

if the initial fluid height is defined as $h(0) = h_0$, then the fluid height as a function of time varies as

$$h(t) = h_0 e^{-t\rho g/RA} \text{ [m]}. \quad (1.31)$$

1.2 Second-order systems

In the previous sections, all the systems had only one energy storage element, and thus could be modeled by a first-order differential equation. In the case of the mechanical systems, energy was stored in a spring or an inertia. In the case of electrical systems, energy can be stored either in a capacitance or an inductance. In the basic linear models considered here, thermal systems store energy in thermal capacitance, but there is no thermal equivalent of a second means of storing energy. That is, there is no equivalent of a thermal inertia. Fluid systems store energy via pressure in fluid capacitances, and via flow rate in fluid inertia (inductance).

In the following sections, we address models with two energy storage elements. The simple step of adding an additional energy storage element allows much greater variation in the types of responses we will encounter. The largest difference is that systems can now exhibit oscillations in time in their natural response. These types of responses are sufficiently important that we will take time to characterize them in detail. We will first consider a second-order mechanical system in some depth, and use this to introduce key ideas associated with second-order responses. We then consider second-order electrical, thermal, and fluid systems.

1.2.1 Complex numbers

In our consideration of second-order systems, the natural frequencies are in general complex-valued. We only need a limited set of complex mathematics, but you will need to have good facility with complex number manipulations and identities. For a review of complex numbers, take a look at the handout on the course web page.

1.2.2 Mechanical second-order system

The second-order system which we will study in this section is shown in Figure 1.19. As shown in the figure, the system consists of a spring and damper attached to a mass which moves laterally on a frictionless surface. The lateral position of the mass is denoted as x . As before, the zero of

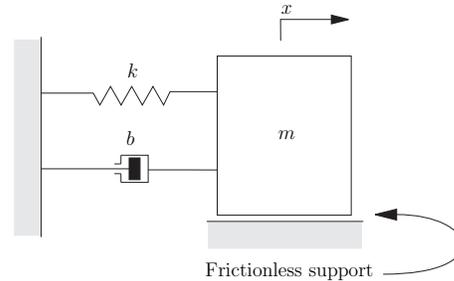


Figure 1.19: Second-order mechanical system.

position is indicated in the figure by the vertical line connecting to the arrow which indicates the direction of increasing x .

A free-body diagram for the system is shown in Figure 1.20. The forces F_k and F_b are identical to those considered in Section 1.1.1. That is, the spring is extended by a force proportional to motion in the x -direction, $F_k = kx$. The damper is translated by a force which is proportional to velocity in the x -direction, $F_b = b \, dx/dt$. As shown in the free-body diagram, these forces have a reaction component acting in the opposite direction on the mass m . The only difference here as compared to the first-order system of Section 1.1.1 is that here the moving element has finite mass m . In Section 1.1.1 the link was massless.

To write the system equation of motion, you sum the forces acting on the mass, taking care to keep track of the reference direction associated with these forces. Through Newton's second law the sum of these forces is equal to the mass times acceleration

$$-F_b - F_k = -b \frac{dx}{dt} - kx = m \frac{d^2x}{dt^2}. \quad (1.32)$$

Rearranging yields the system equation in standard form

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0. \quad (1.33)$$

(As a check on your understanding, convince yourself that the units of all the terms in this equation are force [N].)

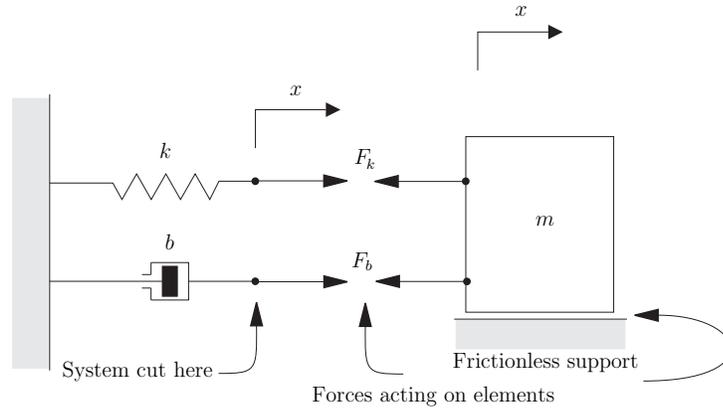


Figure 1.20: Free body diagram for second-order system.

Initial condition response

For this second-order system, initial conditions on both the position and velocity are required to specify the state. The response of this system to an initial displacement $x(0) = x_0$ and initial velocity $v(0) = \dot{x}(0) = v_0$ is found in a manner identical to that previously used in the first order case of Section 1.1. That is, assume that $x(t)$ takes the form $x(t) = ce^{st}$. Substituting this function into (1.33) and applying the derivative property of the exponential yields

$$ms^2ce^{st} + bsce^{st} + kce^{st} = 0. \quad (1.34)$$

As before, the common factor ce^{st} may be cancelled, since it is nonzero for any finite s and t , and with non-rest ($c \neq 0$) initial conditions. Thus we find that s must satisfy the characteristic equation $ms^2 + bs + k = 0$. This second-order polynomial has two solutions

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m} \quad (1.35)$$

and

$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m} \quad (1.36)$$

which are the pole locations (natural frequencies) of the system.

In most cases, the poles are distinct ($b^2 \neq 4mk$), and the initial condition response will take the form

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \quad (1.37)$$

where s_1 and s_2 are given above, and the two constants c_1 and c_2 are chosen to satisfy the initial conditions x_0 and v_0 . If the roots are real ($b^2 > 4mk$), then the response is the weighted sum of two real exponentials. If the roots have an imaginary component ($b^2 < 4mk$), then the exponentials are complex and the response has an oscillatory component. Since in this case $s_1 = s_2^*$, in order to have a real response it must hold that $c_1 = c_2^*$, and thus the response can be expressed as $x(t) = 2\text{Re}\{c_1 e^{s_1 t}\}$, or equivalently as $x(t) = 2\text{Re}\{c_2 e^{s_2 t}\}$.

In the case that the poles are coincident ($b^2 = 4mk$), we have $s_1 = s_2$, and the initial condition response will take the form

$$x(t) = c_1 e^{s_1 t} + c_2 t e^{s_1 t} \quad (1.38)$$

As before, the two constants c_1 and c_2 are chosen to satisfy the initial conditions x_0 and v_0 .

