Kinematics

Professor Sanjay Sarma

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1.0 What is “Dynamics?”

The goal of the field of dynamics is to understand how mechanical systems move under the effect of forces. There are 3 components to the study of dynamics:

- **Kinematics** deals with the motions of bodies. Kinematics has to do with geometry and physical constraints.
- **Kinetics** deals with the evolution of this motion under the effect of forces. Classical Dynamics invoke Newton’s laws.
- **Constitutive relationships** are the relationships which capture the effects of springs, gravitation, electromagnetism, *etc*.

By combining kinematic, kinetic and constitutive relationships, it is possible to generate a complete set of equations collectively called the equations of motion. Solving these equations of motion gives us the motion of the system.

2.0 How this Course is Laid Out

System	Kinematics	Kinetics & Constitutive
Particle		
System of particles		
Rigid Bodies		
Lagrangian formulation		
Oscillations		

3.0 Some Basics on Frames and Derivatives of Vectors

Kinematics is all about *reference frames*, *vectors*, *differentiation*, *constraints* and *coordinates*. Later, we will use *generalized coordinates* and *constraints,* but not yet. Right now, we describe some of the basic terms.

1. A reference frame is a perspective from which a system is observed. The inertial frame of reference is a special frame which is important when we study kinetics, but has no relevance in kinematics *per se*.
2. It is customary to attach three mutually perpendicular unit vectors to frames. You can give them names like: $\underline{\boldsymbol{i}}$, $\underline{\boldsymbol{j}}$, $\underline{\boldsymbol{k}}$ or $\underline{\boldsymbol{e}}_r$, $\underline{\boldsymbol{e}}_\theta$, $\underline{\boldsymbol{e}}_t$ or $\underline{\boldsymbol{a}}_1$, $\underline{\boldsymbol{a}}_2$, $\underline{\boldsymbol{a}}_3$.

- By the way, we underline vectors on the blackboard, but using a **bold** symbol is the equivalent in type-set form. I will do both for some time, but please interpret bold as a vector if you see it by itself in future material.
- Consider a situation in which you have defined a number of frames with unit vectors in each. There is absolutely no reason why you couldn't mix unit vectors. So, you could say: $\boldsymbol{p} = 3\underline{\boldsymbol{i}} + 7\underline{\boldsymbol{e}}_\theta$

  You can differentiate vectors. However, differentiation of vectors has no meaning if you do not specify the frame with respect to which you are differentiating them.

  So it is important to say $\frac{{}^A d}{dt}(3\underline{\boldsymbol{a}} + 7\boldsymbol{b})$ instead of just $\frac{d}{dt}(3\underline{a} + 7b)$ if you are differentiating with respect to frame A. Often, this frame is not stated explicitly, and must be inferred from the context in which the expression is written. I prefer to spell it out explicitly.

Here's the secret: if $\underline{\boldsymbol{a}}_1$, $\underline{\boldsymbol{a}}_2$, $\underline{\boldsymbol{a}}_3$ are the unit vectors associated with a frame A, then

$$\frac{{}^A d\underline{\boldsymbol{a}}_1}{dt} = 0 \; ; \; \frac{{}^A d\underline{\boldsymbol{a}}_2}{dt} = 0 \; ; \; \frac{{}^A d\underline{\boldsymbol{a}}_3}{dt} = 0 \,. \qquad \text{(EQ 1)}$$

In fact any vector which is merely translating in A, even if it is accelerating, has a zero derivative in A. A non-zero derivative occurs when the vector is stretching (in which case it is stretching in any frame) or rotating with respect to A. So in general we can say that:

$$\frac{{}^A d\boldsymbol{b}}{dt} \neq 0 \,. \qquad \text{(EQ 2)}$$

In other words, if you want to differentiate an expression with vectors in it, it is best to express all vector terms in terms of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and then use relationships (I) above. That way, the taking of the derivative becomes trivially easy.

