Illustration of Category Hilb with examples in Atomic and Optical Physics

M.I.T. Student
Department of Physics
MIT
May 13, 2013

Outline

- What is Atomic and Optical physics?

  e.g. atoms and photons
- What is Hilbert space?
- What is Category Hilb?
- Monoidal structure of Hilb
- Feynman diagram in the language of category

  e.g. Quantum Harmonic oscillator
- Conclusions and interests

What is Atomic and Optical Physics about?

Physics: Quantum phenomena of particles

*Atoms, from one to an ensemble

*Photons, from one to a bunch, from lasers, or trapped in a cavity

*simulate other quantum systems

use optical lattice to simulate crystal

use quantum double well to simulate Josephson junction

What is Atomic and Optical Physics about?

Math: Hilbert space

Def. A real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

* One atom in a harmonic trap induced by lasers

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \qquad E_n = (\frac{1}{2} + n)\hbar\omega$$

$$x = \{\sum_i c_i|\phi_i > |c_i \in \mathbb{C}, \sum_i |c_i|^2 = 1\}$$

* A cavity that trapped any number of photons

$$\hat{H} = \frac{1}{2}\epsilon\hat{E}^2 \qquad E_j = j\hbar\omega$$

$$y = \{\sum_j c_j|j > |c_j \in \mathbb{C}, \sum_j |c_j|^2 = 1\}$$

What is Atomic and Optical Physics about?

* a single atom with two hyperfine states

$$z = \{c_s|s> + c_p|p> \, || c_s|^2 + |c_p|^2 = 1\}$$

* a single atom with two hyperfine states
  interacting with photons in a cavity

$$w = z \times y$$

Category Hilb

To view definitions of category Hilb, go to: "Hilb category of Hilbert Spaces" by bc1.
http://planetmath.org/sites/default/files/texpdf/41070.pdf

Category Hilb

- Ob(Hilb)

$$x = \{\sum_i c_i|\phi_i > | c_i \in \mathbb{C}, \sum_i |c_i|^2 = 1\}$$

$$y = \{\sum_j c_j|j > | c_j \in \mathbb{C}, \sum_j |c_j|^2 = 1\}$$

$$z = \{c_s|s > + c_p|p > | |c_s|^2 + |c_p|^2 = 1\}$$

$$w$$

- Morphism

Hom_Hilb (x, x) : $\hat{x}, \hat{p}, \hat{x} + \hat{p} \ldots$

Hom_Hilb (y, y) : $\hat{E}, e^{\alpha \hat{E}^2} \ldots$

Hom_Hilb (x, y) : $\hat{m} : |\phi_i > \rightarrow |i >, i = 0, 1, 2 \ldots$

Physics: phonon – photon mapping, massless, particle – wave duality

Category Hilb is monoidal

- Physics: Interaction of two system -> joint system

  Math: tensor product

  -> monoidal category

$$*(y, z) = w$$

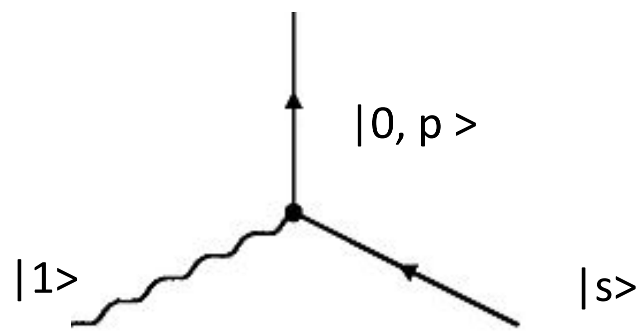
- Monoidal category[2]

i) a category Hilb

ii) a functor

$$* : Hilb \times Hilb \to Hilb$$

Physics: morphism is as important as objects, so that we can form interaction in the Hamiltonian



$$(\hat{\sigma_-} \times \hat{a^\dagger})(|p >, |0 >) = |s, 1 >$$

Category Hilb is monoidal

iii) a unit object

$\mathbb{C} \in Hilb$

iv) left unit law: $l_a : \mathbb{C} \times a \to a$

right unit law: $r_a : a \times \mathbb{C} \to a$

Physics: states in Hilbert space is normalized.

natural isomorphism called the associator:

To view the diagram, go to: "Section 4. The Monoidal Category of Hilbert Spaces" by John Baez.
http://math.ucr.edu/home/baez/quantum/node4.html

such that the following diagrams commute for all objects A, B, C, D,

v)

