18.S66 PROBLEMS #7

Spring 2003

Let S be a subset of $\mathbb{Z}^k$. A *lattice path* of length ℓ from $\alpha \in \mathbb{Z}^k$ to $\beta \in \mathbb{Z}^k$ with steps S may be regarded as a sequence

$$\alpha = v_0, v_1, \ldots, v_\ell = \beta$$

such that each $v_i - v_{i-1} \in S$. A number of lattice path problems have been given already: Problems 8, 137, 138, 139, and 163.

211. [1] The number of lattice paths of length n from $(0,0)$ (ending anywhere) with steps $(1,0)$ and $(0,1)$, is 2^n.

212. [2.5] The number of paths as in #211 above, but never rising above the line $y = x$, is $\binom{n}{\lfloor n/2 \rfloor}$. (The rating [2.5] assumes no knowledge of previous problems.)

213. [1] A *Motzkin path* is a lattice path from $(0,0)$ to $(n,0)$ with steps $(1,0)$, $(1,1)$, and $(1,-1)$, never going below the x-axis. The number of such paths is the *Motzkin number* M_n. Thus $M_0 = 1$, $M_1 = 1$, $M_2 = 2$, $M_3 = 4$, $M_4 = 9$, $M_5 = 21$, $M_6 = 51$. It's not hard to show (though irrelevant here) that

$$\sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

Show that M_n is the number of walks on $\mathbb{N}$ with n steps, with steps -1, 1, or 0, starting and ending at 0.

214. [1.5] The Motzkin number M_n is the number of lattice paths from $(0,0)$ to (n,n), with steps $(2,0)$, $(0,2)$, and $(1,1)$, never rising above the line $y = x$.

215. [1.5] The *(little) Schröder number* s_n is defined to be the number of plane trees with $n+1$ endpoints and no vertex of degree one (i.e., with exactly one child). Thus $s_0 = 1$, $s_1 = 1$, $s_2 = 3$, $s_3 = 11$, $s_4 = 45$,

$s_5 = 197$. The *big Schröder number* r_n is defined by $r_n = 2s_n$. It's not hard to show (though irrelevant here) that

$$\sum_{n\geq 0} s_{n+1}x^n = \frac{1+x-\sqrt{1-6x+x^2}}{4}.$$

Show that s_n is the number of lattice paths in the (x,y) plane from $(0,0)$ to the x-axis using steps $(1,k)$, where $k \in \mathbb{P}$ or $k=-1$, never passing below the x-axis, and with n steps of the form $(1,-1)$.

216. [2] s_n is the number of lattice paths in the (x,y) plane from $(0,0)$ to (n,n) using steps $(k,0)$ or $(0,1)$ with $k \in \mathbb{P}$, and never passing above the line $y=x$.

217. [2] r_n is the number of lattice paths from $(0,0)$ to (n,n), with steps $(1,0)$, $(0,1)$, and $(1,1)$, that never rise above the line $y=x$.

218. [2] The number of lattice paths of length $2n$ from $(0,0)$ to $(0,0)$ with steps $(0,\pm 1)$ and $(\pm 1,0)$ is $\binom{2n}{n}^2$.

219. [1] Let $f(m,n)$ denote the number of lattice paths from $(0,0)$ to (m,n) with steps $(0,1)$, $(1,0)$, $(1,1)$. Then

$$f(m+1,n+1) = f(m,n+1) + f(m+1,n) + f(m,n),\ \ m,n \geq 0.$$

220. [3] Continuing the previous problem, we have

$$(n+2)f(n+2,n+2) = 3(2n+3)f(n+1,n+1) - (n+1)f(n,n),\ \ n \geq 0.$$

221. [2] Let $1 \leq n < m$. The number of lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$ that intersect the line $y=x$ only at $(0,0)$ is given by $\frac{m-n}{m+n}\binom{m+n}{m}$.