Before further analysis, it is helpful to introduce some standard terms. The pole locations are conveniently parameterized in terms of the *damping ratio* ζ , and *natural frequency* ω_n , where

$$\omega_n = \sqrt{\frac{k}{m}} \quad (1.39)$$

and

$$\zeta = \frac{b}{2\sqrt{km}}. \quad (1.40)$$

The natural frequency ω_n is the frequency at which the system would oscillate if the damping b were zero. The damping ratio ζ is the ratio of the actual damping b to the critical damping $b_c = 2\sqrt{km}$. You should see that the critical damping value is the value for which the poles are coincident.

In terms of these parameters, the differential equation (1.33) takes the form

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = 0. \quad (1.41)$$

In the following section we will make the physically reasonable assumption that the values of m , and k are greater than zero (to maintain system order) and that b is non-negative (to keep things stable). With these assumptions, there are four classes of pole locations:

- First, if $b = 0$, the poles are complex conjugates on the imaginary axis at $s_1 = +j\sqrt{k/m}$ and $s_2 = -j\sqrt{k/m}$. This corresponds to $\zeta = 0$, and is referred to as the *undamped* case.
- If $b^2 - 4mk < 0$ then the poles are complex conjugates lying in the left half of the s -plane. This corresponds to the range $0 < \zeta < 1$, and is referred to as the *underdamped* case.
- If $b^2 - 4mk = 0$ then the poles coincide on the real axis at $s_1 = s_2 = -b/2m$. This corresponds to $\zeta = 1$, and is referred to as the *critically damped* case.
- Finally, if $b^2 - 4mk > 0$ then the poles are at distinct locations on the real axis in the left half of the s -plane. This corresponds to $\zeta > 1$, and is referred to as the *overdamped* case.

We examine each of these cases in turn below.

1.2.3 Undamped case ($\zeta = 0$)

In this case, the poles lie at $s_1 = j\omega_n$ and $s_2 = -j\omega_n$. These pole locations are plotted on the s -plane in Figure 1.21.

The homogeneous solution takes the form $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} = c_1 e^{j\omega_n t} + c_2 e^{-j\omega_n t}$. In order for this solution to be real, we must have $c_1 = c_2^*$, and thus this simplifies to

$$x(t) = 2\text{Re}\{c_1 e^{j\omega_n t}\}. \quad (1.42)$$

If we define $c_1 = \alpha + j\beta$ then this becomes

$$x(t) = 2\text{Re}\{(\alpha + j\beta)e^{j\omega_n t}\} \quad (1.43)$$

$$= 2\text{Re}\{(\alpha + j\beta)(\cos \omega_n t + j \sin \omega_n t)\} \quad (1.44)$$

$$= 2(\alpha \cos \omega_n t - \beta \sin \omega_n t). \quad (1.45)$$

The constants α and β in this solution can be used to match specified values of the initial conditions on position x_0 and velocity v_0 . By inspection, we have $x(0) = 2\alpha$, and thus to match a specified initial position, $\alpha = x_0/2$. Taking the derivative yields $\dot{x}(0) = -2\beta\omega_n$, and thus to match a specified initial velocity, we must have $\beta = -v_0/2\omega_n$.

You should not try to memorize this result; rather, internalize the principle which allowed this solution to be readily derived: (possibly complex) exponentials are the natural response of linear time invariant systems.

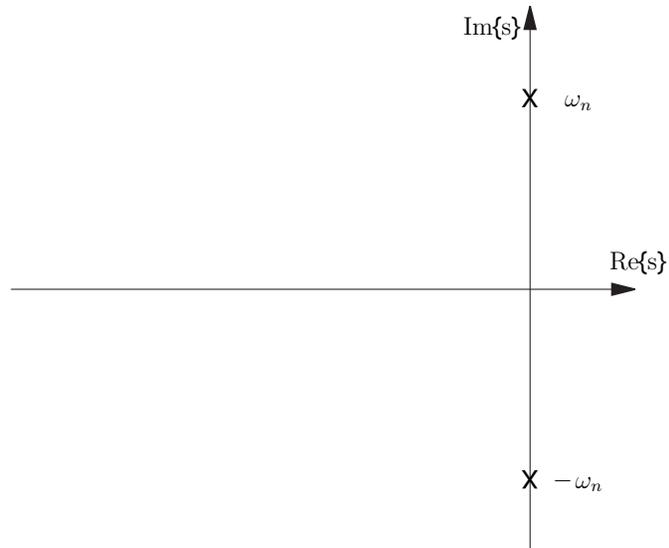


Figure 1.21: Pole locations in the s -plane for second-order mechanical system in the undamped case ($\zeta = 0$).

To show things in another light, suppose that we rewrite the constant c_1 into polar form as $c_1 = Me^{j\phi}$, with $M = |c_1| = \sqrt{\alpha^2 + \beta^2}$ and $\phi = \arg\{c_1\} = \arctan2(\alpha, \beta)$. By the notation $\arctan2$, we mean the two-argument arctangent function which unambiguously returns the angle associated with a complex number, when given the real and imaginary components of that number. Using a single argument arctangent function introduces an uncertainty of π radians into the returned angle ϕ ; be sure to use two-argument arctangent functions in any numerical algorithms that you write. On your calculator, this problem can be avoided by using the rectangular-to-polar conversion function.

With c_1 represented in polar form, the homogeneous response can be written as

$$x(t) = 2\operatorname{Re}\{Me^{j\phi}e^{j\omega_n t}\} \quad (1.46)$$

$$= 2M\operatorname{Re}\{e^{j(\omega_n t + \phi)}\} \quad (1.47)$$

$$= 2M \cos(\omega_n t + \phi). \quad (1.48)$$

The mathematics is notationally cleaner this way, and this more compact form makes clear that the natural response in the undamped case ($\zeta = 0$) is a constant-amplitude sinusoid of frequency ω_n , in which the amplitude M and phase shift ϕ are adjustable to match initial conditions. Note that the solution we have derived is valid for *all* time; in the most general case, the value of the solution in position and velocity could be specified at any given point in time, and the solution constants adjusted to match this constraint.

A picture of this response is shown in Figure 1.22 to make clear the effect of the amplitude and phase parameters; in this figure we have chosen $M = 3$ (and thus a peak value of $2M = 6$) and $\phi = \pi/4$. The period of oscillation is $T = 2\pi/\omega_n$; the plot is shown for $\omega_n = 1$. We have also plotted for reference a unit-amplitude cosine with zero phase shift. Note that positive phase shift corresponds with a forward shift in time in the amount $\Delta t = (\phi/2\pi)T$.