To view the diagram, go to: "Section 4. The Monoidal Category of Hilbert Spaces" by John Baez.
http://math.ucr.edu/home/baez/quantum/node4.html

Category Hilb is monoidal

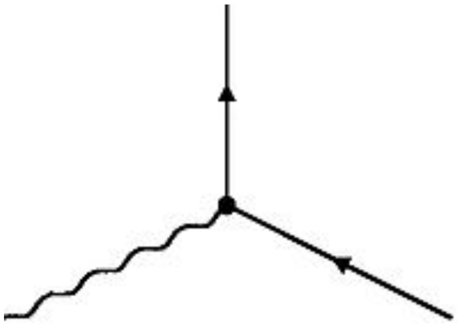
- Associator is natural in a precise sense

Physics:

S, T, L can be viewed as base transformation.

The diagram is commute indicates that associator is defined in a base-independent manner.

Feynman diagram



- states processed by
  evolution operators Exp(-iHt)
  interaction operators $(\hat{\sigma_-}\hat{a}^\dagger + \hat{\sigma_+}\hat{a})$
  evolution operators
  to final states
- This is true for all Hilbert spaces.
- For one Hilbert space that has a specific interaction form, one can also categorize it alone.

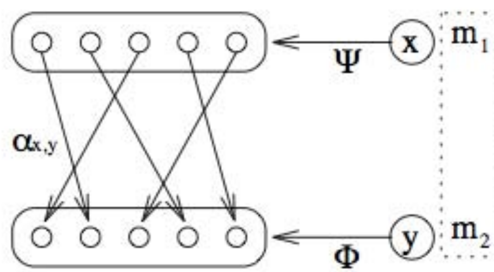
Feynman diagram[3]

- Quantum harmonic oscillator represented in Fock space.

  Categorize Fock space to FinSet_0

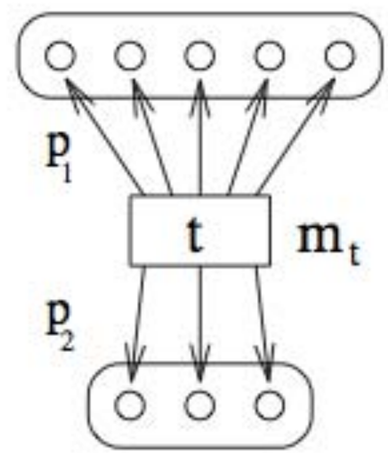
  Categorize states to a functors from a Grp to FinSet_0

Inner Product: a Grp



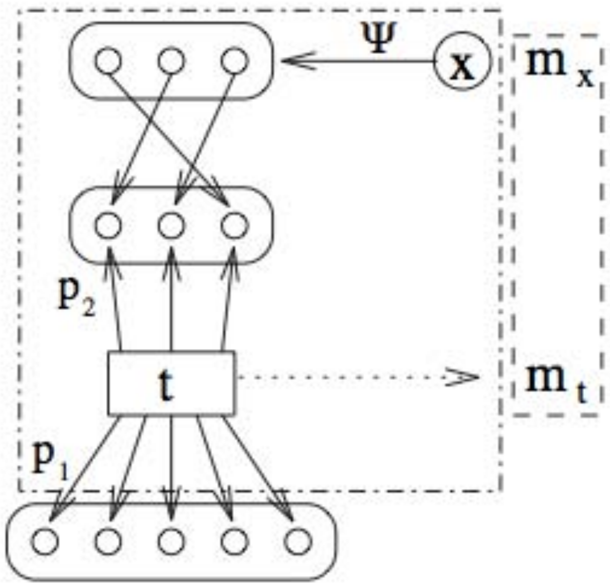
Operators: a Grp with two “projection” functors into FinSet_0.

$$\mathbf{FinSet}_0 \xleftarrow{p_1} T \xrightarrow{p_2} \mathbf{FinSet}_0$$

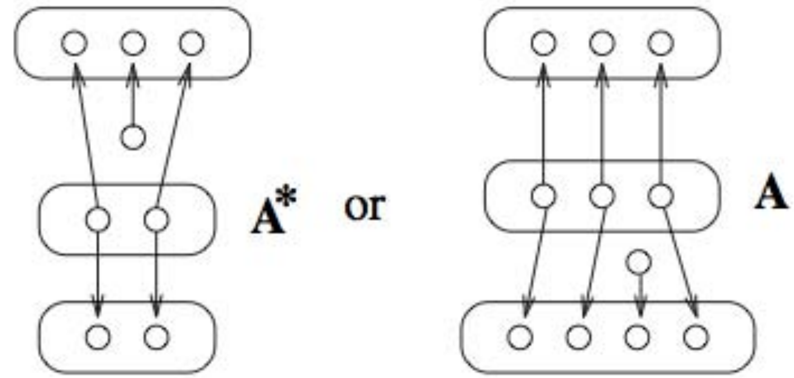


Diagrams are taken from the paper "Categorifying the Quantum Harmonic Oscillator" by Jeffrey Morton and are used by permission.