NOTE. There is an exceptionally elegant proof based on the formula

$$\frac{m-n}{m+n}\binom{m+n}{m} = \binom{m+n-1}{n} - \binom{m+n-1}{m}.$$

222. (*) Let $f(m,n)$ denote the number of triples (α,β,γ), where α,β,γ are lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$, and where β and γ never rise above α. For instance, let $p=q=1$. If α is the path $(0,0),(0,1),(1,1)$, then there are 2^2 choices for (β,γ), while if α is the path $(0,0),(1,0),(1,1)$ there are 1^2 choices for (β,γ). Hence $f(1,1)=5$. In general

$$f(m,n)=\frac{(m+n+1)!\,(2m+2n+1)!}{(m+1)!\,(2m+1)!\,(n+1)!\,(2n+1)!}.$$

223. Let $a_{i,j}(n)$ (respectively, $\bar{a}_{i,j}(n)$) denote the number of lattice paths of length n from $(0,0)$ to (i,j), with steps $(\pm 1,0)$ and $(0,\pm 1)$, never touching a point $(-k,0)$ with $k\geq 0$ (respectively, $k>0$) once leaving the starting point. Then:

(a) (?) $a_{0,1}(2n+1)=4^nC_n$

(b) [3] $a_{1,0}(2n+1)=C_{2n+1}$

(c) (?) $a_{-1,1}(2n)=\frac{1}{2}C_{2n}$

(d) (?) $a_{1,1}(2n)=4^{n-1}C_n+\frac{1}{2}C_{2n}$

(e) (?) $\bar{a}_{0,0}(2n)=2\cdot 4^nC_n-C_{2n+1}$.

224. Let $b_{i,j}(n)$ (respectively, $\bar{b}_{i,j}(n)$) denote the number of walks in n steps from $(0,0)$ to (i,j), with steps $(\pm 1,\pm 1)$, never touching a point $(-k,0)$ with $k\geq 0$ (respectively, $k>0$) once leaving the starting point. Then:

(a) [2.5] $b_{1,1}(2n+1)=C_{2n+1}$

(b) (?) $b_{-1,1}(2n+1)=2\cdot 4^nC_n-C_{2n+1}$

(c) [2.5] $b_{0,2}(2n)=C_{2n}$

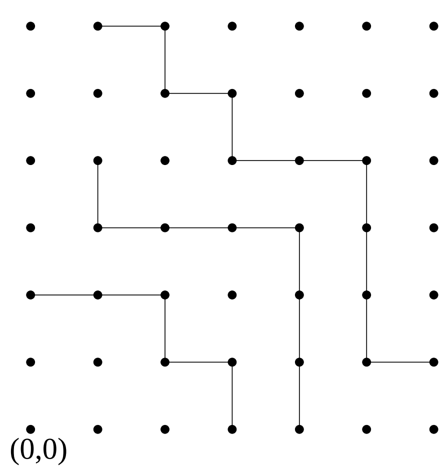
(d) (?) $b_{2i,0}(2n)=\frac{i}{n}\binom{2i}{i}\binom{2n}{n-i}4^{n-i}$, $i\geq 1$. (The case $i=1$ has a known bijective proof.)

(e) (?) $\bar{b}_{0,0}(2n)=4^nC_n$.

225. (*) The number of lattice paths with steps $(-1,0)$, $(0,-1)$, and $(1,1)$ from $(0,0)$ to $(i,0)$ of length $3n+2i$, and staying within the first quadrant (i.e., any point (a,b) along the path satisfies $a,b \geq 0$) is given by
$$\frac{4^n(2i+1)}{(n+i+1)(2n+2i+1)}\binom{2i}{i}\binom{3n+2i}{n}.$$

226. [2] Fix integers $p,n \geq 1$. The number of lattice paths from $(0,0)$ to $(pn,0)$ with steps $(1,p)$ and $(1,-1)$, never falling below the x-axis, is $\frac{1}{pn+1}\binom{(p+1)n}{n}$.

