

TAKING STOCK

In today's lecture, we are going to take stock of where we've come from and discuss where we're going. What were the difficulties in the problems? What were the main things we learned? What is the next challenge?

In the last lecture, we proved the following theorem about 3-rich points for sets of lines in \mathbb{R}^3 :

Theorem 0.1. *Let \mathcal{L} be a set of L lines in \mathbb{R}^3 with $\leq B$ lines in any plane. If $B \geq L^{1/2}$, then $|P_3(\mathcal{L})| \lesssim BL$.*

The proof involved three tools that we developed ahead of time: flat points and lines, degree reduction, and Bezout's theorem. Putting it all together, it is the longest proof we have studied so far in this course. I want to take a little time to put it in context more. We'll look at some examples. Also, we'll try to describe the nature of the difficulty in proving the theorem. Why does it take this much work to prove the theorem?

We begin with a simple example. A collection of B lines in a plane can have $\sim B^2$ 3-rich points. For example, we can take a grid with $B/3$ evenly spaced vertical lines, $B/3$ evenly spaced horizontal lines, and $B/3$ evenly spaced diagonal lines. In this grid, we get $\geq B^2/20$ 3-rich points. Next, if we choose L/B generic planes, and put B lines in each plane, we get an arrangement of lines with $\sim BL$ 3-rich points. We can arrange that there will be $\leq B$ lines in any plane by taking each configuration of B lines and rotating and translating it generically.

1. WHAT MAKES THE THEOREM HARD?

To get a feel for the difficulty, let's consider the following much weaker corollary.

Proposition 1.1. *Suppose that \mathcal{L} is a set of L lines in \mathbb{R}^3 with $\geq L^{1.99}$ 3-rich points. Then, there is a plane that contains ≥ 3 lines of \mathcal{L} .*

To prove the proposition, the key question is "how can we find this plane"? Let's mention one possible way of finding a plane with three lines in it. Let us look at the incidence matrix of $P_3(\mathcal{L})$ with \mathcal{L} . If we find a "triangle" in the incidence matrix, then we automatically get three lines in a plane. A triangle is a set of three lines $l_1, l_2, l_3 \in \mathcal{L}$, and three points $x_1, x_2, x_3 \in P_3$ so that each line contains exactly two of the three points. In this case, the points x_1, x_2, x_3 lie in a unique plane π which contains all three lines.

We can try to find a triangle in the incidence matrix. What do we know about the incidence matrix. By hypothesis, it has dimensions $L \times P$ with $P \geq L^{1.99}$, and each point lies in at least three lines. Also, any two lines intersect in at most one point. Just based on this information, does the matrix need to have a triangle? The answer to this question is no. It comes from an interesting example that was explained to me by Andrew Suk.

Proposition 1.2. *Fix any $\epsilon > 0$. For all sufficiently large L , we can find a set \mathfrak{L} of L lines in $\mathbb{R}^2 \subset \mathbb{R}^3$ and a set of 3-rich points $P \subset P_3(\mathfrak{L})$ so that $|P| \geq L^{2-\epsilon}$ and yet the incidence matrix of P and \mathfrak{L} contains no triangle.*

The construction is based on an important example of Behrend about 3-term arithmetic progressions. Recall that an arithmetic progression of length r is a sequence of numbers $a, a + d, a + 2d, \dots, a + (r - 1)d$. Behrend's example is concerned with the question, "how large is the largest subset of the integers from 1 to N with no 3-term arithmetic progression?"

Theorem 1.3. *(Behrend, 1946) Fix any $\epsilon > 0$. For any N sufficiently large, there is a subset of $[1 \dots N]$ with $\geq N^{1-\epsilon}$ elements and with no 3-term arithmetic progression.*

We'll discuss Behrend's construction some time later... Using it, we now give the proof of Proposition 1.2.

Proof. We describe the lines and the points. The lines are vertical, horizontal, and diagonal lines in a grid. We take vertical lines $x = a$ for $a = 1 \dots S$. We take horizontal lines $y = b$ for $b = 1 \dots S$. And we take diagonal lines $x - y = c$ for $c = -S, \dots, S$. We have a total of $L = 4S + 1$ lines. We let X denote the $S \times S$ grid of lattice points $\{(a, b) \in \mathbb{Z}^2 | 1 \leq a, b \leq S\}$. Each point of X lies in exactly three lines of \mathfrak{L} . The set X is the set of all 3-rich points of \mathfrak{L} . It has size $S^2 \sim L^2$, but the incidence matrix of X with \mathfrak{L} contains many triangles. We will pare down X slightly to a subset $P \subset X$ so that the incidence matrix of P with \mathfrak{L} contains no triangles. The key idea of the proof is that Behrend's construction lets us do this paring.

