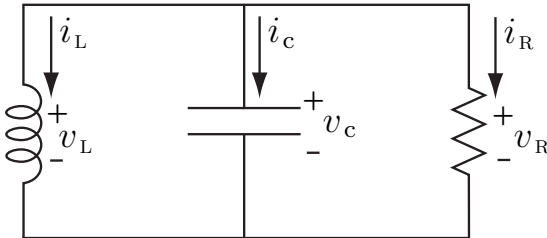


- If  $\zeta > 0$ , then such a second-order system is *stable* in that the natural response decays exponentially to zero in time. This stable case is further subdivided into three possibilities:
- If  $0 < \zeta < 1$ , then such a second-order system is *underdamped*, the poles have imaginary components, and the natural response contains some amount of oscillatory component. Lower values of  $\zeta$  correspond with relatively more oscillatory responses, i.e., are more lightly damped.
- If  $\zeta = 1$ , then such a second-order system is *critically damped*, and the poles are coincident on the negative real axis at a location  $-\omega_n$ .
- If  $\zeta > 1$ , then such a second-order system is *overdamped*, and the poles are at distinct locations on the negative real axis. This case can also be thought of as two independent first-order systems.

### 1.2.8 Electrical second-order system

The electrical circuit shown below can be described by a 2<sup>nd</sup> order homogeneous differential equation. We will see that it exhibits responses which are analogous to the 2<sup>nd</sup> order mechanical system studied earlier.



To develop a differential equation for this circuit in terms of the common voltage across all three elements, first recall the constitutive relationships

$$\begin{aligned} v_R &= i_R R \\ v_L &= L \frac{di_L}{dt} \\ i_C &= C \frac{dv_C}{dt}. \end{aligned}$$

The three elements are in parallel, and thus have equal voltages

$$v_L = v_C = v_R \triangleq v.$$

Another way to say this is that the voltage  $v$  is common to all three elements. Also, by Kirchoff's current law (KCL), currents entering (leaving) a node must sum to zero. As a convention, we will count currents as positive when they are entering a node. Counting entering as positive, the currents at the upper node sum as

$$-i_L - i_C - i_R = 0 \tag{1.83}$$

Rearranging the constitutive equations in terms of the common voltage  $v$  yields

$$\begin{aligned} i_R &= \frac{v}{R} \\ i_C &= C \frac{dv}{dt} \\ i_L &= \frac{1}{L} \int v dt. \end{aligned}$$

Substituting into (1.83) gives

$$\frac{1}{L} \int v dt + C \frac{dv}{dt} + \frac{1}{R} v = 0$$

Taking the derivative of each term and multiplying through by  $L$  yields

$$LC \frac{d^2v}{dt^2} + \frac{L}{R} \frac{dv}{dt} + v = 0$$

If we compare with the canonical 2<sup>nd</sup> order form (1.79), the standard parameters can be identified as

$$\begin{aligned} \omega_n &= \frac{1}{\sqrt{LC}} \\ \zeta &= \frac{1}{2R} \sqrt{\frac{L}{C}} \end{aligned}$$

The associated initial conditions are

$$v(0) = v_C(0)$$

an

$$\dot{v}(0) = \frac{dv_C}{dt}(0) = \frac{i_C(0)}{C} = \frac{1}{C}(-i_L(0) - i_R(0)) = \frac{1}{C}(-i_L(0) - \frac{v_C(0)}{R}).$$

With these expressions, we have the initial conditions in terms of the most typical state variables, which are the inductor current and the capacitor voltage. The solution techniques and graphs of response from the discussion of the second-order mechanical system in Section 1.2.2 are thus applicable, and the modeling portion of this problem is complete.

### 1.2.9 Thermal second-order system

We previously studied the temperature variation of a light bulb with a first-order model of bulb temperature. To gain additional insight, suppose we also include a model of the filament temperature inside the bulb. This requires adding an additional thermal capacitance  $C_f$  with its own temperature  $T_f$  to represent the filament heat energy storage and temperature, respectively. These elements augment the model which previously only accounted for the temperature on the outside of the bulb  $T_b$  and associated thermal capacitance  $C_b$ .

We take some liberties here with the 4<sup>th</sup> power radiative heat transfer relationship and assume that heat flow from the filament to the bulb can be represented by the linear thermal resistance  $R_f$ . This is a significant simplification, in that the filament temperature in a typical incandescent bulb might be on the order of 2500 °C. At this temperature, and given that the filament operates in a vacuum inside the glass envelope, essentially all of the heat transfer from the filament takes place via radiation, and thus depends upon the temperature difference to the 4<sup>th</sup> power. Therefore the heat transfer will be strongly nonlinear with filament temperature. To model this in detail, we would need to include this nonlinear thermal resistance in the model differential equation, and use some nonlinear analysis or simulation technique to find the solution. We will revisit this problem in later sections of these notes after introducing techniques for analyzing and simulating nonlinear systems.

Even with the limitations discussed above, it is still useful to postulate a linear heat transfer mechanism from the filament to the bulb envelope which can be represented by a linear thermal resistance  $R_f$ . This will allow us to gain insight into the rapid time variation associated with the filament temperature when a bulb is turned on or off, and see how this fast time constant couples to the relatively slow thermal time constant associated with the envelope temperature equilibration.

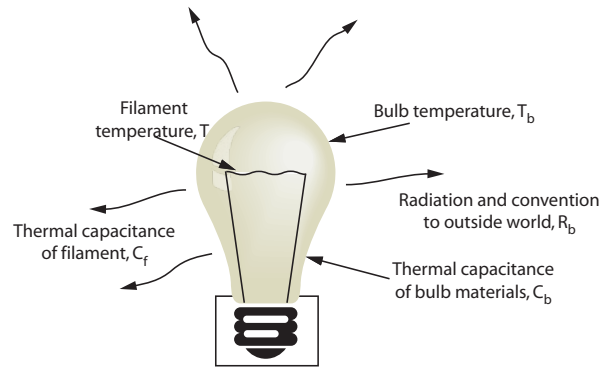


Figure 1.34: Light bulb as a thermal second-order system.

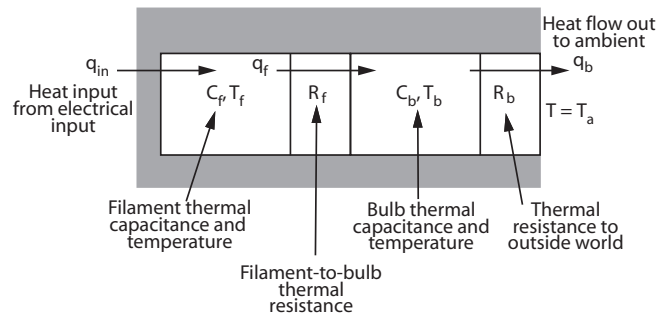


Figure 1.35: Lumped second-order model for light bulb.

The augmented model including filament temperature is shown schematically in Figure 1.34 and in lumped thermal element form in Figure 1.35.

Here the relevant temperature variables are:

$T_f$  = filament temperature [ $^{\circ}K$ ]

$T_b$  = bulb envelope temperature [ $^{\circ}K$ ]

$T_a$  = ambient temperature [ $^{\circ}K$ ]

We also show the heat flows through the system with the reference directions as indicated in the figure:

$q_{in}$  = heat input from filament [ $W$ ] (supplied from electrical source)

$q_f$  = heat flow from filament to bulb [ $W$ ]

$q_b$  = heat flow from bulb to ambient [ $W$ ]

Note that the heat flow into the system is from electrical heating of the filament. In steady-state, this heat flows through the thermal resistances, and finally out into the ambient temperature environment, which is assumed to have a constant temperature  $T_a$  independent of any heat flow value.

The thermal capacitances  $C_f$  and  $C_b$  represent heat storage in the filament and bulb envelope, respectively. These have units of  $[\frac{J}{^\circ K}]$ . The thermal resistances  $R_f$  and  $R_b$  represent the heat-flow connections between the filament/envelope and envelope/ambient, respectively. These have units of  $[\frac{^\circ C}{W}]$ . The heat flows  $q_f$  and  $q_b$  represent the power flow in  $[W]$  through the thermal resistances.