227. [2.5] In this problem all lattice paths have steps $(1,0)$ and $(0,-1)$. An n-*path* is an n-tuple $\mathbf{L} = (L_1,\ldots,L_n)$ of lattice paths. Let $\alpha,\beta,\gamma,\delta \in \mathbb{N}^n$. We say that $\mathbf{L}$ is of *type* $(\alpha,\beta,\gamma,\delta)$ if L_i goes from (β_i,γ_i) to (α_i,δ_i). (Clearly then $\alpha_i \geq \beta_i$ and $\gamma_i \geq \delta_i$.) $\mathbf{L}$ is *intersecting* if for some $i \neq j$, L_i and L_j have a point in common; otherwise $\mathbf{L}$ is *nonintersecting*. The diagram below illustrates a nonintersecting 3-path of type $((3,4,6)$, $(0,1,1)$, $(1,4,6)$, $(0,0,1))$.



Let $B(\alpha,\beta,\gamma,\delta)$ be the number of nonintersecting n-paths of type $(\alpha,\beta,\gamma,\delta)$. Suppose that for any nonidentity permutation π of $1,2,\ldots,n$, there does not exist a nonintersecting n-path whose paths go from (β_i,γ_i) to $(\alpha_{\pi(i)},\delta_{\pi(i)})$. (This is the case e.g. if $\alpha_1 < \cdots < \alpha_n$, $\beta_1 \leq$

$\cdots \leq \beta_n$, $\gamma_1 < \cdots < \gamma_n$, and $\delta_1 \leq \cdots \leq \delta_n$.) Then

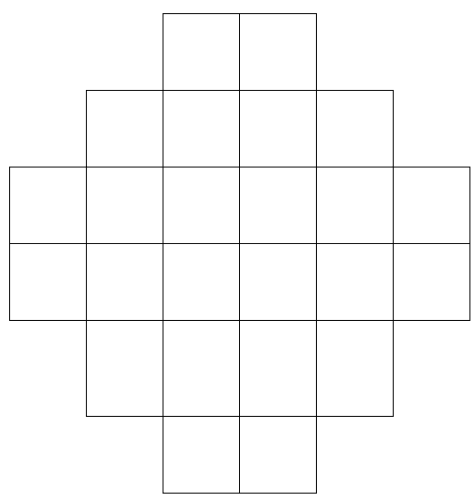
$$B(\alpha, \beta, \gamma, \delta) = \det \left[\binom{\alpha_j - \beta_i + \delta_j - \gamma_i}{\alpha_j - \beta_i} \right]_{i,j=1}^{n}.$$

228. [1.5] The number of ways to tile a $2 \times n$ board with n dominos (two edgewise adjacent squares, oriented either horizontally or vertically) is the Fibonacci number F_{n+1}.

229. [3] Given a finite sequence $\alpha = (2a_1, 2a_2, \ldots, 2a_k)$ of positive even integers, let $B(\alpha)$ be the array of squares (or "board") consisting of $2a_i$ squares in the ith row (read top to bottom), with the centers of the rows lying on a vertical line. The *Aztec diamond* of order n is the board

$$\mathrm{AZ}_n = B(2, 4, 6, \ldots, 2n, 2n, 2n-2, 2n-4, \ldots, 4, 2).$$

For instance, AZ_3 looks like



The number of domino tilings of $\mathrm{AZ}(n)$ is $2^{\binom{n+1}{2}}$.