For the form of the response (1.48) the position at $t = 0$ is $x(0) = 2M \cos \phi$, and the velocity at $t = 0$ is $v(0) = -2M\omega_n \sin \phi$. For the parameter values used in the plot, we then have $x(0) = 4.24$ [m], and $v(0) = -4.24$ [m/sec]. You should check that to graphical accuracy, you can see these values on the plot of Figure 1.22.

In practice it is rare to find a system with truly zero damping, as this corresponds to zero energy loss despite the continuing oscillation. Mechanical systems in vacuum can exhibit nearly lossless behavior. It is fortunate for us that the orbits of planets around a star like our sun are nearly lossless.

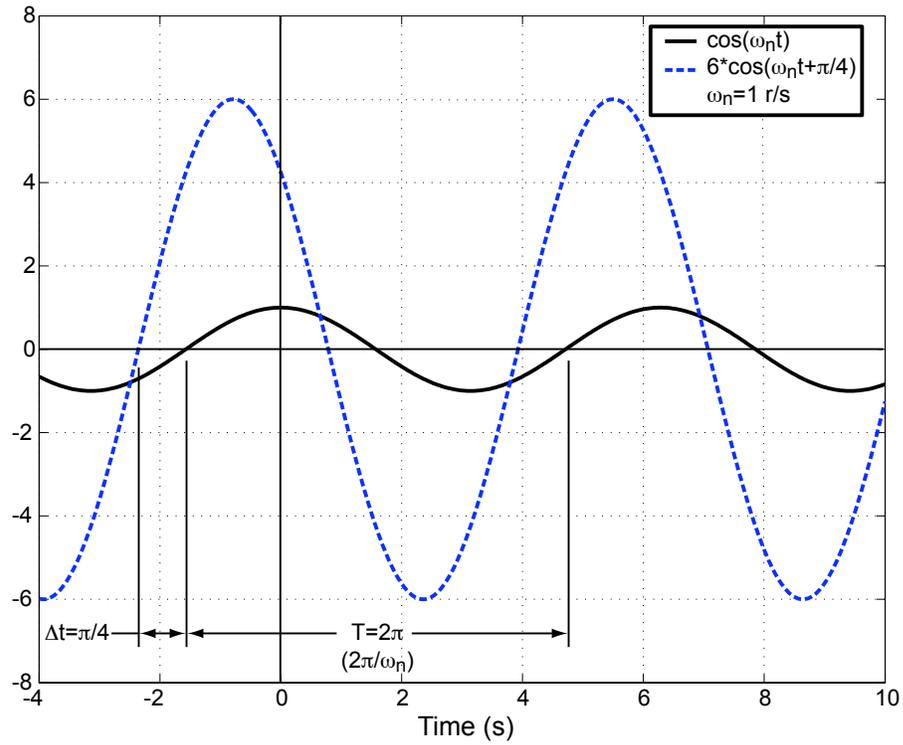


Figure 1.22: Natural response for second-order mechanical system in the undamped case ($\zeta = 0$) with $M = 3$ (and thus a peak value of $2M = 6$) and $\phi = \pi/4$. A reference unit amplitude cosine is also shown.

As a further example, precession and nutation motions of the planets or a spin-stabilized satellite are very close to zero loss. Attitude vibrations of a gravity gradient stabilized satellite are nearly lossless. Micromechanical oscillators in vacuum have nearly zero loss, but their free responses do die out in finite time due to internal dissipation in the constituent materials. Large structures in space (for example solar panels on the Hubble telescope or the main structure of the space station currently in orbit) can exhibit negligible loss, and might well be modeled as having zero damping for many purposes.

So the case we've just studied is idealized, and hard to find in practice. However, with feedback control (to be studied later), it is possible to force a system to net zero loss by providing a driving signal which keeps an oscillation at constant amplitude.

The more typical case is that of finite damping, as studied in the next section. The mathematics with finite damping is slightly more complicated, so keep in mind the overall approach we've just followed in the zero damping case. The approach is essentially the same with finite damping; just keep saying to yourself "complex exponentials are my friend..."

1.2.4 Underdamped case ($0 < \zeta < 1$)

Now we turn attention to solving for the underdamped homogeneous response. Once again, this second-order system has initial conditions given by an initial displacement $x(0) = x_0$ and initial velocity $v(0) = \dot{x}(0) = v_0$. The natural response takes the form as given earlier in (1.37)

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}. \quad (1.49)$$

Since we are considering the underdamped case, then $b^2 < 4mk$, and the roots given by (1.35, 1.36) become

$$\begin{aligned} s_1 &= -\frac{b}{2m} + \frac{j\sqrt{4mk - b^2}}{2m} \\ &\equiv -\sigma + j\omega_d \end{aligned} \quad (1.50)$$

and

$$\begin{aligned} s_2 &= -\frac{b}{2m} - \frac{j\sqrt{4mk - b^2}}{2m} \\ &\equiv -\sigma - j\omega_d. \end{aligned} \quad (1.51)$$

That is, the poles lie at $s = -\sigma \pm j\omega_d$, where

$$\sigma = \zeta\omega_n \quad (1.52)$$

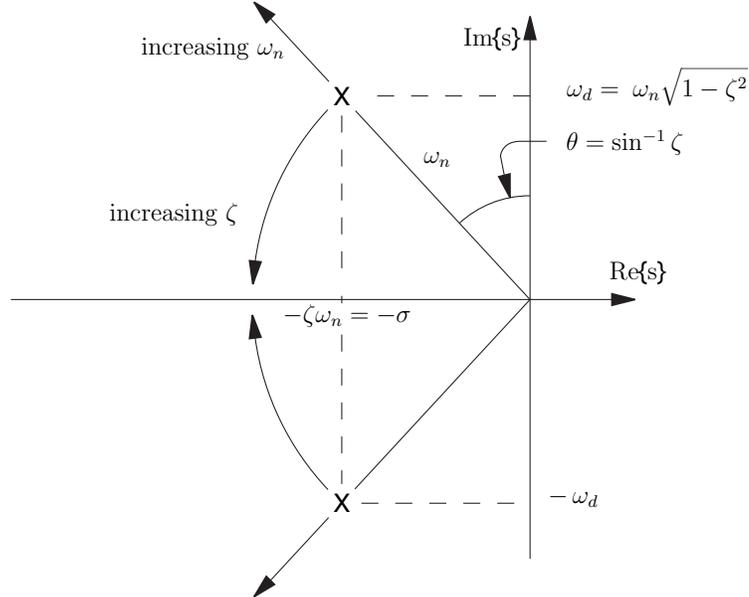


Figure 1.23: Pole locations in the s -plane for second-order mechanical system in the underdamped case ($0 < \zeta < 1$). Arrows show the effect of increasing ω_n and ζ , respectively.

is the *attenuation*, and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (1.53)$$

is the *damped natural frequency*. These pole locations are plotted on the s -plane in Figure 1.23. As shown in the figure, the poles are at a radius from the origin of ω_n and at an angle from the imaginary axis of $\theta = \sin^{-1} \zeta$. The figure also shows the effect of increasing ζ and ω_n . As ζ increases from 0 to 1, the poles move along an arc of radius ω_n from $\theta = 0$ to $\theta = \pi/2$. As ω_n increases, the poles move radially away from the origin, maintaining constant angle $\theta = \sin^{-1} \zeta$, and thus constant damping ratio.