The constitutive relations for the elements are given as follows. For the thermal capacitances, we have

$$C \frac{dT}{dt} = q$$

where  $T$  is the temperature of the capacitance element, and  $q$  is the heat flow *into* the capacitance. For the thermal resistances, we have

$$\Delta T = qR$$

where  $\Delta T$  is temperature drop across resistance and  $qR$  is heat flow through resistance in the defined direction of temperature drop.

To solve thermal problems, we will sum heat flows at the thermal capacitances. At the two thermal capacitances this yields:

$$\begin{aligned} C_f \frac{dT_f}{dt} &= q_{in} - q_f \\ C_b \frac{dT_b}{dt} &= q_f - q_b \end{aligned}$$

Note carefully how the sign convention has been observed for the heat flows with respect to the reference directions defined in Figure 1.35.

Now the constitutive relationships for the resistors give

$$\begin{aligned} q_f &= \frac{1}{R_f}(T_f - T_b) \\ q_b &= \frac{1}{R_b}(T_b - T_a). \end{aligned}$$

Combining these with the earlier pair of equations yields

$$C_f \frac{dT_f}{dt} = -\frac{1}{R_f}(T_f - T_b) + q_{in} \quad (1.84)$$

$$C_b \frac{dT_b}{dt} = \frac{1}{R_f}(T_f - T_b) - \frac{1}{R_b}(T_b - T_a) \quad (1.85)$$

Equations (1.84) and (1.85) give a *state-space* model of the system. The state variables are the two temperatures  $T_f$  and  $T_b$  which represent storage of heat energy in the system. This form of model can be solved directly to find the initial condition response of the system given the initial values  $T_f(0)$  and  $T_b(0)$ ; this model form can also be used to solve for the driven response when we consider the filament heating  $q_{in}$  as an input.

We will discuss state-space models in later sections of these notes. For the present, we will solve the response of this system using classical techniques. To take this classical approach, we need to transform (1.84) and (1.85) into a single 2<sup>nd</sup> order differential equation. This representation can then be solved with the techniques presented in this chapter.

In order to transform (1.84) and (1.85) into a second-order ODE, we need to choose a single variable to represent the system and then write a second-order ODE in this variable. Let's focus on the bulb envelope temperature,  $T_b$ . Thus, we need to eliminate  $T_f$  from (1.84) and (1.85).

To facilitate this, it is helpful to introduce *operator* notation. That is, the operator  $D$  represents differentiation

$$Df(t) \triangleq \frac{df(t)}{dt} = f'(t)$$

$$D^2f(t) \triangleq \frac{d^2f(t)}{dt^2} = f''(t)$$

and so forth.

With this notation and some rearranging, equation (1.84) becomes

$$(R_f C_f D + 1)T_f = T_b + R_f q_{in}.$$

Taking the inverse operation  $(R_f C_f D + 1)^{-1}$  on both sides gives

$$T_f = (R_f C_f D + 1)^{-1}(T_b + R_f q_{in}).$$

Substitution of this result into (1.85) yields

$$[R_b C_b D + 1 + \frac{R_b}{R_f} - \frac{R_b}{R_f}(R_f C_f D + 1)^{-1}]T_b = R_b(R_f C_f D + 1)^{-1}q_{in} + T_a.$$

Now we multiply through by the operator  $(R_f C_f D + 1)$  to give

$$[(R_b C_b D + 1 + \frac{R_b}{R_f})(R_f C_f D + 1) - \frac{R_b}{R_f}]T_b = R_b q_{in} + (R_f C_f D + 1)T_a,$$

and thus

$$[R_b C_b R_f C_f D^2 + (R_f + R_b)C_f D + R_b C_b D + 1]T_b = R_b q_{in} + (R_f C_f D + 1)T_a.$$

Next we convert back into explicit differential equation form to give a 2<sup>nd</sup> order differential equation

$$R_b C_b R_f C_f \frac{d^2 T_b}{dt^2} + ((R_f + R_b) C_f + R_b C_b) \frac{dT_b}{dt} + T_b = R_b q_{in} + R_f C_f \frac{dT_a}{dt} + T_a$$

where  $q_{in}$ ,  $dT_a/dt$ , and  $T_a$  are inputs.

For the purposes of this problem we will view the ambient temperature as a constant, and thus  $dT_a/dt = 0$ . Further, it is useful to define the temperature *difference*  $T \triangleq T_b - T_a$ . With these definitions

$$\frac{dT_a}{dt} = 0$$

and thus

$$\frac{dT}{dt} = \frac{dT_b}{dt}$$

and

$$\frac{d^2 T}{dt^2} = \frac{d^2 T_b}{dt^2}.$$

With  $dT_a/dt = 0$ , and rewriting in terms of  $T$ , we have

$$R_b C_b R_f C_f \frac{d^2 T}{dt^2} + ((R_f + R_b) C_f + R_b C_b) \frac{dT}{dt} + T = R_b q_{in}.$$

Now suppose we set the system parameters as

$$\begin{aligned} R_b &= 2 \text{ [}^\circ\text{K/W]} \\ R_f &= 40 \text{ [}^\circ\text{K/W]} \\ C_b &= 65 \text{ [J/}^\circ\text{C]} \\ C_f &= 10^{-3} \text{ [J/}^\circ\text{C]} \end{aligned}$$

(These have been picked to reasonably fit the response of a real 60W bulb.)

With these parameters, note that  $R_f \gg R_b$ , and thus the term  $(R_f + R_b) \simeq R_f$ , and the differential equation can be written approximately as

$$R_b C_b R_f C_f \frac{d^2 T}{dt^2} + (R_f C_f + R_b C_b) \frac{dT}{dt} + T = R_b q_{in}.$$

The characteristic equation associated with this differential equation is

$$R_b C_b R_f C_f s^2 + (R_f C_f + R_b C_b) s + 1 = 0.$$

This can be directly factored as

$$(R_b C_b s + 1)(R_f C_f s + 1) = 0.$$

showing that the system has two real axis poles in the left half plane. (An exact analysis will show that the approximation  $R_f \gg R_b$  used above gives a reasonable estimate of the pole locations, and allows us to write these pole locations in the simple form given above.) Plugging in numbers gives

$$\begin{aligned} R_b C_b &= 130 \text{ sec} \triangleq \tau_b \\ R_f C_f &= 0.04 \text{ sec} \triangleq \tau_f \end{aligned}$$

Notice the two widely different time constants. The long-time scale  $\tau_b = 130$  sec is associated with the thermal transient on the bulb envelope. It takes on the order of  $5\tau_b \simeq 10$  minutes for the bulb temperature to equilibrate. The short-time scale  $\tau_a = 0.04$  sec is associated with the thermal transient of the filament. This small piece of coiled wire heats-up/cool-down far more rapidly than the glass bulb envelope. It takes on the order of  $5\tau_a = 0.2$  sec for the filament temperature to equilibrate. This fits with our experience in watching the filament temperature collapse rapidly when an incandescent bulb is turned off.

Note that this system is highly overdamped. From the approximate differential equation we can identify  $\omega_n = 1/\sqrt{R_b C_b R_f C_f}$  and  $2\zeta/\omega_n = R_f C_f + R_b C_b$ . On substitution of numerical values we find that  $\omega_n = 0.44$  rad/sec and  $\zeta = 57$ . The system poles are then at

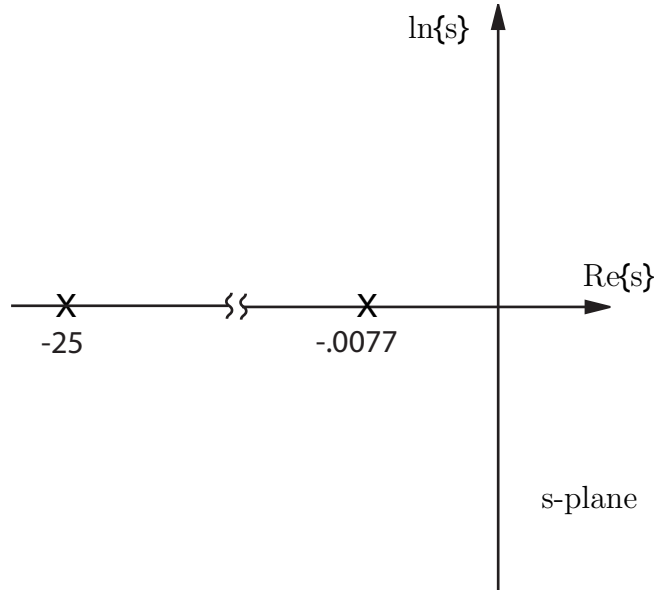
$$\begin{aligned} s_b &= -\frac{1}{\tau_b} = -7.7 \times 10^{-3} \text{ sec}^{-1} \\ s_f &= -\frac{1}{\tau_f} = -25 \text{ sec}^{-1}. \end{aligned}$$

The natural response will thus be of the form

$$T(t) = c_1 e^{-\frac{t}{\tau_b}} + c_2 e^{-\frac{t}{\tau_f}}$$

which can be set to match any given initial conditions by appropriate choice of the constants  $c_1$  and  $c_2$ . This completes the modeling of the bulb thermal system with linear resistances.