By Behrend's construction, we can find a subset $P_0 \subset [S/2, \dots, 3S/2]$ so that $|P_0| \geq S^{1-\epsilon}$ and yet P_0 contains no 3-term arithmetic progression. We define the set $P := \{(a, b) \in X | a + b \in P_0\}$. For each $d \in [S/2, \dots, 3S/2]$, the set of $(a, b) \in X$ so that $a + b = d$ has $\geq S/2$ elements, and so $|P| \geq (1/2)S^{2-\epsilon} \geq cL^{2-\epsilon}$.

Consider a triangle in the incidence matrix of X and \mathfrak{L} . The horizontal lines are pairwise disjoint, as are the vertical lines and the diagonal lines. Therefore, the triangle must consist of one horizontal line, one vertical line, and one diagonal line. Let $x_i = (a_i, b_i) \in X$ be the vertices of the triangle. We have to show that the three vertices are not all in P . It suffices to show that $d_i = a_i + b_i$ form a 3-term arithmetic progression. This follows by the geometry of the triangle.

It's probably best at this moment to draw your own picture. But for completeness, we write down the details. Suppose that x_1 is the lower-left vertex, x_2 is the right angle, and x_3 is the upper-right vertex. We have $(a_2, b_2) = (a_1, b_1 + d)$. And $(a_3, b_3) = (a_2 + e, b_2)$. But because the diagonal line is at a 45 degree angle, we see that the triangle is isosceles and so $e = d$. A short computation shows that $a_1 + b_1, a_2 + b_2, a_3 + b_3$ make a 3-term arithmetic progression. \square

Therefore, we probably need a different idea to locate a plane with three lines in it. We can formulate this issue more precisely using the axioms of incidence theory (for points, lines, planes in three dimensions). In these axioms, we have a set of points, and each line or plane is a subset of the points, and the whole structure obeys a list of axioms. We don't give the whole list of axioms here, but we give the flavor by mentioning two examples. 1. For any two points, there is a unique line containing the two points. 2. If three points don't all lie on a line, then there is a unique plane containing the three points. Etc. Now we may ask whether Theorem 0.1 or Proposition 1.1 hold more generally in the axioms of incidence theory. I believe that the answer is 'no' and that Suk's construction can be modified to prove the following

Conjecture 1.4. *Fix any $\epsilon > 0$. Then for arbitrarily large numbers L , the following holds: there is a set of points, lines, and planes obeying the incidence axioms, and a subset \mathfrak{L} of the lines, so that $|\mathfrak{L}| = L$, $|P_3(\mathfrak{L})| \geq L^{2-\epsilon}$ and yet each plane contains ≤ 2 lines of \mathfrak{L} .*

Theorem 0.1 depends on some other structure about lines in \mathbb{R}^3 which is not captured in the incidence axioms. What structure is it? Our proof is based on algebraic structure.

There's a fairly short proof of Proposition 1.1 using reguli. If \mathfrak{L} has $L^{1.99}$ 3-rich points, then it follows from Problem Set 2 that there is a regulus or plane containing $\gtrsim L^{.99}$ lines of \mathfrak{L} . Since the lines inside a regulus cannot make any 3-rich points, it's not too hard to push a bit farther and prove that there is a plane containing $\gtrsim L^{.99}$ lines of \mathfrak{L} . Reguli provide an additional structure which is not included in the incidence axioms. Basically this structure amounts to including degree 2 surfaces as well as planes.

The technique of reguli cannot easily push all the way down to $L^{3/2}$ 3-rich points. To try to find a regulus with many lines, we can look at the intersection matrix of the lines of \mathfrak{L} . If this matrix has a $3 \times A$ minor of all 1's, then we can find $\sim A$ lines which lie in a common plane, lie in a common regulus, or pass through a common point. But by Brown's construction, the intersection matrix may have no 3×3 minor of all 1's and still have $L^{5/3}$ 1's. It's hard to rule out that we may have $\sim L^{5/3}$ 3-rich points but the intersection matrix may have no 3×3 minor of all 1's.

In our proof with the polynomial method, we include in the story not just surfaces of degree 2 but surfaces of all degrees. With this algebraic structure, we are able to

prove Theorem 0.1, which holds as long as the number of 3-rich points is at least a large constant times $L^{3/2}$. It's actually not clear what happens below this threshold (i.e. for $B < L^{1/2}$ in the statement of the theorem). The polynomial method (as we've been using it) stops working, but I don't know any examples with P_3 significantly larger than BL .