To be more specific, the effect of ζ is shown in Figure 1.24 as ζ takes on the values $\zeta = 0, 0.1, 0.3, 0.7, 0.8$, and 1.0 for a system with $\omega_n = 1$.

Now for the details of developing a solution which meets given initial conditions. Since we have $s_1 = s_2^*$, the solution (1.49) will be real if $c_1 = c_2^*$.

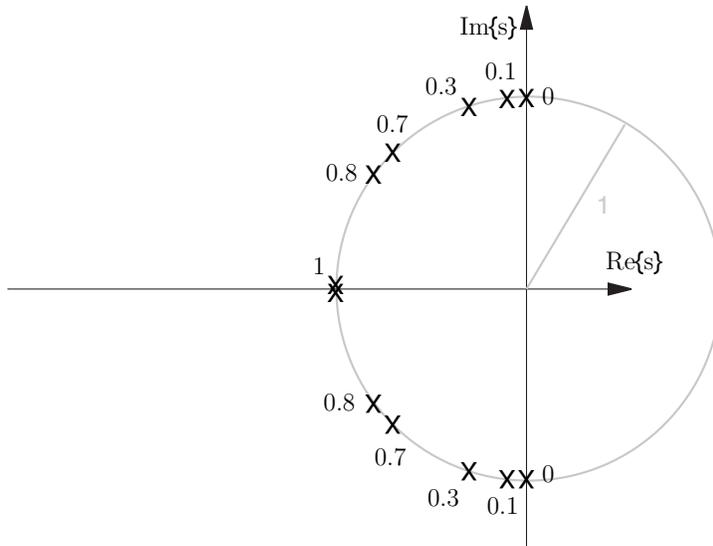


Figure 1.24: Pole locations for $\omega_n = 1$ and $\zeta = 0, 0.1, 0.3, 0.7, 0.8,$ and 1 .

With this constraint, as before, the solution simplifies to

$$x(t) = 2\text{Re}\{c_1 e^{(-\sigma + j\omega_d)t}\}. \quad (1.54)$$

As before, we define $c_1 = \alpha + j\beta$; then the response becomes

$$x(t) = 2\text{Re}\{(\alpha + j\beta)e^{(-\sigma + j\omega_d)t}\} \quad (1.55)$$

$$= 2e^{-\sigma t}\text{Re}\{(\alpha + j\beta)(\cos \omega_d t + j \sin \omega_d t)\} \quad (1.56)$$

$$= 2e^{-\sigma t}(\alpha \cos \omega_d t - \beta \sin \omega_d t). \quad (1.57)$$

The position at $t = 0$ is given by $x_0 = 2\alpha$, and thus to match a specified x_0 we require

$$\alpha = \frac{x_0}{2}. \quad (1.58)$$

Taking the derivative with respect to time gives the velocity as

$$\dot{x}(t) = -2\sigma e^{-\sigma t}(\alpha \cos \omega_d t - \beta \sin \omega_d t) + 2e^{-\sigma t}(-\alpha\omega_d \sin \omega_d t - \beta\omega_d \cos \omega_d t) \quad (1.59)$$

Collecting terms gives

$$\dot{x}(t) = 2e^{-\sigma t}((-\sigma\alpha - \beta\omega_d) \cos \omega_d t + (\sigma\beta - \alpha\omega_d) \sin \omega_d t), \quad (1.60)$$

and thus the initial velocity is

$$v_0 = \dot{x}(0) = 2(-\sigma\alpha - \beta\omega_d) \quad (1.61)$$

Substituting in the earlier result $\alpha = x_0/2$ gives

$$v_0 = 2\left(-\frac{\sigma x_0}{2} - \beta\omega_d\right) \quad (1.62)$$

and thus we can solve for β as

$$\beta = -\frac{(v_0 + \sigma x_0)}{2\omega_d} \quad (1.63)$$

Alternately, the initial condition constant can be expressed in polar notation as we did in the undamped case. That is, let $c_1 = Me^{j\phi}$, with $M = |c_1| = \sqrt{\alpha^2 + \beta^2}$ and $\phi = \arg\{c_1\} = \arctan2(\alpha, \beta)$. With c_1 represented in polar form, the underdamped homogeneous response can be written as

$$x(t) = 2\text{Re}\{Me^{j\phi}e^{(-\sigma + j\omega_d)t}\} \quad (1.64)$$

$$= 2Me^{-\sigma t}\text{Re}\{e^{j(\omega_d t + \phi)}\} \quad (1.65)$$

$$= 2Me^{-\sigma t}\cos(\omega_d t + \phi). \quad (1.66)$$

This more compact form may be more suitable for some analyses, and is also more helpful when hand-sketching this waveform.

To make things specific, consider the response to initial position $x_0 = 0$ and initial velocity $v_0 = 1$.⁵ This yields the values $\alpha = 0$ and $\beta = -1/2\omega_d$. The polar representation is thus $M = 1/2\omega_d$ and $\phi = -\pi/2$. Substituting into either form of the homogeneous solution (1.57, 1.66) gives the response as

$$x(t) = \frac{1}{\omega_d} e^{-\sigma t} \sin \omega_d t. \quad (1.67)$$

Note that this solution is valid for all time, and satisfies the initial conditions imposed at $t = 0$.

The reason for previously defining σ and ω_d is now more clear since the time response is naturally expressed in terms of these variables. Note that since both σ and ω_d scale linearly with ω_n , the response characteristic time-scale decreases as $1/\omega_n$. Also note that for a constant initial velocity, the response amplitude decreases with ω_n . This is so because assuming that m is a fixed value, increasing ω_n while holding ζ constant requires increasing the values of both k and b . Thus, the mass with initial velocity 1 “runs into” a stiffer system, and is returned to rest more rapidly.