### 1.2.10 Fluidic second-order system

We now consider a fluidic second-order example which gives some insight into the phenomenon of water-hammer. In your home you may sometimes have noticed a brief banging sound in the walls at the moment when an open faucet valve is closed suddenly. This water-hammer effect is due to the sudden deceleration of water moving in the pipes. The situation can be represented schematically as shown in Figure 1.36.

The pressure tank is a fluid accumulator which can be modeled as a fluidic capacitance sufficiently large, in concert with the well pump and pressure controller, to be modeled as a pressure source with a pressure of about  $3 \times 10^5$  Pa (6895 Pa = 1 psi). Note: Pressure in this problem is measured with respect to atmosphere.

A long run of pipe carries the water upstairs to a shower, where flow is controlled by a valve. We will model the pipe as having a fluidic resistance,  $R$ , and a fluidic inertance,  $L$ . The valve will be modeled as an on/off switch. A schematic of this connection might appear as shown in Figure 1.37.

The fluidic capacitance shown represents the compliance of the pipes and valve, as well as the compressibility of the water plus any entrained air. The flow through the pipe is  $q(t)$  [ $\text{m}^3/\text{sec}$ ].

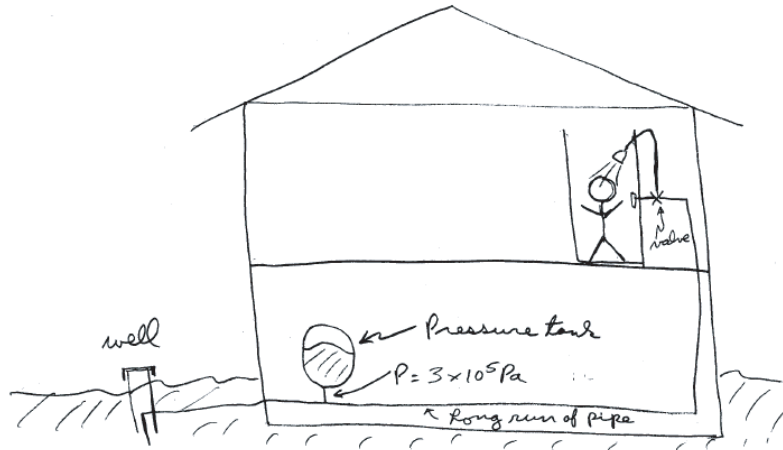


Figure 1.36: Simple model of home water supply system and upstairs shower.

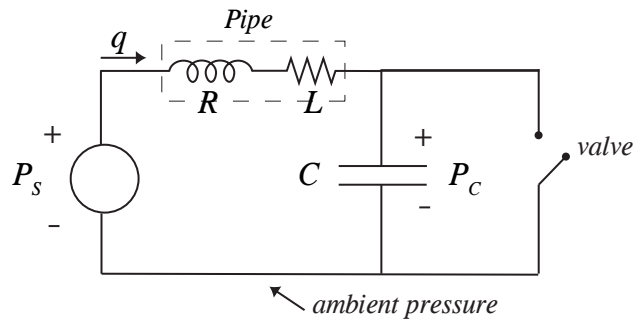


Figure 1.37: Electrical circuit equivalent of plumbing system. The shower valve is modeled as a switch which can be opened or closed with no in-between state. Note that a closed shower valve corresponds to an open electrical switch; no flow is permitted.

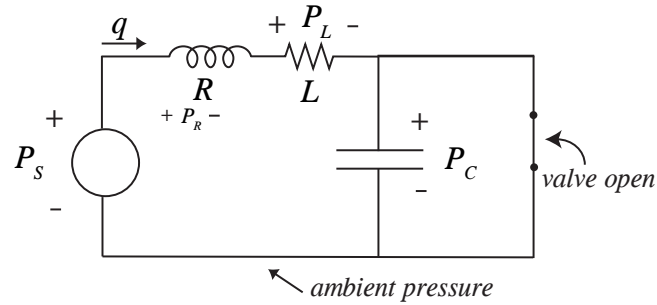


Figure 1.38: Electrical equivalent to fluid system with switch closed to allow flow.

We will assume the valve is initially allowing full flow, that is the valve is open. This full-flow condition corresponds with the electrical equivalent shown in Figure 1.38 in which the electrical switch is closed to allow current to flow.

Thus, the initial state of the system is given by

$$\begin{aligned} q(t < 0) &= \frac{P_s}{R} \\ P_c(t < 0) &= 0. \end{aligned}$$

At  $t = 0$ , the valve opens (switch closes), and the system configuration changes to that shown in Figure 1.39. The initial conditions for this system are  $q(0) = \frac{P_s}{R}$  and  $P_c(0) = 0$ . Note also that with the valve closed (switch open), the flow  $q$  is common to all three elements, and thus

$$\begin{aligned} P_R &= qR \\ P_L &= L \frac{dq}{dt} \\ q &= C \frac{dP_c}{dt} \Rightarrow P_c = \frac{1}{C} \int q dt \end{aligned}$$

The connection also imposes  $P_c + P_L + P_R = P_s$ , and thus

$$qR + L \frac{dq}{dt} + P_c = P_s$$

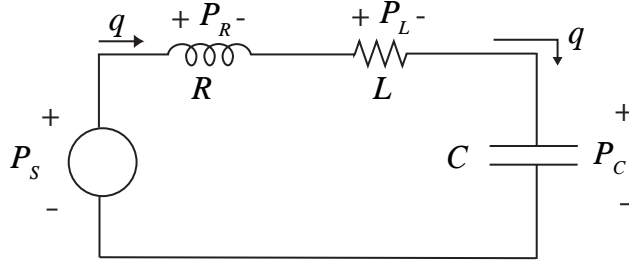


Figure 1.39: Electrical equivalent to fluid system with switch open, corresponding to blocking flow. With switch open, that leg of the circuit has been removed to clarify the diagram.

Substituting from above gives

$$RC \frac{dP_c}{dt} + LC \frac{d^2 P_c}{dt^2} + P_c = P_s$$

If we define  $P = P_c - P_s$ , and recognizing that  $P_s$  is a constant yields  $\frac{dP_s}{dt} = 0$ , this can be rewritten as a homogeneous equation:

$$LC \frac{d^2 P}{dt^2} + RC \frac{dP}{dt} + P = 0$$

Comparing with the standard second-order form, we see that:

$$LC = \frac{1}{\omega_n^2} \Rightarrow \omega_n = \frac{1}{\sqrt{LC}}$$

$$RC = \frac{2\zeta}{\omega_n} \Rightarrow \zeta = \frac{1}{2} RC \omega_n = \frac{R}{2} \sqrt{\frac{C}{L}}$$