Feynman diagram



time evolution

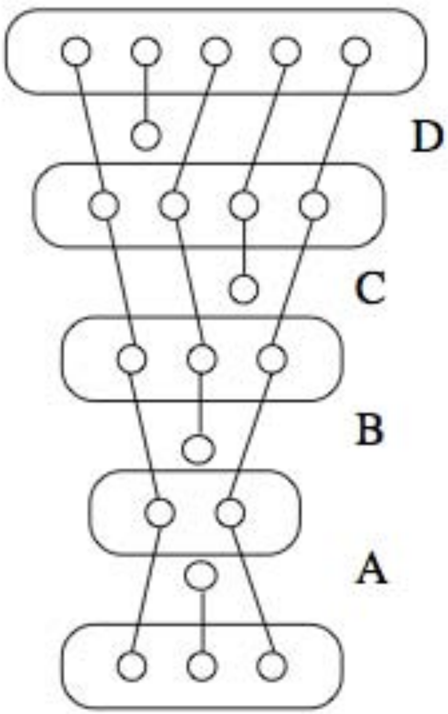


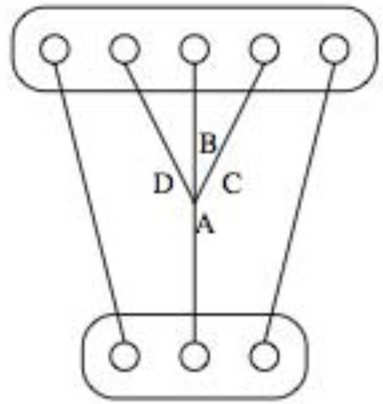
phonon creation and annihilation

Diagrams are taken from the paper "Categorifying the Quantum Harmonic Oscillator" by Jeffrey Morton and are used by permission.

Feynman diagram

$\hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}$: atom-atom short range interaction in our lab





Diagrams are taken from the paper "Categorifying the Quantum Harmonic Oscillator" by Jeffrey Morton and are used by permission.

Conclusions and interests

- Hilb has Hilbert space as objects, operators as morphisms
- Hilb is a monoidal category

  Quantum systems can have interaction with each other, joining together as a new system
- More than one way to categorize quantum systems

  Categorize Fock space into FinSet_0, due to observation of its relationships with enumerative combinatorics.
- Feynman diagram can be translated to diagrams in category.

* Non-trivial morphism from finite Hilbert space to an infinite one or vice versa?

* Benefit of Feynman diagram in category language?

* Similarity of nCob and Hilb[4], relation of general relativity and quantum mechanics? Physics predication?

* Topological field theory, quantum gravity…

Bibliography

[1] bci1, Hilb category of Hilbert spaces
http://planetmath.org/sites/default/files/texpdf/41070.pdf

[2] John Baez, The Monoidal Category of Hilbert Spaces, 2004
http://math.ucr.edu/home/baez/quantum/node4.html

[3] Jeffrey Morton, Categorifying the Quantum Harmonic Oscillator, 2006
http://math.ucr.edu/home/baez/morton_ct06talk.ps

[4] John Baez, Quantum Quandaries: A Category -Theoretic Perspective, 2004
http://arxiv.org/abs/quant-ph/0404040v2

[5] David Spivak, Category Theory for Scientists, 2013
http://arxiv.org/pdf/1302.6946.pdf

[6] Bob Coecke, Kindergarten Quantum Mechanics, 2005
http://arxiv.org/abs/quant-ph/0510032

[7] John Baez, James Dolan, From Finite Sets to Feynman Diagrams, 2000
http://arxiv.org/abs/math/0004133

Bibliography

[8] John Baez, Mike Stay, Physics, Topology, Logic and Computation: A Rosetta Stone, 2009

http://arxiv.org/pdf/0903.0340v3.pdf

[9] Chris Heunen, Categorical quantum models and logics, 2010

http://books.google.com/books?id=nsd8EExdKIwC&printsec=frontcover#v=onepage&q&f=false

[10] nLab

http://ncatlab.org/nlab/show/HomePage

Thank you!