This is shown in Figure 1.25 where the response for four values $\omega_n = 10, 20, 50, 100$ is shown with damping ratio held constant at $\zeta = 0.2$. Note that the initial slope of each of the four responses is identical and equal to 1 which, of course, is the initial velocity specified above. The point to retain from Figure 1.25 is that ω_n sets the response time scale with larger values of ω_n corresponding to faster time scales.

Viewed another way, the response can be plotted on axes normalized to $t' = \omega_n t$; this response will then only depend upon the value of ζ , which determines the relative damping. With time normalized as above, the effect of varying values of $\zeta = 0.1, 0.2, 0.5, 0.9$ is shown in the plot of the initial condition response (1.67) in Figure 1.26. In this response, the term $\sin \omega_d t$ provides the oscillatory part. Multiplying by the term $e^{-\sigma t}$ yields the decaying exponential amplitude on the oscillation seen in the figure. As shown in the figure, as ζ (and thus σ) approaches zero the response becomes more lightly damped, due to the fact that the exponential envelope decays more slowly.

The effect of the parameters k , b , and m can also be understood in terms of the s -plane pole locations, as shown in Figure 1.27. Referring to

⁵This initial condition can be established by an impulse in force F_c of area equal to m N-sec.

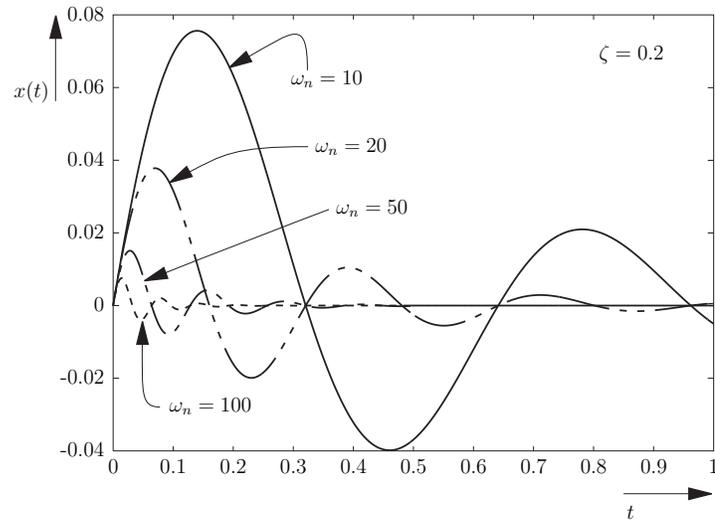


Figure 1.25: Initial condition response ($x_0 = 0$, $v_0 = 1$) for second-order mechanical system in the underdamped case ($0 < \zeta < 1$), with varying values of $\omega_n = 10, 20, 50, 100$, and constant damping ratio $\zeta = 0.2$.

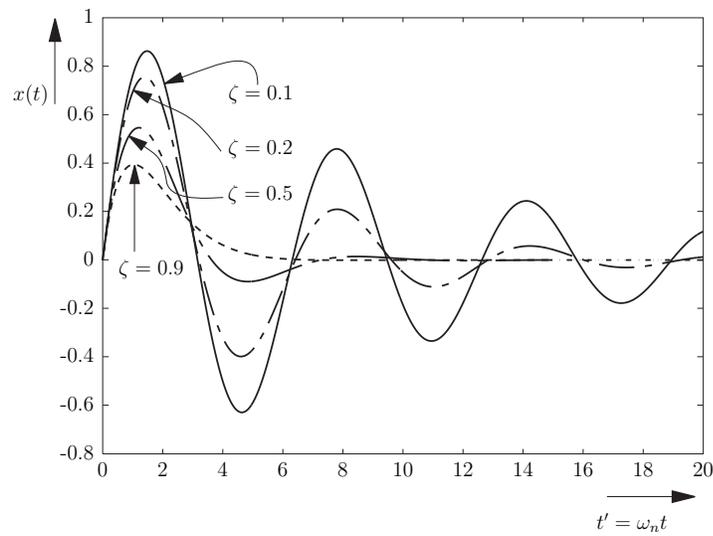


Figure 1.26: Initial condition response for second-order mechanical system in the underdamped case ($0 < \zeta < 1$), with varying values of $\zeta = 0.1, 0.2, 0.5, 0.9$. Time axis is normalized to $t' = \omega_n t$.

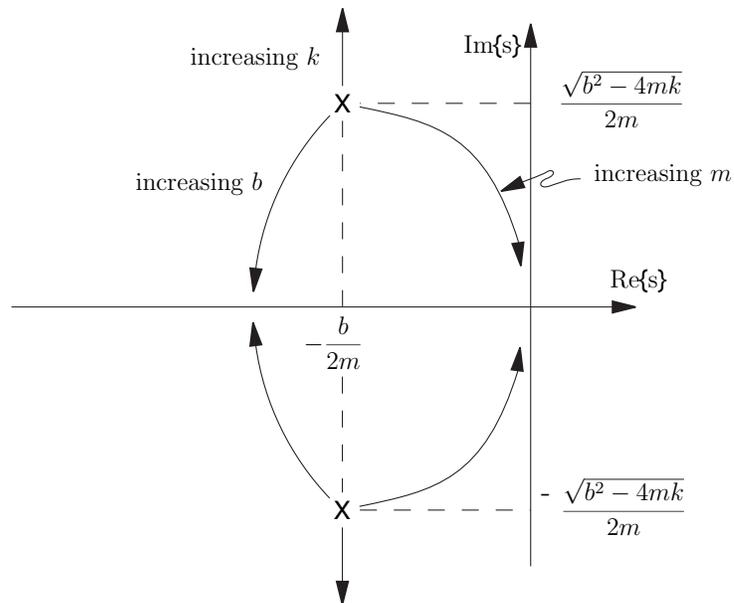


Figure 1.27: Pole locations in the s -plane for second-order mechanical system in the underdamped case ($0 < \zeta < 1$). Arrows show the effect of increasing k , b , and m , respectively.