We also need the associated initial conditions. With the valve initially open (shower flowing), and with zero valve resistance, the pressure at the valve at  $t = 0$  will be equal to the ambient outside pressure. The initial valve pressure is thus

$$P(0) = P_c(0) - P_s = 0 - P_s = -P_s.$$

We also need  $\frac{dP}{dt}(0)$ . This comes from  $q(0)$  as follows. We know that

$$q = C \frac{dP_c}{dt}$$

and thus

$$\frac{dP_c}{dt} = \frac{dP}{dt} = \frac{1}{C}q(0) = \frac{P_s}{RC}$$

and thus

$$\frac{dP}{dt}(0) = \frac{P_s}{RC}.$$

Assuming distinct roots, the homogeneous solution is

$$P(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

where the eigenvalues are

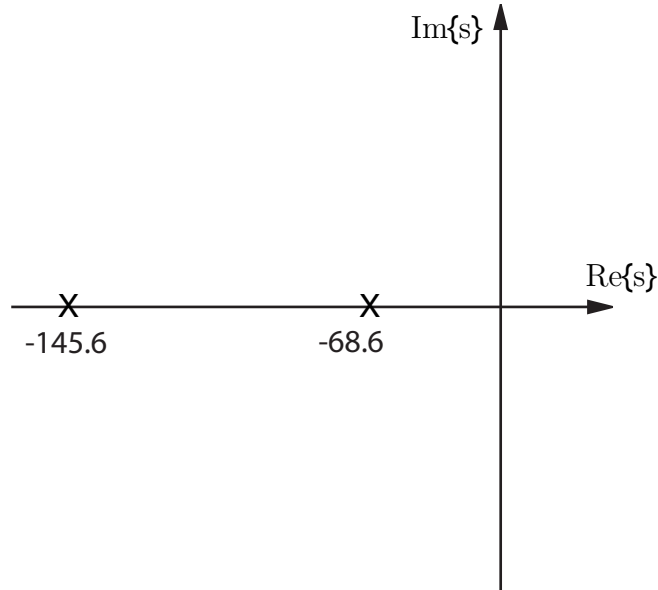
$$\begin{aligned} s_1 &= -\frac{R}{2L} + \frac{\sqrt{R^2 C^2 - 4LC}}{2LC} \\ s_2 &= -\frac{R}{2L} - \frac{\sqrt{R^2 C^2 - 4LC}}{2LC} \end{aligned}$$

A reasonable set of numerical parameters might be

$$\begin{aligned} R &= 1.5 \times 10^9 \left[ \frac{\text{Pa sec}}{\text{m}^3} \right] \\ L &= 7 \times 10^6 \left[ \frac{\text{Pa sec}^2}{\text{m}^3} \right] \\ C &= \frac{1}{7} \times 10^{-10} \left[ \frac{\text{m}^3}{\text{Pa}} \right] \end{aligned}$$

and thus  $\omega_n = 100 \frac{\text{rad}}{\text{sec}}$ , and  $\zeta = 1.07$ , which corresponds to an overdamped system. The roots are thus located at:

$$\begin{aligned} s_1 &= -107.1 + 38.5 = -68.6 \\ s_2 &= -107.1 - 38.5 = -145.6 \end{aligned}$$



The numerical values of the initial conditions are

$$P(0) = -P_S = -3 \times 10^5 = c_1 + c_2 \text{ [Pa]}$$

$$\frac{dP}{dt}(0) = \frac{P_S}{RC} = \frac{3 \times 10^5}{1.5 \times 10^9 \cdot \frac{1}{7} \times 10^{-10}} = 1.4 \times 10^7 \left[ \frac{\text{Pa}}{\text{sec}} \right] = c_1 s_1 + c_2 s_2$$

From the first relationship

$$c_2 = P(0) - c_1.$$

Plugging into the second gives

$$\dot{P}(0) = c_1 s_1 + (P(0) - c_1) s_2.$$

and thus

$$\dot{P}(0) - P(0) s_2 = c_1 (s_1 - s_2)$$

and so

$$c_1 = \frac{\dot{P}(0) - P(0) s_2}{s_1 - s_2} = -3.85 \times 10^5 \text{ [Pa]}$$

$$c_2 = P(0) - c_1 = 8.5 \times 10^4 \text{ [Pa]}$$

Thus the pressure as a function of time is

$$P(t) = -3.85 \times 10^5 e^{-68.6t} + 8.5 \times 10^4 e^{-145.6t}.$$

A plot of this time function and the two constituent modes is shown in Figure 1.40.

Note that pressure  $P(t)$  varies rapidly from an initial value of  $P(0) = -3 \times 10^5$  [Pa] to exponentially approach a final value of  $P(\infty) = 0$ . Since we have defined  $P(t) = P_c(t) - P_s$ , this means that the pressure at the valve varies rapidly from an initial value of  $P_c(0) = 0$  to a final value of  $P_c(\infty) = P_s = 3 \times 10^5$  Pa. This sudden variation in pressure exerts a near step in force on the end of the pipe of  $F = P_c \cdot A$ , where  $A$  is the pipe area. If the pipe inner diameter is  $2 \times 10^{-2}$  m then  $A = \frac{\pi D^2}{4} \cong 3 \times 10^{-4}$  m, and thus  $F = 3 \times 10^5 \cdot 3 \times 10^{-4} = 90$  N. This significant force is applied to the valve body which is supported by the wall framing members. The step load on the wall results in the characteristic banging sound. (Similar forces also occur where the pipe makes turns in its path through the walls, and thus you hear banging throughout the plumbing system when you suddenly shut off the shower.) Be kind to your plumbing and turn the valve off a bit slowly!

### 1.3 Higher-order systems

For higher-order linear time invariant systems, the response to initial conditions will be of the form  $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \dots + c_n e^{s_n t}$  if the system poles are distinct. The case of coincident poles is discussed in the next section. The system order  $n$  is equal to the number of state variables required to represent the system, or alternatively the order of the system characteristic equation. The (possibly complex) values  $s_1, \dots, s_n$  are referred to as the *characteristic frequencies*, or *eigenvalues*, or equivalently as the *poles* of the system, and the (possibly complex) values  $c_1, \dots, c_n$  are determined by the system initial conditions. The poles of a system are often plotted in the *s-plane* where the horizontal axis is the real part of the pole, and the vertical axis is the imaginary part of the pole. The advantage of this plot is that the system time response characteristics can be deduced from a single graphical representation.

#### 1.3.1 Coincident roots

When the number of distinct solutions of the characteristic equation is less than the system order  $n$ , some of the poles will be coincident. When there

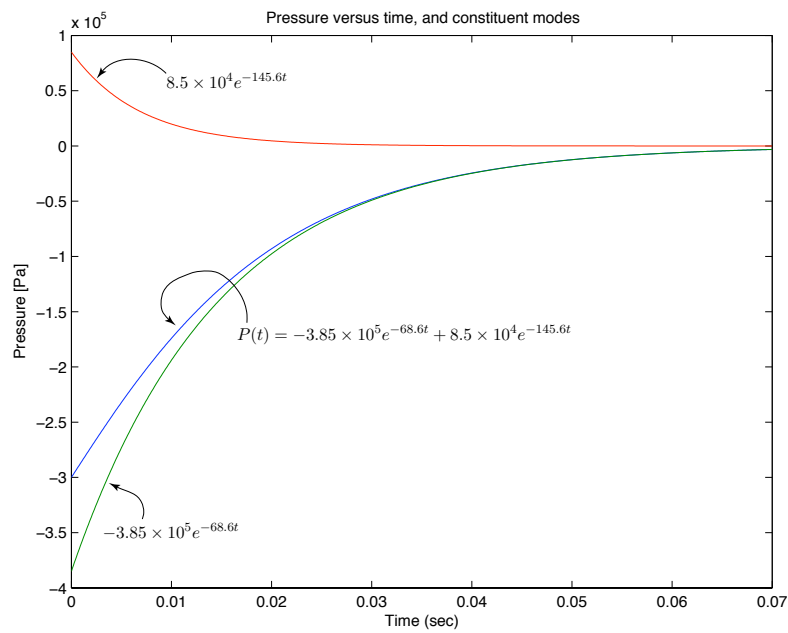


Figure 1.40: Natural response for water hammer problem. The two constituent modes which sum to give this response are also shown.



are coincident poles, the number of poles at a given location is referred to as the *multiplicity* of the pole. A pole  $s_i$  of multiplicity 1 has associated with it a solution  $e^{s_i t}$ . Poles of multiplicity 2 have associated with them solutions  $e^{s_i t}$  and  $te^{s_i t}$ . Poles of multiplicity 3 have associated with them solutions  $e^{s_i t}$ ,  $te^{s_i t}$  and  $t^2 e^{s_i t}$ . In the general case, poles of multiplicity  $m$  have associated solutions  $e^{s_i t}$ ,  $te^{s_i t}$ ,  $\dots$ ,  $t^{m-1} e^{s_i t}$ . This result holds for complex poles as well as real-axis poles. In satisfying initial conditions, each associated term is multiplied by a (possibly complex) weighting coefficient.

