2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

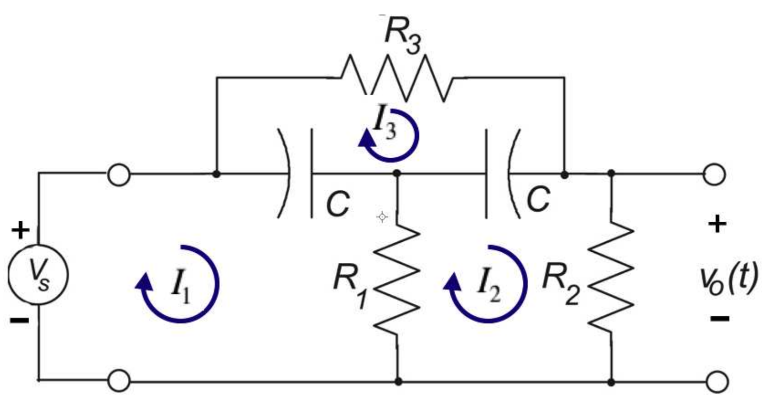
2.004 *Dynamics and Control II*
Spring Term 2008

Solution of Problem Set 3

Assigned: Feb. 22, 2008 Due: Feb. 29, 2008

Problem 1:

3 loops are defined according to below figure and then KVL equations are derived for each loop:



$$\begin{cases} \text{Loop 1:} & -V_s + \frac{1}{Cs}(I_1 - I_3) + R_1(I_1 - I_2) = 0 \\ \text{Loop 2:} & R_1(I_2 - I_1) + \frac{1}{Cs}(I_2 - I_3) + R_2 I_2 = 0 \\ \text{Loop 3:} & \frac{1}{Cs}(I_3 - I_1) + R_3 I_3 + \frac{1}{Cs}(I_3 - I_2) = 0 \end{cases}$$

Above set of equations can be solved and then $H(s)$ could be derived from $V_o = R_2 I_2$ as:

$$H(s) = \frac{V_o(s)}{V_s(s)} = \frac{(R_1 R_3 C^2 s^2 + 2R_1 Cs + 1)R_2}{R_1 R_3 R_2 C^2 s^2 + (2R_1 R_2 + 2R_1 R_3 + R_3 R_2)Cs + R_2 + R_3}$$

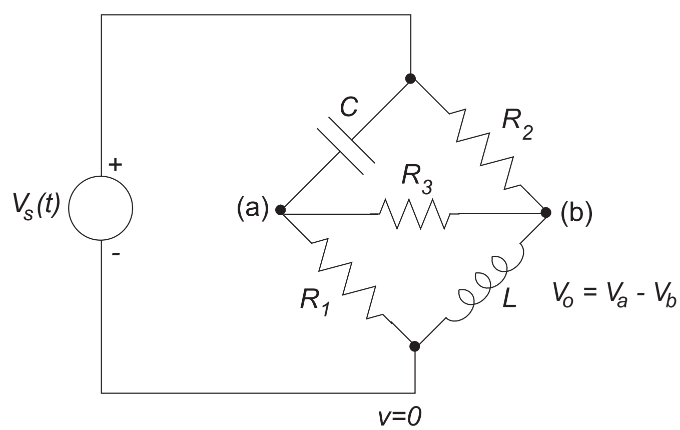
The algebraic manipulations can be done with a symbolic math software. Here we have used MATLAB to solve the equations.

MATLAB Code :