- Furthermore for any two vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$,

$$\frac{{}^A d}{dt}(\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{b}}) = \underline{\boldsymbol{a}} \bullet \frac{{}^A d}{dt}(\underline{\boldsymbol{b}}) + \frac{{}^A d}{dt}(\underline{\boldsymbol{a}}) \bullet \underline{\boldsymbol{b}}$$

and

$$\frac{{}^A d}{dt}(\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}) = \underline{\boldsymbol{a}} \times \frac{{}^A d}{dt}(\underline{\boldsymbol{b}}) + \frac{{}^A d}{dt}(\underline{\boldsymbol{a}}) \times \underline{\boldsymbol{b}} \,.$$

4.0 Calculating Velocities and Accelerations of Given Points

The last and the next homework are all about calculating velocities and accelerations of points defined in various frames of references. There are three approaches to this problem.

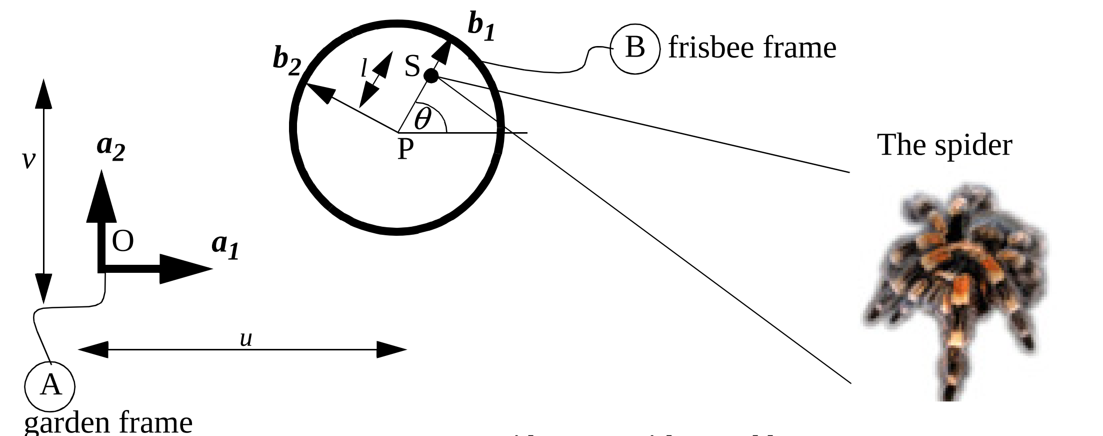

FIGURE 1. Spider on a Frisbee Problem

We will use the spider-on-the-frisbee problem used in class to illustrate the approaches. Spider photo courtesy of B. Smith. See Figure 1.

4.1 The really tedious but conceptually simple approach (RTBCSA)

1. Write down the position vector of the spider with respect to point O: $\mathbf{r}^{OS}. = \mathbf{r}^{OP} + \mathbf{r}^{PQ}$. For convenience, you can write it in mixed form if you would like:

$$\mathbf{r}^{OS} = u\boldsymbol{a}_1 + v\boldsymbol{a}_2 + l\boldsymbol{b}_1$$

2. Now, say you want to calculate velocities and accelerations with respect to frame A. Then you are looking for:

$$^{A}\boldsymbol{V}^{S} = \frac{^{A}d}{dt}\boldsymbol{r}^{OS} = \frac{^{A}d}{dt}(u\boldsymbol{a}_1 + v\boldsymbol{a}_2 + l\boldsymbol{b}_1). \quad \text{(EQ 3)}$$

3. Because of Equations 1, this simplifies to:

$$^{A}\boldsymbol{V}^{S} = \boldsymbol{a}_1\frac{^{A}d}{dt}(u) + \boldsymbol{a}_2\frac{^{A}d}{dt}(v) + \frac{^{A}d}{dt}(l\boldsymbol{b}_1). \quad \text{(EQ 4)}$$

The first two terms are done (!) but how do we take care of the third term? Here we use the trick alluded to earlier. We simply rewrite $\boldsymbol{b}_1$ in terms of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$:

$$\boldsymbol{b}_1 = \boldsymbol{a}_1\cos\theta + \boldsymbol{a}_2\sin\theta. \quad \text{(EQ 5)}$$