### 1.3.2 Linearity

The differential equations we've been studying in this chapter are *linear*. This is so because they have terms which are weighted sums of time derivatives of the signal of interest. The operation of arbitrary order differentiation followed by multiplication by a constant is linear. By linear we mean that if operating on signal  $u_1$  produces output signal  $y_1$ , and operating on input signal  $u_2$  produces output signal  $y_2$ , then the weighted combination of input signals  $\alpha u_1 + \beta u_2$  produces the output signal which is the weighted combination  $\alpha y_1 + \beta y_2$  of the individual responses, for all signals and weighting coefficients. Said another way, *superposition* applies; the summation of input signals does not produce new sorts of output responses.

In the context of homogeneous or natural responses, this means that if we have calculated the response of a system to two sets of initial conditions, then the response to a weighted sum of these initial conditions is the weighted sum of the individual responses. That is, suppose we have one set of initial conditions  $C_1$  which produces a homogeneous response signal  $x_1(t)$ , and another set of initial conditions  $C_2$  which produces a homogeneous response signal  $x_2(t)$ . Then if and only if the system is linear the response to initial conditions  $\alpha C_1 + \beta C_2$  is  $\alpha x_1 + \beta x_2$ . This property holds for the linear constant coefficient differential equations we have been studying in this chapter.

As a counterexample, suppose one of the differential equations contains a term  $x^2$ . Linearity does not hold here, since  $(x_1 + x_2)^2 \neq x_1^2 + x_2^2$ . Squaring is a *nonlinear* operation. We will discuss some aspects of nonlinear systems in subsequent chapters. However, it is not possible to provide broadly applicable exact analytical solutions for nonlinear systems. The tremendous power of linear system theory comes in large part from the generality of the results.

## Chapter 2

# Driven Response

### 2.1 First-order mechanical system step response

Suppose we now drive the first-order mechanical system of Figure 1.6 with a force  $F$ , as shown in Figure 2.1. Now let the force take the form of a step function  $F(t) = F_0 u_s(t)$ . If the initial value for  $x$  is assumed equal to zero, then the system position response is given by

$$x(t) = \frac{F_0}{k}(1 - e^{-t/\tau}) \quad (t \geq 0) \quad (2.1)$$

where  $\tau = b/k$  as before. The response begins at the initial value of zero and exponentially approaches the final value with a time constant  $\tau$ . It is interesting to note that the final value is given by  $x(\infty) = F_0/k$ , that is, the steady-state force  $F_0$  divided by the spring constant  $k$ . This makes physical sense, since the damper force depends upon velocity and it thus supplies no force in steady-state. As before, the response becomes faster as  $\tau$  decreases, as shown in Figure 2.2 where the response is plotted for  $F_0 = k$ , and with four values  $\tau = 2, 1, 0.5, 0.1$ .

Two more features of the step response bear noting. First, the difference between the current and final values decreases by a multiplicative factor of  $e^{-1} \approx 37\%$  in each interval of one time constant. Second, the slope at any point when projected to the final value defines an interval of one time constant. These features are illustrated in Figures 2.3 and 2.4, respectively.

For most practical purposes, the step response is considered complete in about four or five time constants (i.e., within 1–2% of final value). Note however, that if very high accuracy is required, a much larger number of time constants are needed in order to consider the response to be settled

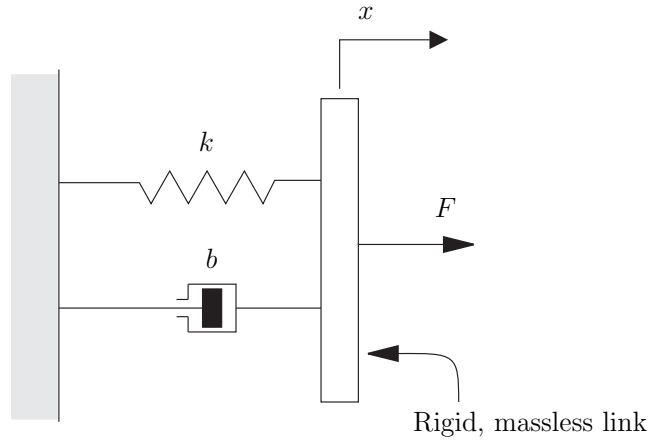


Figure 2.1: First-order mechanical system model with driving force  $F$ .

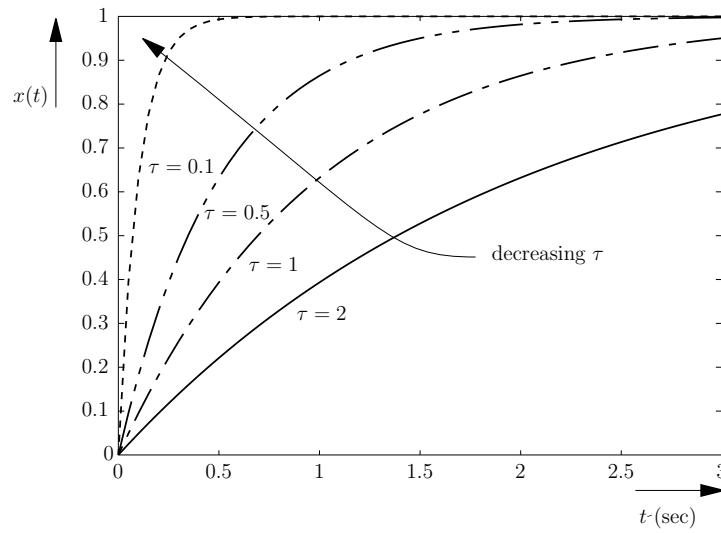


Figure 2.2: First-order mechanical system response to a step in applied force under varying time constant  $\tau = 2, 1, 0.5, 0.1$ . The arrow shows the effect of decreasing  $\tau$ .

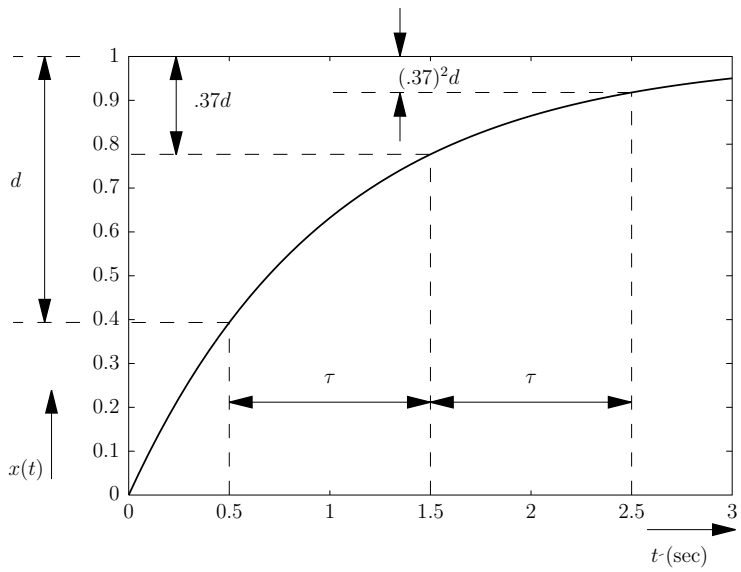


Figure 2.3: Illustrating the features of a first-order system step response. The response decays by a multiplicative factor of 0.37 in each interval of time equal to one time constant.

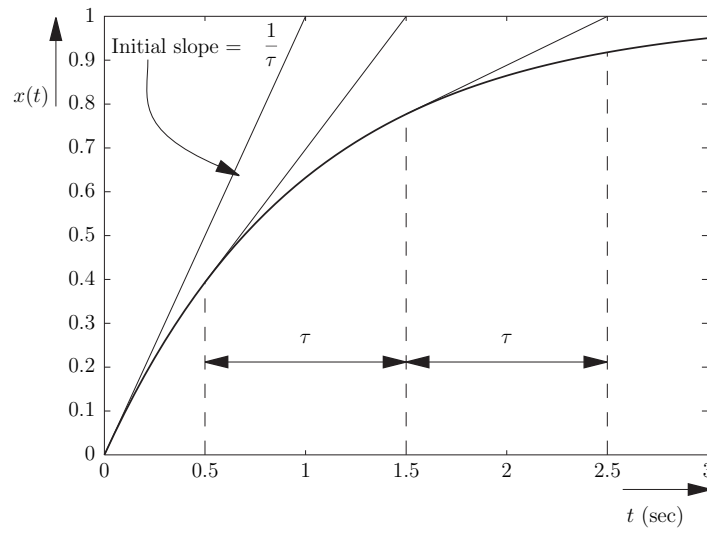


Figure 2.4: Illustrating features of a first-order system step response. The slope at any point when projected to the final value defines an interval of one time constant.