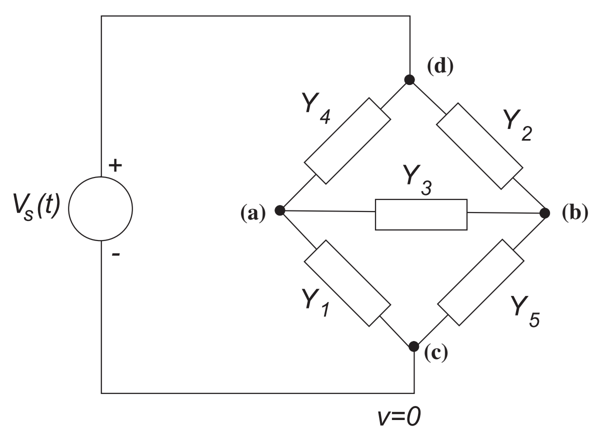
```
>> syms R1 R2 R3 C Vs s I1 I2 I3 real
>> eq1=-Vs+1/(C*s)*(I1-I3)+R1*(I1-I2);
>> eq2=R1*(I2-I1)+1/(C*s)*(I2-I3)+R2*I2;
>> eq3=1/(C*s)*(I3-I1)+R3*I3+1/(C*s)*(I3-I2);
>> [I1,I2,I3]=solve(eq1,eq2,eq3,I1,I2,I3);
>> H_s=simplify(I2*R2/Vs)
```

```
H_s =

(1+2*R1*C*s+R1*C^2*s^2*R3)/(C*s*R2*R3+R2*R3*R1*s^2*C^2+2*C*s*R1*R3
...+R3+R2+2*R2*R1*s*C)*R2


>> H_s=collect(H_s,s)

H_s =

(1+2*R1*C*s+R1*C^2*s^2*R3)/(R2*R3*R1*s^2*C^2+(2*R2*R1*C+2*C*R1*R3+
...C*R2*R3)*s+R2+R3)*R2
```

Problem 2:



(a)



(b) KCL equations for node (a) and (b) are derived. Note that $\mathrm{I}_{\vec{ab}}$ stands for the current in branch ab from (a) toward (b).

$$\begin{cases} \text{(a)}: & \mathrm{I}_{\vec{ad}} + \mathrm{I}_{\vec{ab}} + \mathrm{I}_{\vec{ac}} = 0 \\ \text{(b)}: & \mathrm{I}_{\vec{bd}} + \mathrm{I}_{\vec{ba}} + \mathrm{I}_{\vec{bc}} = 0 \end{cases} \Rightarrow \begin{cases} \text{(a)}: & Y_4(V_a - V_d) + Y_3(V_a - V_b) + Y_1(V_a - V_c) = 0 \\ \text{(b)}: & Y_2(V_b - V_d) + Y_3(V_b - V_a) + Y_5(V_b - V_c) = 0 \end{cases}$$

(c) To simplify the equations, we can assume that $V_c = 0, V_d = V_s$.

$$\begin{cases} \text{(a)}: & Y_4(V_a - V_s) + Y_3(V_a - V_b) + Y_1(V_a) = 0 \\ \text{(b)}: & Y_2(V_b - V_s) + Y_3(V_b - V_a) + Y_5(V_b) = 0 \end{cases}$$

(d) This part and the next part are solved with MATLAB by the attached code. We solve the equations for V_a and V_b and then we form a transfer function relating $V_o(s)$ to $V_s(s)$ from $V_o(s) = V_a - V_b$:

$$H(s) = \frac{Y_5Y_4 - Y_2Y_1}{Y_2Y_4 + Y_2Y_3 + Y_2Y_1 + Y_3Y_4 + Y_3Y_1 + Y_5Y_4 + Y_5Y_3 + Y_5Y_1}$$

(e) Substitute $Y_1 = \frac{1}{R_1}, Y_2 = \frac{1}{R_2}, Y_3 = \frac{1}{R_3}, Y_4 = Cs, Y_5 = \frac{1}{Ls}$ to the former relation:

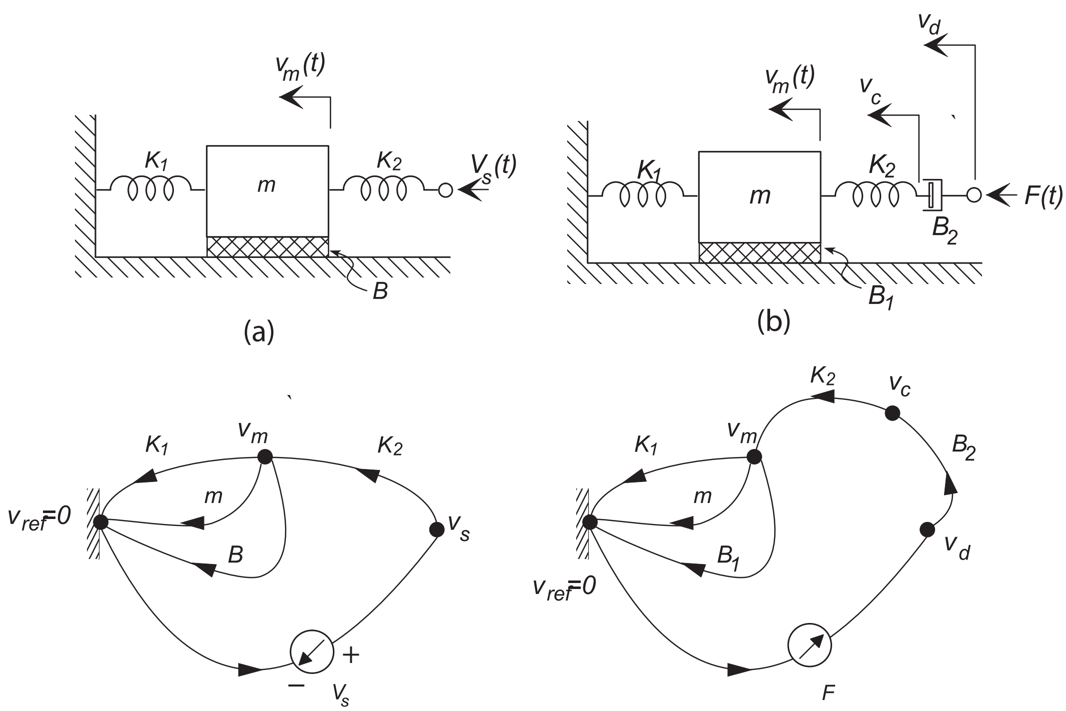
$$H(s) = \frac{(CR_2R_1 - L)R_3s}{CLR_1(R_3 + R_2)s^2 + ((R_1 + R_2 + R_3)L + CR_1R_2R_3)s + R_2(R_1 + R_3)}$$

MATLAB Code :

```
>> clear all
>> syms Y1 Y2 Y3 Y4 Y5 Vs Va Vb real
>> eq_a=Y4*(Va-Vs)+Y3*(Va-Vb)+Y1*Va;
>> eq_b=Y2*(Vb-Vs)+Y3*(Vb-Va)+Y5*Vb;
>> [Va,Vb]=solve(eq_a,eq_b,Va,Vb)

Va =

Vs*(Y2*Y3+Y3*Y4+Y2*Y4+Y5*Y4)/(Y2*Y4+Y2*Y3+Y2*Y1+Y3*Y4+Y3*Y1+Y5*Y4+Y5*Y3+Y5*Y1)



Vb =

Vs*(Y2*Y4+Y2*Y3+Y3*Y4+Y2*Y1)/(Y2*Y4+Y2*Y3+Y2*Y1+Y3*Y4+Y3*Y1+Y5*Y4+Y5*Y3+Y5*Y1)


>> H=simplify((Va-Vb)/Vs)

H =

-(-Y5*Y4+Y2*Y1)/(Y2*Y4+Y2*Y3+Y2*Y1+Y3*Y4+Y3*Y1+Y5*Y4+Y5*Y3+Y5*Y1)


>> syms s L C R1 R2 R3
>> Hs=simplify(subs(H,{Y1 Y2 Y3 Y4 Y5},{1/R1 1/R2 1/R3 C*s 1/(L*s)}));
```

```
>> Hs=collect(Hs,s)

Hs =

(C*R2*R1-L)*R3*s/((C*R3*R1*L+C*R2*R1*L)*s^2+(R2*L+L*R1+C*R2*R3*R1+L*R3)*s
...+R2*R1+R2*R3)
```

Problem 3:



(a) The linear graph models of the system are shown in the above figure. We write the KCL for node (m) and then substitute for admittances. The positive sign for current (force) corresponds to the current out of a node:

$$F_m + F_B + F_{K_1} - F_{K_2} = 0$$

$$Y_m(V_m - V_{ref}) + Y_B(V_m - V_{ref}) + Y_{K_1}(V_m - V_{ref}) - Y_{K_2}(V_s - V_m) = 0$$

$$ms(V_m - 0) + B(V_m - 0) + \frac{K_1}{s}(V_m - 0) - \frac{K_2}{s}(V_s - V_m) = 0$$