By the way, even though we don't need it here,

$$\boldsymbol{b}_2 = -\boldsymbol{a}_1\sin\theta + \boldsymbol{a}_2\cos\theta \quad \text{(EQ 6)}$$

Anyway, we can insert the expressions for $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ into Equation 4 above, take the derivative, and we are done! We get:

$$ {}^{A}\boldsymbol{V}^{S} = \boldsymbol{a_1}\dot{u} + \boldsymbol{a}_2\dot{v} - \boldsymbol{a_1}\dot{\theta}\sin\theta + \boldsymbol{a_2}\dot{\theta}\,\text{co }\theta + \dot{l}(\boldsymbol{a_1}\cos\theta + \boldsymbol{a_2}\sin\theta)\,. \qquad \text{(EQ)} $$

4. The acceleration, ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{S}}$ is simply the derivative of ${}^{\mathbf{A}}\boldsymbol{V}^{\mathbf{S}}$ done exactly as before. The good news is that since we have already down-converted everything to $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$, the task is simple. When everything is done, we get:

$$ {}^{A}\boldsymbol{a}^{S} = {}^{A}\frac{d}{dt}{}^{A}\boldsymbol{V}^{S} = (\ddot{u}\boldsymbol{a_1} + \ddot{v}\boldsymbol{a_2}) + (\ddot{l}\cos\theta\boldsymbol{a_1} + \ddot{l}\sin\theta\boldsymbol{a_2}) + 2(-\dot{l}\dot{\theta}\sin\theta\boldsymbol{a_1} + \dot{l}\dot{\theta}\cos\theta\boldsymbol{a_2}) $$
$$ (-l\ddot{\theta}\sin\theta\boldsymbol{a_1} + l\ddot{\theta}\cos\theta\boldsymbol{a}_2) + (-l\dot{\theta}^2\cos\theta\boldsymbol{a_1} - l\dot{\theta}^2\sin\theta\boldsymbol{a_2})\,. \qquad \text{(EQ)} $$

Keep in mind that we can now rewrite the $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ in terms of $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ if we want to (it's a free country, and the unit vectors need to be lined up with the frame of reference only during differentiation!). So we choose to rewrite some terms in the right-hand-side of Equation 8 above using the following reverse transformations:

$$ \boldsymbol{a_1} = \boldsymbol{b_1}\text{co }\theta - \boldsymbol{b_2}\sin\theta \qquad \text{(EQ 9)} $$

$$ \boldsymbol{a_2} = \boldsymbol{b_1}\text{si }\theta + \boldsymbol{b_2}\cos\theta \qquad \text{(EQ 10)} $$

To get:

$$ {}^{A}\boldsymbol{a}^{S} = (\ddot{u}\boldsymbol{a_1} + \ddot{v}\boldsymbol{a_2}) + \ddot{l}\boldsymbol{b}_1 + 2\dot{l}\dot{\theta}\boldsymbol{b_2} + l\ddot{\theta}\boldsymbol{b_2} - l\dot{\theta}^2\boldsymbol{b}_1\,. \qquad \text{(EQ 1)} $$

Let's introduce ourselves our friends in Equation 11, many of whom crept out of the woodwork under the harsh light of unrelenting vector calculus (and to the surprise of Messrs. Euler, Coriolis and Centripetal[1] during the course of history as you will have read in the text):

$$ {}^{A}\boldsymbol{a}^{S} = \underline{(\ddot{u}\boldsymbol{a_1} + \ddot{v}\boldsymbol{a_2})} + \underline{\ddot{l}\boldsymbol{b}}_1 + \underline{2\dot{l}\dot{\theta}\boldsymbol{b_2}} + \underline{l\ddot{\theta}}\boldsymbol{b_2} - \underline{l\dot{\theta}}^2\boldsymbol{b}_1 $$

- ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{P}}$: acceleration of point P with respect to frame A.
- ${}^{\mathbf{B}}\boldsymbol{a}^{\mathbf{S}}$: acceleration of point S with respect to frame B. In other words the view of a person sitting on the frisbee.
- Coriolis effect
- Eulerian acceleration. Basically the angular acceleration scaled by the lever arm *l*.
- Centripetal effect

1. That's a joke. I'm sure you know that "centripetal" means "towards the center" in Latin.

4.2 The Less Tedious Approach (LTA)
Alternatively: Taking Derivatives with the Angular Velocity

We now introduce the concept of an angular velocity. In 2D, if frame B is rotating with respect to frame A at a rate $\dot{\theta}$ then we say that the angular velocity of B with respect to A is ${}^A\omega^B$ and ${}^A\omega^B = \dot{\theta}\boldsymbol{a}_3 = \dot{\theta}\boldsymbol{b}_3$. Some key trivia about angular velocities:

1. First, note that you don't need the concept at all. As we have seen, we can calculate everything we want to from straight vector calculus.
2. Angular velocities are convenient for two reasons:
   - First, they can be used to calculate derivatives more easily, as we will see shortly.
   - Second, motions of rigid bodies can be captured as translations in a certain direction and a rotation. All the points in rigid bodies travel together. So the angular velocity is useful to capture the motion of the entire rigid body. In fact a frame is a rigid body too, just a transparent, fictitious one. So angular velocities also capture the motions of frames of reference.
   - It might seem that angular velocities need to be defined about a certain point or axis. This is **not** the case. The text has a very nice explanation of why not. Every point on a frame or a rigid body has the exact same angular velocity, and there is not point that needs to be specified.
   - We will continue our development in 2D but all results will also apply "seamlessly" in 3D!!

Now for the magic formula. In class I showed you that for any vector $\boldsymbol{r}$ which you might seek to take a derivative in frame of reference A,

$$\frac{{}^A d\boldsymbol{r}}{dt} = \frac{{}^B d\boldsymbol{r}}{dt} + {}^A\omega^B \times \boldsymbol{r}. \qquad \text{(EQ 12)}$$

The implication is very simple. As you know from the previous section and from problems in PSet 1, you can always take derivatives by down-converting everything to the unit vectors attached to the frame of reference (say A) with respect to which you seek to take derivatives. however, this can be tedious if some vectors are better specified in some intermediate frame, say B. Wouldn't it be wonderful if you could simply do all your business in the more convenient frame B and just throw in an "adjustment term" to convert stuff to frame A? Well, look no further than Equation 12. Basically, Equation 12 lets you do your business in a frame attached to the frisbee, and to simply add in a ${}^A\omega^B \times \boldsymbol{r}$ term to take care of the fact that you wanted to compute a derivative in frame A. How do we use it?

Consider the frisbee problem again, and look at Equation 4, reproduced below:

$${}^A\boldsymbol{V}^S = \boldsymbol{a_1}\frac{{}^A d}{dt}(u) + \boldsymbol{a}_2\frac{{}^A d}{dt}(v) + \frac{{}^A d}{dt}(l\boldsymbol{b_1})$$

Earlier we were compelled to "down-convert"[1] that third term in terms of $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$. That created a lot of math that it would be convenient to avoid. So using Formula 12 we can instead calculate it as:

$$\frac{{}^A d}{dt}(l\boldsymbol{b_1}) = \frac{{}^B d}{dt}(l\boldsymbol{b_1}) + {}^A\omega^B \times l\boldsymbol{b_1}.$$

Remember that we defined ${}^A\omega^B = \dot{\theta}\boldsymbol{a}_3 = \dot{\theta}\boldsymbol{b}_3$, so the equation above simplifies to:

$$\frac{{}^A d}{dt}(l\boldsymbol{b_1}) = \dot{l}\boldsymbol{b_1} + l\dot{\theta}\boldsymbol{b}_2. \qquad \text{(EQ 13)}$$