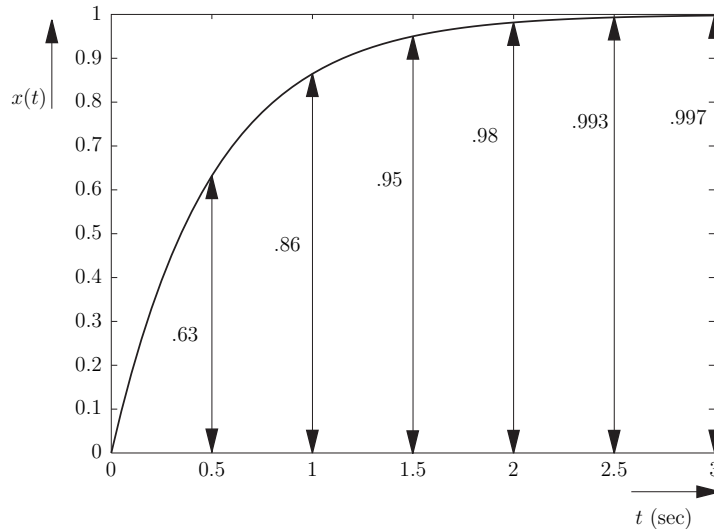


Figure 2.5: Illustrating the settling time of a first order system ( $\tau = 0.5$ ).

to final value. For instance, if an error of one part in  $10^5$  is required, then  $n = \log_e 10 \log_{10} 10^5 = 2.3 \cdot 5 = 11.5$  time constants are required. The settling behavior over six time constants ( $\tau = 0.5$ ) is shown in Figure 2.5.

## 2.2 First-order thermal system driven response

The temperature of the envelope of an incandescent light bulb can be reasonably well modeled as a first-order lumped thermal system. To investigate this, I measured the temperature of the bulb in a desk lamp, as shown in Figure 2.6.

The temperature of the bulb is sensed with a *thermocouple* which measures temperature by means of the variation of voltage generated across a junction between two dissimilar metals. The thermocouple used in this experiment is shown in Figure 2.7. The thermocouple junction is attached to the light bulb with tape under an insulating patch. The insulating patch is used so that the thermocouple temperature is equal to the bulb envelope temperature.

With the bulb equilibrated to room temperature, I turned on the lamp, waited about 10 minutes for the bulb to reach nearly steady state tempera-

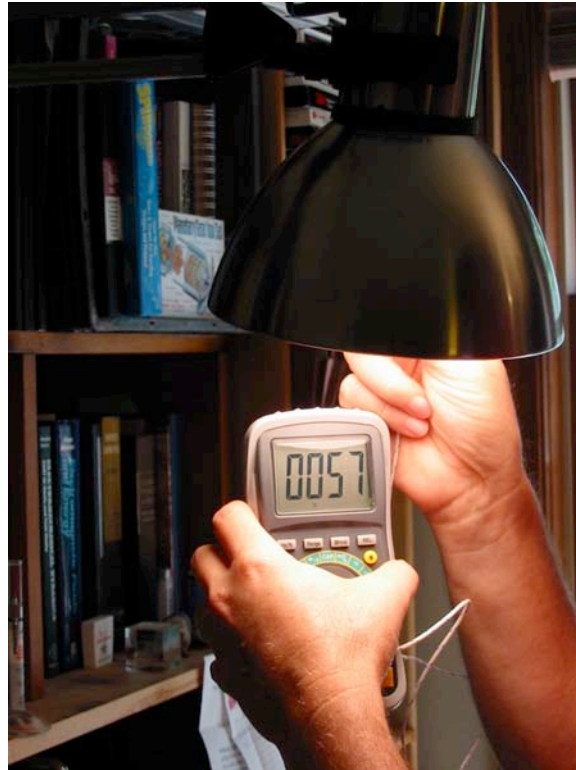


Figure 2.6: Instrumentation for measuring temperature of the incandescent bulb in a desk lamp. Display shows temperature in  $^{\circ}\text{C}$ .

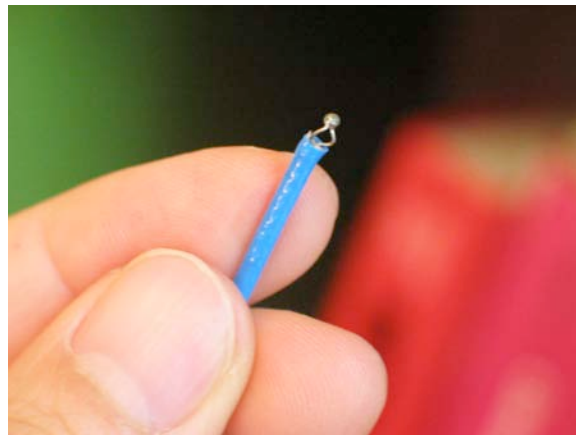


Figure 2.7: Thermocouple used to measure bulb temperature.

ture, and then turned off the lamp. I recorded the bulb temperature every five seconds over the course of the experiment as the bulb heated and cooled. This data is shown in Figure 2.8. To graphical accuracy, the data looks very close to first-order. The small glitch in the data at about 1130 sec is due to bumping the test lead, which resulted in movement of the thermocouple, and thus a transient in temperature. Experimental data commonly has such extra effects included, and we need to exercise judgement as to what aspects of the data to attempt to include in our models. It is also important to properly report the sources of known anomalies in your own work.

In order to fit the data to an exponential response from a first-order model, I wrote a Matlab script to plot the data and a model response. The fitted response appears as shown in Figure 2.9. The temperature data is in terms of rise above ambient; on the day this data was taken, the ambient temperature was  $29^\circ\text{C}$ . The model fit is reasonably good with the functions shown. Thus we could use a first-order model to make predictions of the bulb temperature as a function of other driving functions than the long on/off cycle shown.

### 2.3 Step response of second-order mechanical system

Consider again the second-order mechanical system shown in Figure 1.19. We modify this system to include a driving force  $F$  acting in the  $x$ -direction as shown in Figure 2.10. A free-body diagram for the system is given in Figure 2.11. The differential equation for this system is

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F. \quad (2.2)$$

We now let the input force take the form of a step  $F(t) = F_0 u_s(t)$  with amplitude  $F_0$ , and further assume the system has zero initial conditions. The homogeneous part of the solution will take the forms that we have discussed previously, depending upon the damping ratio. The underdamped case is of interest here, so we will further assume  $0 < \zeta < 1$ .

With the step input, we can calculate the response by thinking in terms of a particular solution plus homogeneous solution. In steady-state, the input force has the constant value  $F_0$ . This force is balanced in steady-state by the deflection of the spring such that the net force on the mass is zero. That is, we postulate the particular solution  $x_p(t) = F_0/k$ . As this solution has zero velocity and acceleration, the damper and mass do not contribute



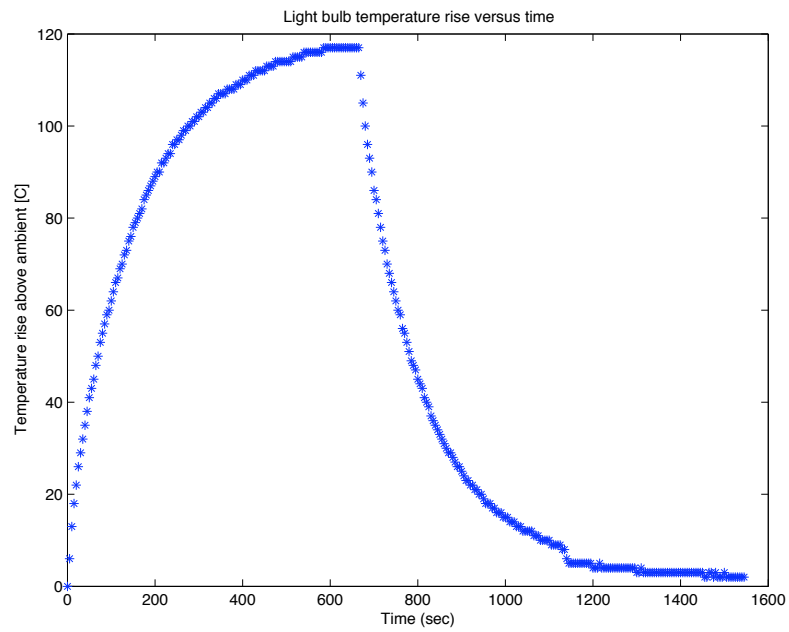


Figure 2.8: Bulb temperature data in rise above ambient [C] as a function of time. Lamp is turned on at  $t = 0$ , and turned off at  $t = 665$  [sec].