$$\frac{V_m(s)}{V_s(s)} = \frac{K_2}{ms^2 + Bs + (K_1 + K_2)}$$

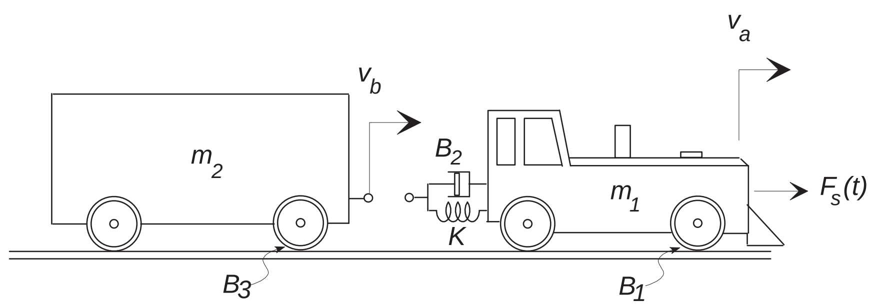
(b) We write the KCL for node (m), (c) and (d) and follow the same procedure. Note that K_2 and B_2 as series elements have equal force in their terminals; because they have no inertial force (they have no mass). Consequently as series element connecting mass m to force source $F(t)$, they do not contribute our transfer function.

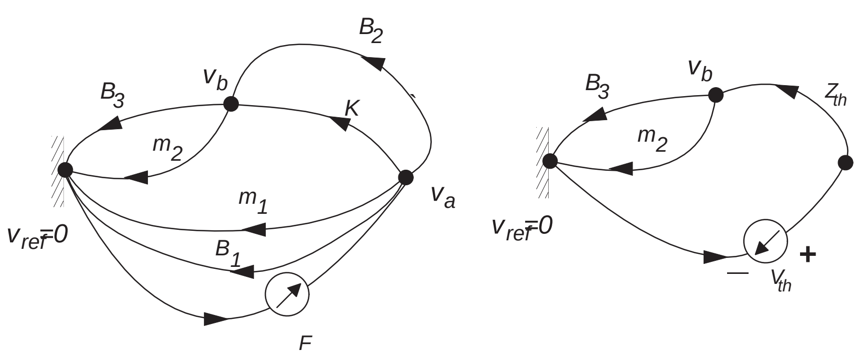
$$\begin{cases} \text{node (m)} & F_m + F_{B_1} + F_{K_1} - F_{K_2} = 0 \\ \text{node (c)} & F_{K_2} - F_{B_2} = 0 \\ \text{node (d)} & F_{B_2} - F = 0 \end{cases} \Rightarrow F_m + F_{B_1} + F_{K_1} - F = 0$$

$$ms(V_m - 0) + B_1(V_m - 0) + \frac{K_1}{s}(V_m - 0) - F = 0$$

$$\frac{V_m(s)}{F(s)} = \frac{s}{ms^2 + B_1 s + K_1}$$

Problem 4:





(a) The linear graph model of the system is shown above. The locomotive section is equivalently represented by a Thévenin source and impedance. The Thévenin impedance is computed by removing the force (current) source and looking to the system from node (b) and (ref):

$$Z_{th} = (Z_{B_2}||Z_K) + (Z_{B_1}||Z_{m_1}) = (\frac{1}{B_2}||\frac{s}{K}) + (\frac{1}{B_1}||\frac{1}{ms}) = \frac{s}{K + B_2 s} + \frac{1}{ms + B_1}$$

The Thévenin source is computed by open circuiting node (b) and (ref). In that case $v_b = v_a$ and $V_{th} = V_{b-ref} = V_{a-ref}$:

$$V_{th} = F_s(Z_{B_1}||\ Z_{m_1}) = \frac{F_s}{ms + B_1}$$

(b) Look at the previous figure.

<u>Problem 5:</u>

(a)

$$P = RI^2 = R(\frac{V_{th}}{R + R_o})^2 = V_{th}^2 \frac{R}{(R + R_o)^2}$$

(b)

$$\frac{dP}{dR} = V_{th}^2 \frac{1}{(R + R_o)^2}(1 - 2\frac{R}{R + R_o}) = V_{th}^2 \frac{R_o - R}{(R + R_o)^3}$$

$$\text{Maximum Power} \Rightarrow \frac{dP}{dR} = 0 \Rightarrow R = R_o$$