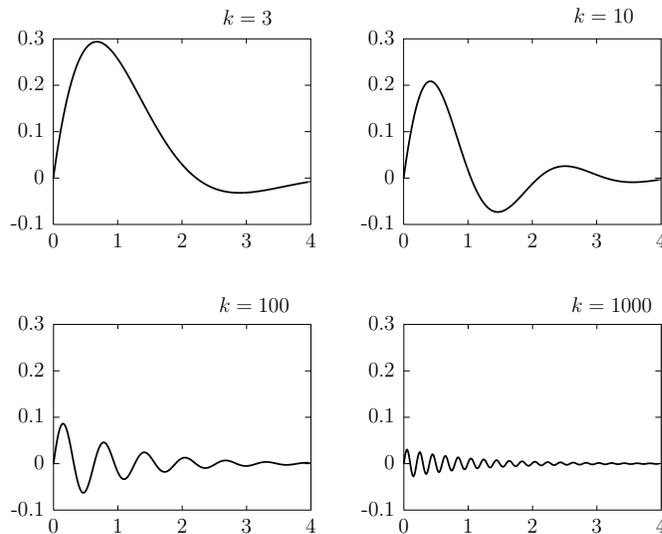


Figure 1.28: Initial condition $v_0 = 1$ response for second-order system with $m = 1$ kg, $b = 2$ N-sec/m, and four values $k = 3, 10, 100, 1000$ N/m.

equations (1.50) and (1.51) we can see that the real part of the poles is unaffected by the choice of spring rate k , and that the magnitude of the imaginary parts grows with increasing k . This effect is shown by the arrows in Figure 1.27 which indicate that the poles move vertically away from the origin with increasing k while maintaining constant real parts.

Thus with increasing k the system natural frequency increases while the damping ratio decreases. Since the real part of the poles remains unchanged (σ stays constant), the decay-rate of the response remains unchanged. This effect is shown in Figure 1.28 where four responses are shown for initial conditions $x_0 = 0, v_0 = 1$ which have constant values of $m = 1$ kg, $b = 2$ N-sec/m. In the four figures, k takes the values $k = 3, 10, 100, 1000$. Note that each response has the same decay rate, as we have argued above. As k is increased, the response amplitude decreases, since the stiffer spring turns the mass around from its initial velocity in decreasing amounts of time. The conclusion to be reached is that increasing k makes the system respond faster, but does not affect the time required to settle to its final value. This is

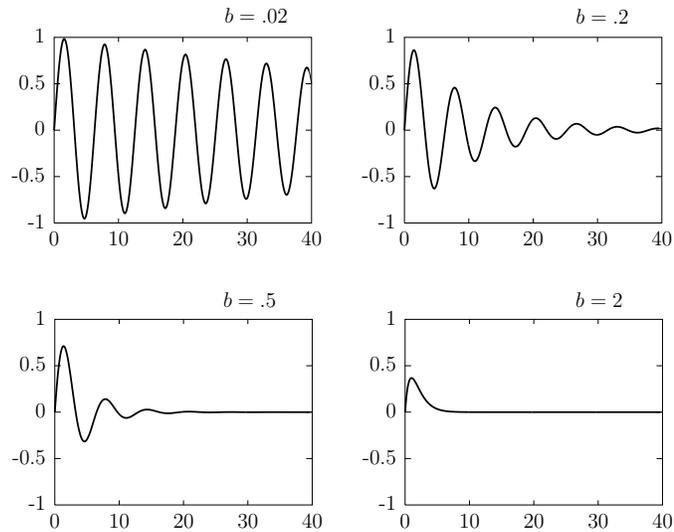


Figure 1.29: Initial condition ($x_0 = 0$, $v_0 = 1$) response for second-order system with $m = 1$ kg, $k = 1$ N/m, and four values $b = .02$, $.2$, $.5$, 2 N-sec/m.

an important lesson relating to the design of mechanical structures. Simply stiffening a structure without adding damping helps in the sense that the amplitude response to disturbances is reduced, but does not help in the sense that the characteristic time required to settle is *not* reduced.

Referring to equations (1.39) and (1.40) we can see that the natural frequency of the poles is unaffected by the choice of damping constant b , and that ζ grows with increasing b . This effect is shown by the arrows in Figure 1.27 which indicate that the poles move toward the real axis with increasing b along a circular arc of constant radius ω_n from the origin. This effect is shown in Figure 1.29 where initial condition ($x_0 = 0$, $v_0 = 1$) responses are shown which have constant values of $m = 1$ kg, $k = 1$ N/m, and thus $\omega_n = 1$. In the four figures, b takes the values $b = .02$, $.2$, $.5$, 2 N-sec/m. The conclusion to be reached is that increasing b makes the system better damped, but does not affect the natural frequency. It does, however, affect the *damped* natural frequency. For relatively light damping, the damped

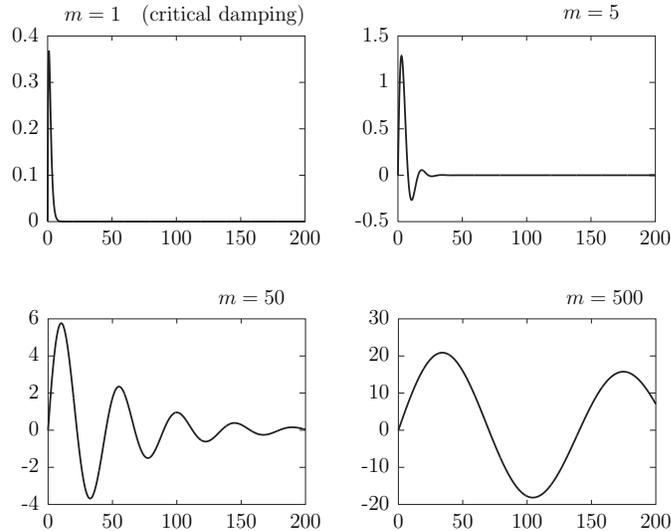


Figure 1.30: Initial condition response ($x_0 = 0$, $v_0 = 1$) for second-order system with $k = 1$ N/m, $b = 2$ N-sec/m, and where m takes the values $m = 1, 5, 50, 500$ kg.

natural frequency is very close to the natural frequency, and thus the period of oscillation does not change materially in the first three plots. However, as the poles approach the real axis, the damped natural frequency approaches zero, and thus is significantly different from the natural frequency which remains constant as b varied. The last trace in the figure shows the critically damped case, in which $\omega_d = 0$ and there is thus no oscillation in the response.