We can insert that back into Equation 4 and get Equation 7 without all the intermediate drama, and in a much more convenient form:

$${}^A\boldsymbol{V}^S = \boldsymbol{a_1}\frac{{}^A d}{dt}(u) + \boldsymbol{a}_2\frac{{}^A d}{dt}(v) + \dot{l}\boldsymbol{b_1} + l\dot{\theta}\boldsymbol{b}_2. \qquad \text{(EQ 14)}$$

Voila, eh?!! Now to get ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{S}}$ we can calculate the second derivative; and we use our magic formula again every time we are confronted with a a $\boldsymbol{b_1}$ or a $\boldsymbol{b_2}$ in the term being differentiated with respect to frame of reference A. So in other words, our magic formula, Formula 12, is a very simple tool of convenience.

4.3 A Really Simple Formulaic Approach (RSFA)

We have solved the spider-on-the-frisbee problem using two approaches now. It turns out that the problem has a certain attributes that repeat in a lot of other problems:

1. There are many situations in which a rigid body travels through space either in a ballistic trajectory or under its own power or attached to a mechanism or system by a link or a cable. The rigid body in these situations in moving with respect to a frame A. Let's say that Point O is fixed in Frame A. Let's assume that the rigid body bas frame B attached to it.
2. In many cases, there is a point on rigid body by, Point P, whose position is known as a function of time. In other words, $\boldsymbol{r}^{OP}(t)$ is known, from which we can calculate ${}^{\mathbf{A}}\boldsymbol{V}^{\mathbf{P}}$ and ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{P}}$. For example, we might know the exact position of a reference point on a satellite, or the exact location of a navigation unit on a fighter plane, or the exact location of a joint on a link in a robotic arm, or a GPS unit attached to a giant peanut hurtling through space. This is shown in Figure 2.

1. The term "down-convert" is not technical. I am using it informally.

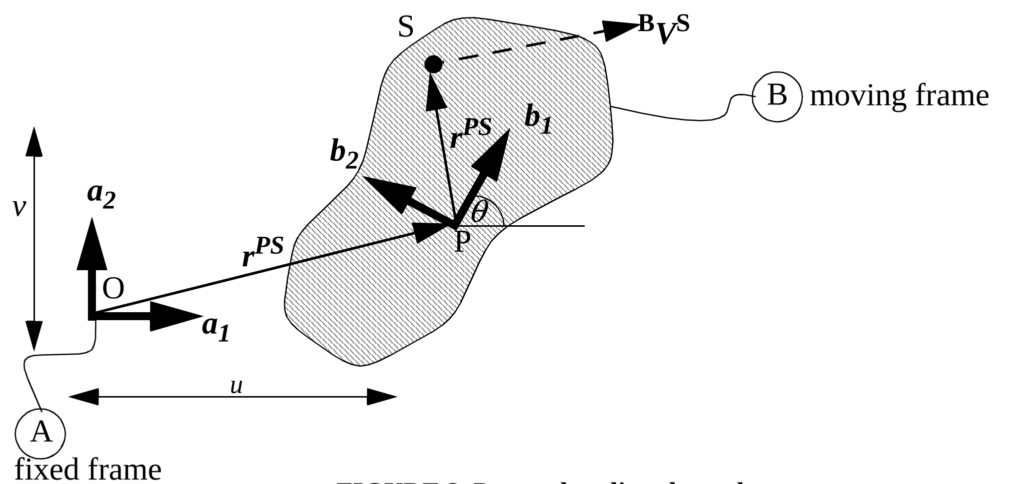

FIGURE 2. Peanut hurtling through space

3. Assume that the angular velocity of B in A is also known, and we call it ${}^{A}\omega^{B}$. By the way, we can define the *angular acceleration* of a rigid body B with respect to a frame A as:

$$ {}^{A}\alpha^{B} = \frac{{}^{A}d}{dt}{}^{A}\omega^{B}. $$

   This is simply a definition. We might just as well have called it ${}^{A}\dot{\omega}^{B}$, but we prefer ${}^{A}\alpha^{B}$.