2. THE BIG PICTURE

We have mostly been talking about estimates for the incidences of lines in \mathbb{R}^2 or \mathbb{R}^3 . We can usually begin on any given problem by thinking about basic facts about incidences, such as "two points lie on a unique line". These facts lead to some basic estimates, but in many cases the basic estimates are far from sharp. To improve them, we need some subtler facts about lines. We have followed two main approaches.

- (1) Use the topological structure of Euclidean space. This approach leads to the crossing number lemma, the Szemerédi-Trotter theorem, and other applications.
- (2) Use the algebraic structure of Euclidean space. This approach leads to the joints theorem and Theorem 0.1 above.

How can we recognize/guess which tool is good for which problem? In the case of the Szemerédi-Trotter theorem, the need for topological considerations is motivated by the example of lines in finite fields. The Szemerédi-Trotter theorem fails badly if we let \mathcal{L} be the set of all lines in \mathbb{F}_q^2 . Finite fields have most of the algebraic structure that we see in \mathbb{R}^2 , but they're very different topologically.

It's less clear to me how to recognize the need for algebraic structure. For example, I still find it kind of surprising that there is not a very different proof of the joints theorem - and such a proof may indeed exist. I think one can probably demonstrate that these theorems don't follow just from 'incidence axioms'. (Of course Szemerédi-Trotter also does not follow just from incidence axioms.) In practice, if a certain question seems similar to the joints theorem or finite field Kakeya..., then it's a candidate for the polynomial method. Also, if it's possible to do some degree reduction, then the problem is a good candidate for the polynomial method.

So far, these two techniques have been complementary. We can't prove the Szemerédi-Trotter theorem with just the polynomial method. If we try to find a low-degree polynomial on the points, then we get a degree which is larger than the number of points on each line, and then we can't do anything with it. If we try to find a low degree polynomial that vanishes on the lines, since we are in the plane, we just get a degree L polynomial and it doesn't lead to any interesting information about the points. There is no possibility of doing degree reduction - any set of L lines in

the plane has degree exactly L . This may suggest that the polynomial method is not well suited to study questions about lines in the plane.

On the other hand, the topological methods have had only limited success in proving estimates for the joints problem. The basic issue is that curves in \mathbb{R}^3 do not divide space into components, and so the whole set up is totally different. There are papers using the topological approach to prove interesting estimates about the joints problem – the best estimate proven this way is something like $J \leq L^{1.62\dots}$. The method involves taking lines or curves in space and projecting them onto planes, and then using the crossing number lemma to study the projections. It seems difficult to capture all the 3-dimensional structure that we're interested in with these two-dimensional projections...

So we have studied two methods. They are useful in different situations – in some sense they deal with different difficulties. However, there are problems that involve both types of difficulties.

3. THE NEXT GOAL

Our next goal is the following theorem. It was conjectured by Elekes and Sharir and proven by Katz and G.

Theorem 3.1. *Suppose that \mathcal{L} is a set of lines in \mathbb{R}^3 with $\leq L^{1/2}$ lines in any plane. Suppose that $3 \leq k \leq L^{1/2}$. Then $|P_k| \lesssim L^{3/2}k^{-2}$.*

This theorem involves both types of difficulties. For large values of k , it is false over finite fields. In particular, let us consider the set of all lines in \mathbb{F}_q^3 . We have $|\mathcal{L}| \sim q^4$. The number of lines in each plane is $\sim q^2 \leq L^{1/2}$. Each point lies in $\geq q^2$ lines. Therefore, taking $k = q^2 \leq L^{1/2}$, we have $|P_k| = q^3 \sim L^{3/2}k^{-3/2}$. We see indeed that our theorem is false over finite fields. The example is reminiscent of the Szemerédi-Trotter theorem, and it suggests we need to use the topological structure of \mathbb{R}^3 . If we try to adapt the algebraic proof of Theorem 0.1 to large k , then the method gives the upper bound $|P_k| \lesssim L^{3/2}k^{-3/2}$, matching the example in finite fields. Moreover it looks plausible that the proof of Theorem 0.1 can be extended to finite fields, and that the same results hold there.

On the other hand, if we look for a purely topological proof, it seems hard to prove the case $k = 3$ that we already proved with the polynomial method.

Our next goal is to prove this theorem by combining the polynomial method and the topological method.

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