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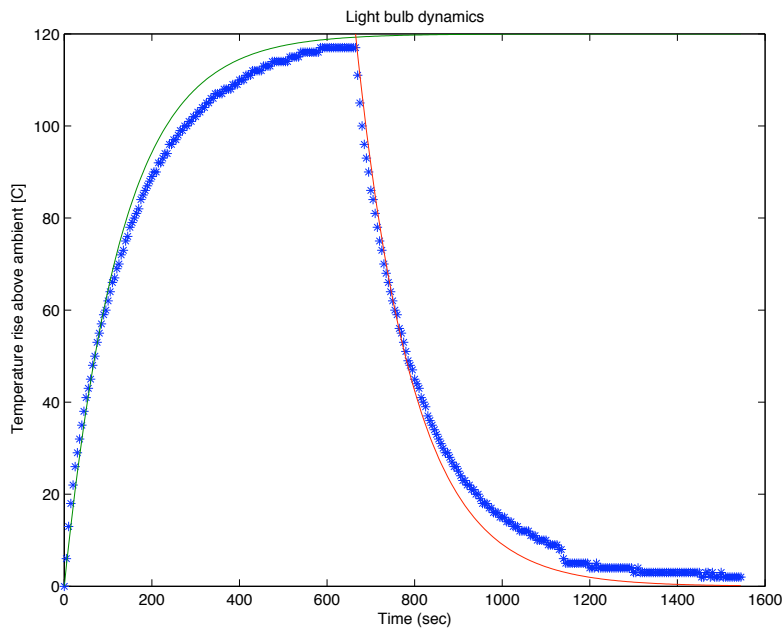


Figure 2.9: Bulb temperature data as a function of time and fitted data from reasonably close first-order model.

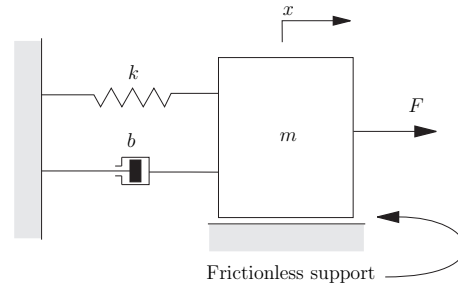


Figure 2.10: Second-order mechanical system driven by a force  $F$ .

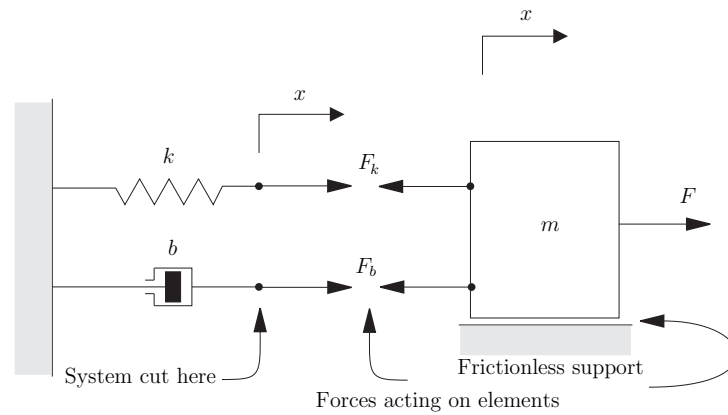


Figure 2.11: Free body diagram for second-order system driven by a force  $F$ .

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any term. To satisfy the condition of zero initial velocity and position, a suitable homogeneous response  $x_h$  must be added to the particular response found above.

The homogeneous response must be such as to satisfy initial conditions when it is added to the particular solution. The homogeneous response will take the form developed in the previous chapter. Since  $x_p(0) = F_0/k$ , and we assume zero initial conditions, then we must have from the homogeneous response that  $x_h(0) = -F_0/k$ , and  $\dot{x}_h(0) = 0$ .

For the underdamped case considered here, the homogeneous solution will thus take either the rectangular form (1.57) or the polar form (1.66). Applying the initial condition relations from that section yields the homogeneous solution as

$$x_h(t) = -(F_0/k)e^{-\sigma t}(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t) \quad (2.3)$$

$$= -(F_0/k)\frac{\omega_n}{\omega_d}e^{-\sigma t} \sin(\omega_d t + \phi) \quad (2.4)$$

Here two equivalent forms of the solution are given, where we define  $\phi = \arctan(\sqrt{1 - \zeta^2}/\zeta)$  and recall the definitions  $\sigma = \zeta\omega_n$  and  $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ . The complete solution is

$$x(t) = x_p(t) + x_h(t) = (F_0/k)(1 - e^{-\sigma t}(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t)) \quad (2.5)$$

$$= (F_0/k)(1 - \frac{\omega_n}{\omega_d}e^{-\sigma t} \sin(\omega_d t + \phi)) \quad (2.6)$$

Since the system is linear, the character of the response does not depend upon the driving amplitude, and we thus can normalize the problem by choosing  $F_0 = k$  (or alternately plot all response magnitudes in units of  $F_0/k$ ). With this choice, the normalized response is

$$x_n(t) = 1 - e^{-\sigma t}(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t) \quad (2.7)$$

$$= 1 - \frac{\omega_n}{\omega_d}e^{-\sigma t} \sin(\omega_d t + \phi) \quad (2.8)$$

A plot of this solution is shown in Figure 2.12 for various values of  $\zeta$ . In this plot, we have normalized the amplitude as discussed above, and also normalized time to  $t' = \omega_n t$ . Note that in this plot,  $\zeta$  affects both the overshoot (with a maximum of 100% overshoot for the case  $\zeta = 0$ ) and the duration of the ringing in the response.

Second-order systems occur frequently in practice, and so standard parameters of this response have been defined. These include the maximum

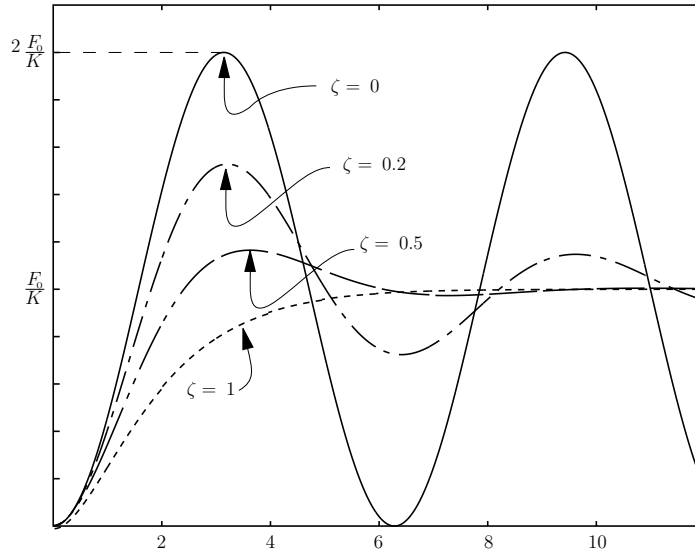


Figure 2.12: Second-order system response to a step in force of  $F_0 u_s(t)$

amount of overshoot  $M_p$ , the time at which this occurs  $t_p$ , the settling time  $t_s$  to within a specified tolerance band, and the 10-90% rise time  $t_r$ . These parameters are defined with respect to a standard second-order response as shown in Figure 2.13.

