sequence to be one of these points. So let z1 be the first, just out of assumption. And what we're going to do is continue constructing these zi's. Well, we know then, furthermore, is that there exists a ball of radius 1 of yi 2 such that-- sorry, 1/2 such that the intersection of these 2 ball of radius 1 over 2 around yi 2 intersection the ball of radius 1 yj1 has infinitely many xi.

How do we know that this is true? Well, we know that infinitely many of them already lie in this ball. And so we already know if we take the intersection of this with all the other balls of radius 1/2, there has to be infinitely many there.

And to see this, I'll draw a little bit more of a picture. So here is our metric space x. And we start with covering it with balls of radius 1, which I'll draw pretty big, just so that we don't have to make a bunch of small pictures.

So here are the balls of radius 1. Well, what we know then is that one of these-- let's say this one-- contains infinitely many terms. So we choose our z1 to be here.

Furthermore, we have the intersection of this with balls of radius 1/2. So I make these a little bit smaller. Sorry, that's not quite good.

Yeah, these will be my balls of radius 1/2. What I know then is that the intersection of the balls of radius 1/2 and the balls of radius 1-- there's only going to be finitely many of them, by assumption. And the fact that there are finitely many of them implies that there exists infinitely many of them that lie in one of these balls. One of these balls of radius 1/2 intersected one of the balls of radius 1.

So what we can do is choose z2 to be in this intersection of them. And we're going to reiterate this process. We're going to assume that z3 is one of the infinitely many points in the intersection of balls of radius 1/3, 1/2, and 1, and so on and so forth.

And what we hope to show is that the sequence is going to be Cauchy. But that's not, at this point, going to be too hard to do. OK, let me write this down more fully. We're going to choose zk to be in the intersection of balls of radius nk of n from n equals 1 to k.

Yeah, balls of radius 1 to k. And now we want to show that this sequence is going to be Cauchy. Well, what's the distance-- or for all epsilon bigger than 0, choose n in the natural numbers such that 1 over n is less than epsilon.

Then what do we know? Well, for all n and m bigger than or equal to n, we know then that zn and zm are both going to be contained in the ball of radius 1 over n of one of the yi's. I won't describe which one because it's a little bit annoying. But we know that they both must lie in 1 over n because they're in this intersection of all these terms.

So what we know then is that the distance from zn to zm must be less than 1 over n or less than or equal to 1 over n, which is less than epsilon, which implies that the zi's must be Cauchy. So therefore, the zi's are Cauchy-- and by Cauchy completeness, we know that there exists a limit point. zi converges to some z in x.

But recall again that our zi's are constructed from our original sequence. They're a subsequence of our original sequence. So what we've shown is that there is a convergent subsequence of our sequence in x. And this is exactly what we sought out to show.

And this shows that totally bounded and Cauchy complete implies sequentially compact. And this is the end of our four-part proof. So again, to reiterate, we've shown that topologically compact is the same as sequentially compact, where here we used the Lebesgue number lemma and totally boundedness.

Then from here we show that sequentially compact is the same as totally bounded and Cauchy complete. Cauchy complete didn't take too much more work. That was pretty short. The other direction required quite a bit of maneuvering to get that convergent subsequence, but again, the notes contain this more fully if you want to read through it slowly and at your own pace.

And furthermore, lastly, we showed that 1 is the same as 4, where topologically compact is the same as closed subsets with the finite intersection property having nonempty intersection. And now, let's think carefully about how each of these proofs went because each have their own purpose in a proof about metric spaces. To show that 1 was the same as 4, we looked at open covers, right?

We took the open cover definition and looked at complements of closed sets and being open, and complements of open sets being closed. And we used all these different properties like that. And that makes a lot of sense because both definitions are in terms of closed and open sets. But to show sequentially compact was the same as totally bounded and Cauchy complete, we used definitions of sequential compactness, right?

Because it's so much easier to look at convergent subsequences and such when you're looking at sequential compactness. So we could have proven that 1 implies 3, but it would have been a little bit more difficult and a little bit harder to maneuver around. And in fact, that's why I've shown these four properties today because if you're in a class where you're proving facts about metric spaces, you need to use the property that makes the most sense to use in a given moment.

For instance, on your problem set, you're going to be asked to show that the union of finitely many compact sets is going to be compact. How do you know that this is going to be true? Well, you can use sequential compactness.

You can use topological compactness. You can use any of these for properties that you still wish to choose, but be sure to think carefully about which one because one of them might be a little bit easier. So I won't discredit you if you use any of the other ones.

And this is true in general when you're going into proofs about metric spaces. Having the intuition to flip from one definition to the other is deeply important and very useful to know. So that will conclude today's lecture.

On next Thursday, we're going to prove some new facts about compact spaces. And in particular, what we're going to show or talk about is the history of compact metric spaces and why they were developed in the first place, which comes from what is known as the Dirichlet problem, but I'll leave that discussion until next Thursday. Until then have a great weekend. Thank you.