230. [2.5] The *augmented Aztec diamond* of order n is the board

$$\mathrm{AAZ}_n = B(2, 4, 6, \ldots, 2n, 2n, 2n, 2n-2, 2n-4, \ldots, 4, 2).$$

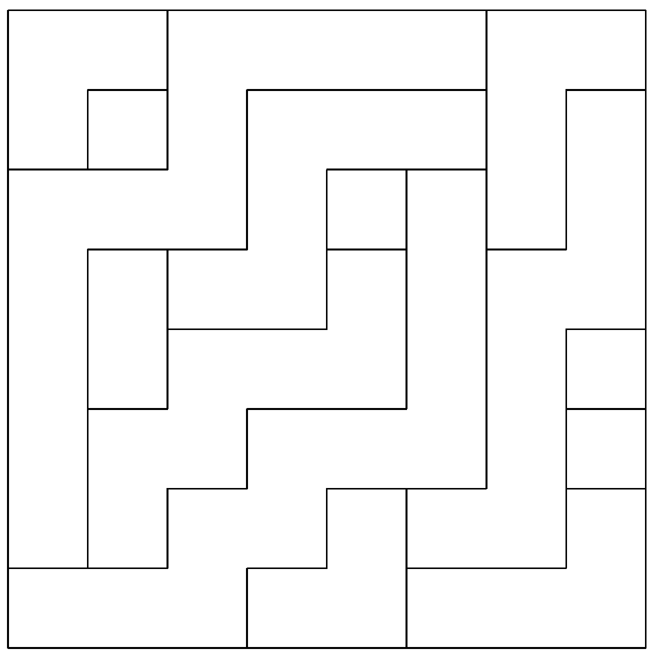
In other words, AAZ_n is obtained from AZ_n by adding a new row of length $2n$ in the middle. The number of domino tilings of AAZ_n is equal to the number $f(n)$ of lattice paths with steps $(1,0)$, $(0,1)$, and $(1,1)$ from $(0,0)$ to (n,n), as defined in Problem 218.

231. [2.5] The *half Aztec diamond* of order n is the board

$$\mathrm{HAZ}_n = B(2, 4, 6, \ldots, 2n, 2n).$$

The number of domino tilings of HAZ_n is the number of lattice paths with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$ from $(0, 0)$ to (n, n) that never rise above the line $y = x$.

232. [2.5] A *snake* on the $m \times n$ chessboard is a nonempty subset S of the squares of the board with the following property: Start at one of the squares and continue walking one step up or to the right, stopping at any time. The squares visited are the squares of the snake. Here is an example of the 8×8 chessboard covered with disjoint snakes.



The total number of ways to cover an $m \times n$ chessboard (and many other nonrectangular boards as well, such as the Young diagram of a partition) with disjoint snakes is a product of Fibonacci numbers.

233. [3] The *Somos-4 sequence* $a_0, a_1, \ldots$ is defined by

$$a_n a_{n+4} = a_{n+1} a_{n+3} + a_{n+2}^2, \quad n \geq 0, \tag{9}$$

with the initial conditions $a_0 = a_1 = a_2 = a_3 = 1$. Show that each a_n is an integer by finding a combinatorial interpretation of a_n and verifying combinatorially that (9) holds. The known combinatorial interpretation of a_n is as the number of matchings (vertex-disjoint sets of edges covering all the vertices) of certain graphs G_n.

NOTE. A similar interpretation is known for the terms of the *Somos-5 sequence*, defined by

$$a_n a_{n+5} = a_{n+1}a_{n+4} + a_{n+2}a_{n+3}, \quad n \geq 0,$$

with $a_0 = a_1 = a_2 = a_3 = a_4 = 1$. It is known the terms of the Somos-6 and Somos-7 sequences are integers, but no combinatorial proof (or simple proof in general) is known. These sequences are defined by

$$a_n a_{n+6} = a_{n+1}a_{n+5} + a_{n+2}a_{n+4} + a_{n+3}^2, \quad n \geq 0,$$

with $a_0 = \cdots = a_5 = 1$, and

$$a_n a_{n+7} = a_{n+1}a_{n+6} + a_{n+2}a_{n+5} + a_{n+3}a_{n+4}, \quad n \geq 0,$$

with $a_0 = \cdots = a_6 = 1$. By now it should be obvious what the definition of the *Somos-k* sequence is for any $k \geq 2$. Somewhat surprisingly, the terms of the Somos-8 sequence (and presumably Somos-k for all $k > 8$, though I'm not sure whether this is known) are not all integers; the first noninteger for Somos-8 is $a_{17} = 420514/7$.