### 2.3.1 Overshoot

Let's start with a discussion of the overshoot. The time at which the peak in (3.8) occurs can be determined by setting the derivative to zero. That is

$$\begin{aligned} \frac{dx}{dt} &= \sigma e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \\ &\quad - e^{-\sigma t} (-\omega_d \sin \omega_d t + \sigma \cos \omega_d t) \end{aligned} \quad (2.9)$$

$$= e^{-\sigma t} \left( \frac{\sigma^2}{\omega_d} \sin \omega_d t + \omega_d \sin \omega_d t \right). \quad (2.10)$$

This has a value of zero for  $\omega_d t = \pi \pm n2\pi$ ;  $n = 0, 1, 2, \dots$  and thus the first peak in the response is found where  $\omega_d t = \pi$ , and thereby the peak time is given by

$$t_p = \frac{\pi}{\omega_d}. \quad (2.11)$$

### 2.3. STEP RESPONSE OF SECOND-ORDER MECHANICAL SYSTEM 79

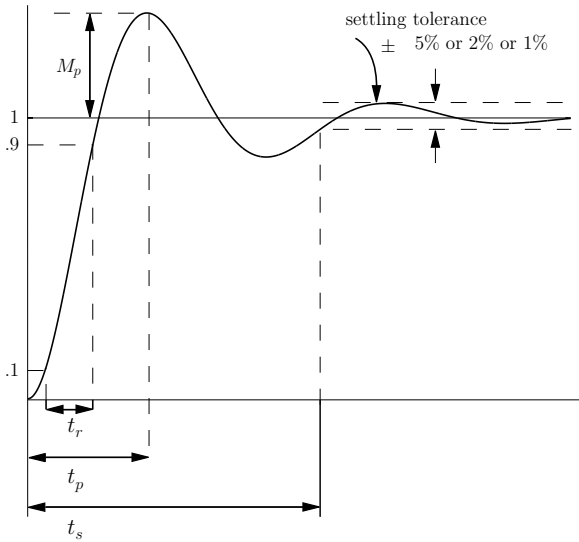


Figure 2.13: Standard second-order step response parameters.

By definition, the overshoot  $M_p$  is the amount that the normalized response exceeds unity. That is, the peak value is  $x_{peak} = 1 + M_p$ . At the value of time  $t_p$  for the peak, we can calculate the magnitude of the response as

$$x(t_p) = 1 - e^{-\sigma\pi/\omega_d} \left( \cos \pi + \frac{\sigma}{\omega_d} \sin \pi \right) \quad (2.12)$$

$$= 1 + e^{-\sigma\pi/\omega_d}. \quad (2.13)$$

With the definition of the peak value given above, we thus have

$$M_p = e^{-\sigma\pi/\omega_d} \quad (2.14)$$

$$= e^{-\pi\zeta/\sqrt{1-\zeta^2}}; \quad 0 \leq \zeta < 1. \quad (2.15)$$

Recall that this formula only applies in the underdamped case, as indicated explicitly above.

While the result above is exact, it is a bit cumbersome to remember and to manipulate for design calculations. Thus, we introduce the approximation

$$M_p \approx 1 - \frac{\zeta}{0.6}. \quad (2.16)$$

This linear approximation is easier to compute with and for many design problems is sufficiently accurate.

### 2.3.2 Rise time

We define the rise time  $t_r$  as indicated in Figure 2.13 as the time required for the response to transit between 10% and 90% of the final value.<sup>1</sup> Inspection of Figure 2.12 shows that the rise time is not strongly dependent upon  $\zeta$ , whereas it scales linearly with  $\omega_n$ . For intermediate values of  $\zeta$ , the rise time is approximately given by

$$t_r \approx \frac{1.8}{\omega_n}. \quad (2.17)$$

This approximation is very useful for determining the approximate natural frequency based upon step response measurements, or in sketching the step response of a known system.

### 2.3.3 Settling time

Finally, we define the settling time. As shown in Figure 2.13, the settling time is the time required for the response to stay within a defined error band  $\pm\Delta$  of the final normalized value of unity. The exact criterion is difficult to express in analytic form, and so we use the conservative approximation of calculating the time at which the *envelope* of the response reaches values within the error band. This approximation is conservative, since the response itself never lies outside the envelope. Since the envelope decays to the final value of unity with an error magnitude of  $e^{-\sigma t}$ , it reaches the tolerance band for

$$e^{-\sigma t_s} = \Delta, \quad (2.18)$$

and thus we have  $-\sigma t_s = \ln \Delta$ , or

$$t_s = -\frac{1}{\sigma} \ln \Delta. \quad (2.19)$$

In order to calculate the settling time to a given value, it helps to remember that  $e^{2.3} \approx 10$ , and thus, for example  $\ln 0.01 \approx -4.6$ . With this insight, the 1% settling time is given by

$$t_s \approx \frac{4.6}{\zeta \omega_n}; \text{ (1\% settling)}. \quad (2.20)$$

---

<sup>1</sup>Some texts define rise time as the time required to transit between 0% and 100%, since this is easier to calculate explicitly. However, in engineering practice, such a measure is difficult to implement, and the dependence on  $\zeta$  is much stronger. Thus, in engineering practice we use the 10–90% rise time. Where these need to be distinguished, we can insert the subscript 10–90 or 0–100 to be clear which is being discussed.

### 2.3. STEP RESPONSE OF SECOND-ORDER MECHANICAL SYSTEM 81

There is, of course, nothing special about settling to 1%. For example, if a large machine tool takes a move of 1 m, and if it can be modelled as a second order system, and if we require it to settle to 1  $\mu\text{m}$ , then the settling time will be much longer. This specification requires settling to one part in  $\Delta = 10^6$  of the move, and thus for this specification we will require a time of about  $6 * 2.3/\zeta\omega_n$  to settle within tolerance.

#### 2.3.4 Summary

The results developed above are summarized as:

$$\begin{aligned} t_r &\simeq \frac{1.8}{\omega_n} && \text{(medium values of } \zeta) \\ M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} \simeq 1 - \frac{\zeta}{0.6} \\ t_p &= \frac{\pi}{\omega_d} \\ t_s &\simeq \frac{4.6}{\zeta\omega_n} && (\pm 1\% \text{ error tolerance}) \end{aligned}$$

#### 2.3.5 Application to design

For the purposes of design, we can use the results derived above to place constraints on the location of the poles of a second order system. For example, if the rise time must be below a given required value  $t_{rr}$ , then we must have

$$\omega_n > \frac{1.8}{t_{rr}}. \quad (2.21)$$

Further, if we require that the overshoot be less than some required value  $M_{pr}$ , then we must have approximately

$$\zeta > 0.6(1 - M_{pr}). \quad (2.22)$$

Finally, if we require that the settling time be less than some required value  $t_{sr}$ , then we must have

$$\zeta\omega_n > \frac{4.6}{t_{sr}}. \quad (2.23)$$

Recall that  $\zeta\omega_n = \sigma$  is the real part of the system poles.

These constraints can be expressed graphically by requiring that the system poles lie to the left of the shaded regions in the  $s$ -plane shown in Figure 2.14.



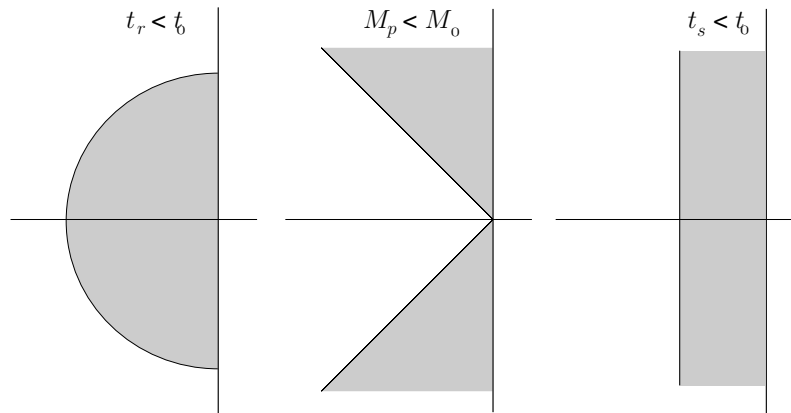


Figure 2.14: Grey areas show prohibited regions for poles to meet design specifications.