The mass m affects all the system parameters. As m is increased, the natural frequency decreases, and the damping ratio also decreases. Thus, as m increases, the poles move toward the origin along the arc shown in Figure 1.27. This effect is shown in the time-domain in Figure 1.30 where initial condition ($x_0 = 0$, $v_0 = 1$) responses are shown which have constant values of $k = 1$ N/m, $b = 2$ N-sec/m, and where m takes the values $m = 1, 5, 50, 500$ kg. The conclusion to be reached is that as the mass is increased the response characteristic time becomes longer, and simultaneously the response becomes more poorly damped. Note also that the amplitude

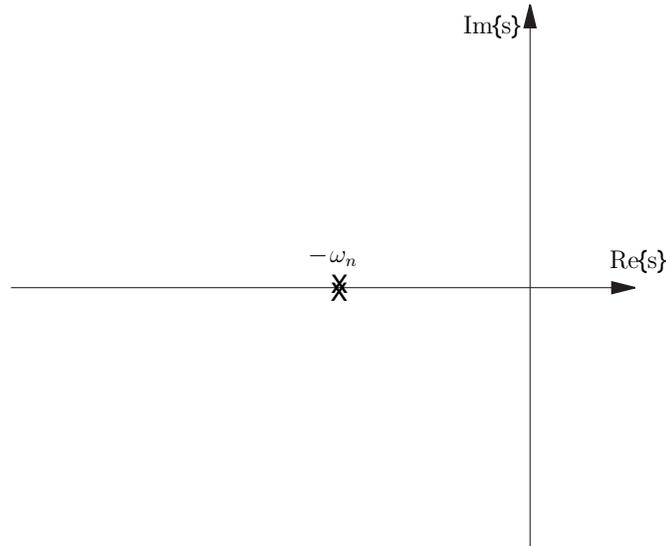


Figure 1.31: Pole locations in the s -plane for second-order mechanical system in the critically-damped case ($\zeta = 1$).

of the transient increases strongly with the increase in mass. This effect demonstrates why an overloaded car can begin to show poor suspension response.

1.2.5 Critically-damped case

In the critically damped case, $\zeta = 1$ and the two poles coincide at $s_1 = s_2 = -\omega_n$. These pole locations are plotted on the s -plane in Figure 1.31. The homogeneous solution takes the form

$$x(t) = c_1 e^{s_1 t} + c_2 t e^{s_1 t} \quad (1.68)$$

$$= c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t} \quad (1.69)$$

where c_1 and c_2 are real numbers. (Compare this expression to the equivalent result for the undamped case; these look nearly identical, but now the expression is composed of pure real terms.)

In what is now a familiar theme, we use the initial condition specifications to set the values of the parameters c_1 and c_2 as follows. At $t = 0$, the position is $x_0 \equiv x(0) = c_1$; this sets the first parameter. Taking the time derivative gives the velocity as

$$\dot{x}(t) = -\omega_n c_1 e^{-\omega_n t} + c_2 e^{-\omega_n t} - \omega_n t c_2 e^{-\omega_n t} \quad (1.70)$$

$$= (c_2 - \omega_n c_1 - \omega_n t c_2) e^{-\omega_n t}. \quad (1.71)$$

At $t = 0$, the velocity is $v_0 \equiv \dot{x}(0) = c_2 - \omega_n c_1$. Substituting in with $c_1 = x_0$ and rearranging gives the second parameter as $c_2 = v_0 + \omega_n x_0$.

To make things specific, consider the response to an initial position $x_0 = 0$ and initial velocity $v_0 = 1$. The parameters then become $c_1 = 0$ and $c_2 = 1$, and thus the homogenous response is

$$x(t) = t e^{-\omega_n t}. \quad (1.72)$$

You've already seen a plot of this response with $\omega_n = 1$ in the last panel of Figure 1.29, and the first panel of Figure 1.30. Take a look at these again. For a given ω_n this is the fastest natural response which exhibits no overshoot as it returns to zero. For this reason, mechanisms and control systems are sometimes tuned for critical damping, or to be only slightly underdamped as it is desirable to have a system which responds quickly, but with little or no overshoot.

1.2.6 Overdamped case

In both the critically damped case and the overdamped case, the description of the pole locations in terms of ζ and ω_n , while mathematically consistent, is not of as great utility as in the underdamped case. Since the poles are real for $\zeta \geq 1$, they can most readily be described in terms of their time constants and viewed as two separate first-order systems. However, for purposes of understanding, we continue the description in terms of the second-order parameters.

In the overdamped case, $\zeta > 1$ and the two poles are at separate locations on the real axis: $s_1 = -(\zeta - \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = -(\zeta + \sqrt{\zeta^2 - 1})\omega_n$. These pole locations are plotted on the s -plane in Figure 1.32. Note that $|s_2| > |s_1|$, and thus the pole at s_1 is closer to the origin in the s -plane than s_2 . In the limit as ζ approaches infinity, the root s_1 will approach the origin, while the root s_2 approaches infinity.⁶

⁶As another way of looking at things, it is interesting to note that if the mass m

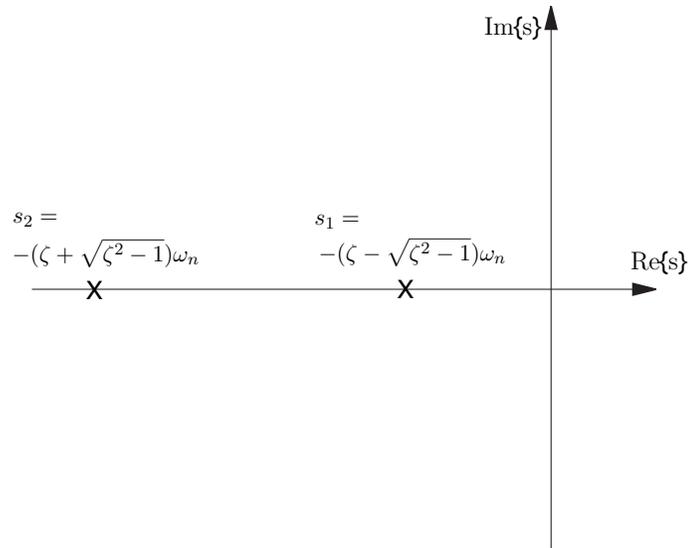


Figure 1.32: Pole locations in the s -plane for second-order mechanical system in the overdamped case ($\zeta > 1$).

The homogeneous solution takes the familiar form

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \quad (1.73)$$

$$= c_1 e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} + c_2 e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \quad (1.74)$$

where c_1 and c_2 are real numbers.