4. Now consider another point S moving with respect to rigid body B. Say an is astronaut walking on the International Space Station as it hurtles through space. Let us say that you are an observer sitting immobile on rigid body B (the Space Station). In your frame of reference, the point S has a velocity and an acceleration respectively ${}^{\mathbf{B}}\boldsymbol{V}^{\mathbf{S}}$ and ${}^{\mathbf{B}}\boldsymbol{a}^{\mathbf{S}}$.

5. <u>Key objective:</u> You want to know what the velocity and acceleration of point S with respect to frame A, ${}^{\mathbf{A}}\boldsymbol{V}^{\mathbf{S}}$ and ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{S}}$, are. Look at this carefully: we are given ${}^{\mathbf{B}}\boldsymbol{V}^{\mathbf{S}}$ and ${}^{\mathbf{B}}\boldsymbol{a}^{\mathbf{S}}$ and we want ${}^{\mathbf{A}}\boldsymbol{V}^{\mathbf{S}}$ and ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{S}}$.

6. It turns out that ${}^{\mathbf{A}}\boldsymbol{V}^{\mathbf{S}}$, by a process very similar to our solving the frisbee problem, comes to:

$$ {}^{A}\boldsymbol{V}^{S} = {}^{A}\boldsymbol{V}^{P} + {}^{B}\boldsymbol{V}^{S} + {}^{A}\omega^{B} \times \boldsymbol{r}^{PS}. \qquad \text{(EQ 15)} $$

   Judicious use of this formula saves you a bunch of differentiation.

7. It turns out that the expression for ${}^{\mathbf{A}}\boldsymbol{a}^{\mathbf{S}}$ can be derived rather trivially, by simply applying magic formula 12 to Equation 15 to get:

$$ {}^{A}\boldsymbol{a}^{S} = {}^{A}\boldsymbol{a}^{P} + {}^{B}\boldsymbol{a}^{S} + {}^{A}\alpha^{B} \times \boldsymbol{r}^{PS} + {}^{A}\omega^{B} \times ({}^{A}\omega^{B} \times \boldsymbol{r}^{PS}) + 2\,{}^{A}\omega^{B} \times {}^{B}\boldsymbol{V}^{S}. \qquad \text{(EQ 16)} $$

   We refer to this as the *super-magic formula*. Judicious use of this formula saves you a bunch of differentiation for a second time. Using this formula, our frisbee problem can be solved almost trivially. Hurray. Oh, and the last three terms are respectively the Eulerian Acceleration, the Centripetal Acceleration and the Coriolis Acceleration.

Note though that when you use the super-magic formula, you must be very careful in specification of frames of reference.

4.4 Summary

There are several key points here.

1. I have shown you how to calculate velocities and accelerations in moving systems using frames of reference.
2. You can build a very complex system with parts moving on parts moving on parts, and calculate the velocities and accelerations of all the parts by going through this process repetitively. These are called intermediate frames. The angular velocities have a wonderful property:

$$ {}^{A}\omega^{B} = {}^{A}\omega^{P} + {}^{P}\omega^{Q} + \ldots + {}^{U}\omega^{V} + {}^{V}\omega^{B}. \qquad \text{(EQ 17)} $$

   In other words, angular velocities add over intermediate frames. Don’t worry, we won’t need Equation 19 in class.
3. While this class is limited to 2D, the formulae extend seamlessly to 3D!! So what you have learnt in kinematics works well outside class.
4. Connection to the Williams Text: Everything I have said maps directly to the Williams text. Only my notation is different. Specifically,
   - What we refer to as ${}^{\mathbf{A}}\boldsymbol{V}^{\mathbf{S}}$, Professor Williams would refer to as $\boldsymbol{V}^{\mathbf{S}}$, and similarly for acceleration.
   - What we refer to as ${}^{\mathbf{B}}\boldsymbol{V}^{\mathbf{S}}$, Professor Williams would refer to as $\boldsymbol{V}_{\boldsymbol{rel}}{}^{\mathbf{S}}$, and similarly for acceleration.