We use the initial condition specifications to set the values of the parameters c_1 and c_2 as follows. At $t = 0$, the position is $x_0 \equiv x(0) = c_1 + c_2$. Taking the time derivative in (1.73) gives the velocity as

$$\dot{x}(t) = c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t}. \quad (1.75)$$

At $t = 0$, the velocity is $v_0 \equiv \dot{x}(0) = c_1 s_1 + c_2 s_2$. Substituting in with $c_2 = x_0 - c_1$ and rearranging gives

$$c_1 = \frac{v_0 + x_0 s_2}{s_2 - s_1}, \quad (1.76)$$

and

$$c_2 = -\frac{v_0 + x_0 s_1}{s_2 - s_1}. \quad (1.77)$$

To make things specific, consider the response to an initial position $x_0 = 0$ and initial velocity $v_0 = 1$. With these initial conditions, the parameters become $c_1 = -c_2 = 1/(s_2 - s_1) = 1/(2\omega_n \sqrt{\zeta^2 - 1})$. The response is thus given by

$$x(t) = \frac{1}{2\omega_n \sqrt{\zeta^2 - 1}} (e^{s_1 t} - e^{s_2 t}), \quad (1.78)$$

valid for all time. Plots of this response for $\omega_n = 1$ and $\zeta = 1, 2, 5, 10$ are shown in Figure 1.33.

Note that the response for large ζ is approximately first-order with a time constant of $-1/s_1$. This happens because the exponential with the larger pole magnitude decays more quickly than the exponential with the smaller pole magnitude. Thus we see that for stable systems, poles with smaller real part magnitude in the s -plane dominate the time response. As a practical matter, we can ignore in the time response any poles with real parts greater than a factor of 5–10 larger in magnitude than the magnitude of the real part of the dominant poles. Of course, any unstable poles can never be ignored, no matter how far they are from the origin, since their natural response grows exponentially with time.

is allowed to approach zero, and the limits on s_1 and s_2 are properly taken, then s_1 approaches the value $-k/b$, and s_2 approaches infinity. Thus the second-order system in this limit of zero mass properly devolves to the first order case studied in Section 1.1.1.

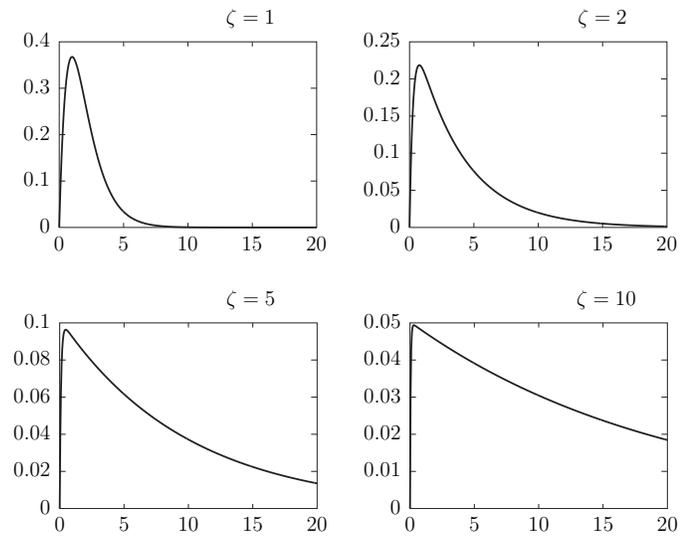


Figure 1.33: Initial condition response for second-order system in the overdamped case, with $\omega_n = 1$ and $\zeta = 1, 2, 5, 10$.

1.2.7 Canonical second-order form

The results of the previous section are generalizable to other systems which can be modeled with second-order linear constant coefficient differential equations. To summarize these results in a general way:

The canonical form of the second-order homogeneous system (1.41) is repeated here for reference

$$\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = 0. \quad (1.79)$$

Any second-order linear constant coefficient homogeneous differential equation can be written in this form and then the parameters ζ and ω_n read off by inspection. Then the types of responses plotted in the previous sections for the mechanical system will be directly applicable to many types of physical systems including electrical, thermal, and fluidic as described in the next few sections.

We will also find it helpful to collect here for reference the definition of the real and imaginary parts of the poles. The characteristic equation associated with (1.79) is

$$\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1 = 0. \quad (1.80)$$

The roots of this characteristic equation are $s = -\sigma \pm j\omega_d$, where

$$\sigma = \zeta\omega_n \quad (1.81)$$

is the *attenuation*, and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (1.82)$$

is the *damped natural frequency*. Again, these results are applicable to any system which can be expressed in the form (1.79).

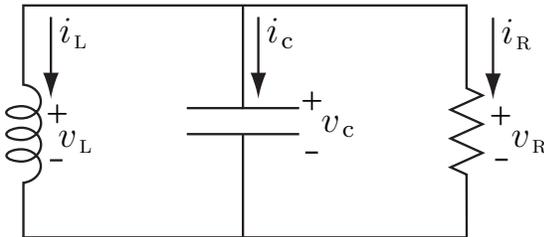
The natural frequency ω_n determines the time-scale of the response; we will assume without loss of generality that $\omega_n > 0$. Then the damping ratio ζ determines much of the character of the natural response:

- If $\zeta < 0$, then such a second-order system is unstable in that the natural response grows in time without bound.
- If $\zeta = 0$, then such a second-order system is marginally stable in that the natural response is of constant amplitude in time. This is the *undamped* case studied earlier.

- If $\zeta > 0$, then such a second-order system is *stable* in that the natural response decays exponentially to zero in time. This stable case is further subdivided into three possibilities:
- If $0 < \zeta < 1$, then such a second-order system is *underdamped*, the poles have imaginary components, and the natural response contains some amount of oscillatory component. Lower values of ζ correspond with relatively more oscillatory responses, i.e., are more lightly damped.
- If $\zeta = 1$, then such a second-order system is *critically damped*, and the poles are coincident on the negative real axis at a location $-\omega_n$.
- If $\zeta > 1$, then such a second-order system is *overdamped*, and the poles are at distinct locations on the negative real axis. This case can also be thought of as two independent first-order systems.

1.2.8 Electrical second-order system

The electrical circuit shown below can be described by a 2^{nd} order homogeneous differential equation. We will see that it exhibits responses which are analogous to the 2^{nd} order mechanical system studied earlier.



First, recall the constitutive relationships

$$\begin{aligned} v_R &= i_R R \\ v_L &= L \frac{di_L}{dt} \\ i_C &= C \frac{dv_C}{dt} \end{aligned}$$

The three elements are in parallel, and thus have equal voltages.

$$v_L = v_C = v_R \triangleq v$$