Review: step response of 1st order systems

Step response in the s–domain

$$\frac{a}{s(s+a)};$$

in the time domain

$$\left(1-\mathrm{e}^{-at}\right)u(t);$$

time constant

$$\tau=\frac{1}{a};$$

rise time (10%→90%)

$$T_r=\frac{2.2}{a};$$

settling time (98%)

$$T_s=\frac{4}{a}.$$

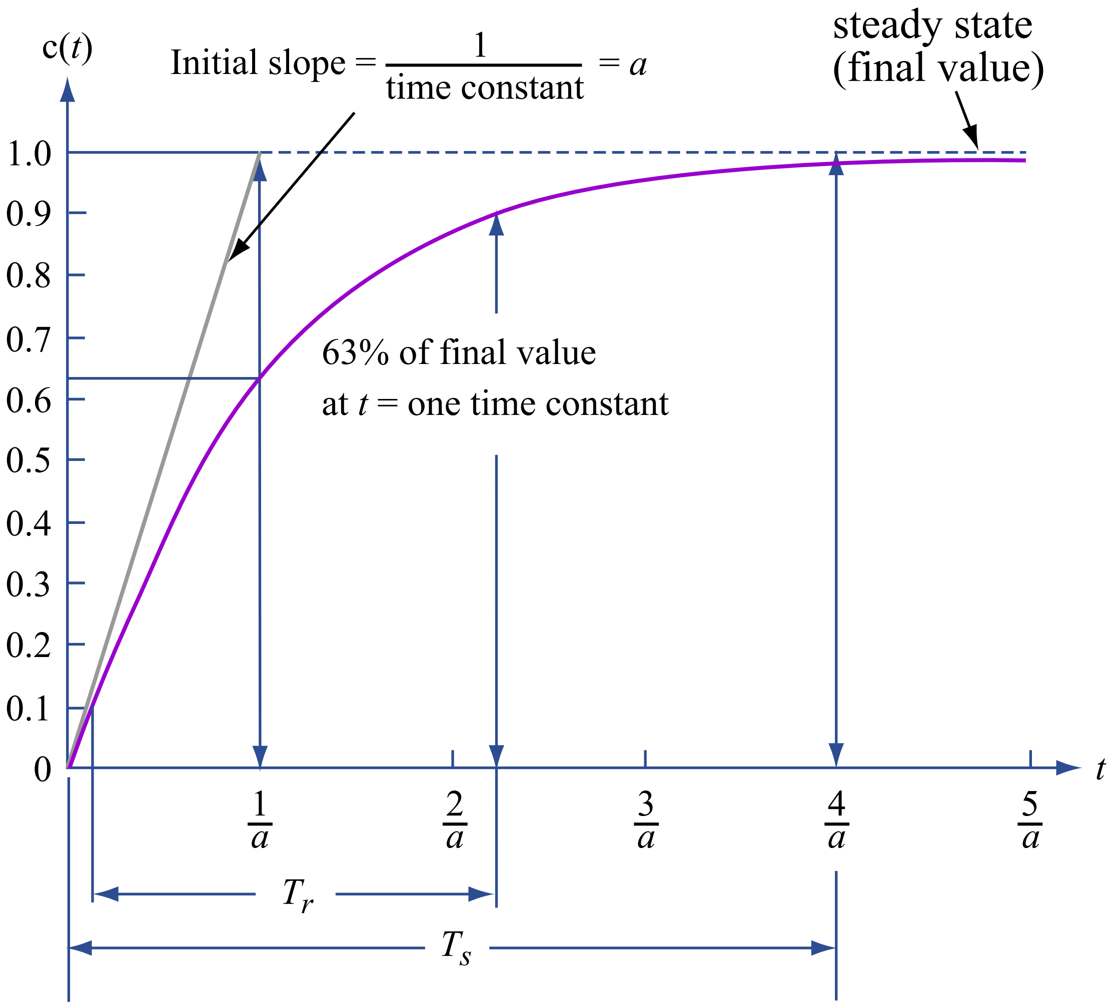
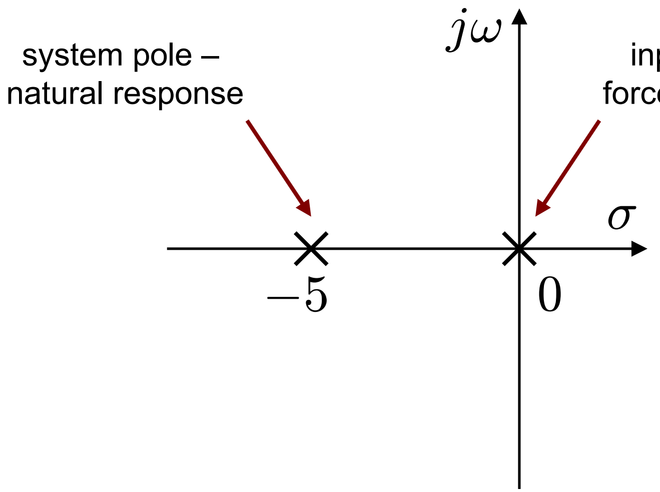
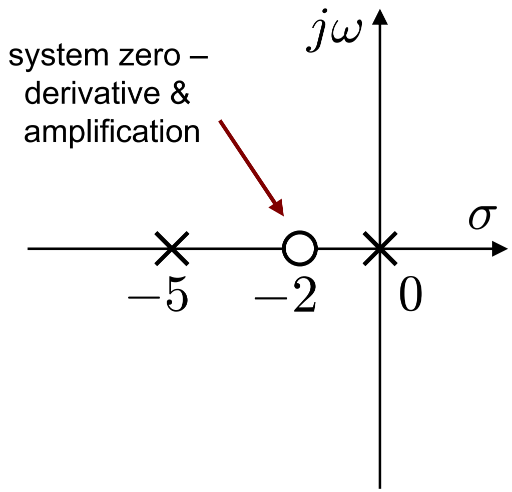


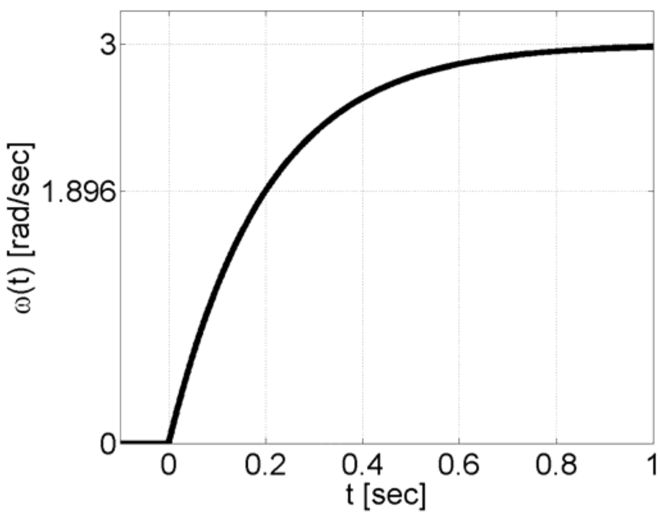
Figure 4.3

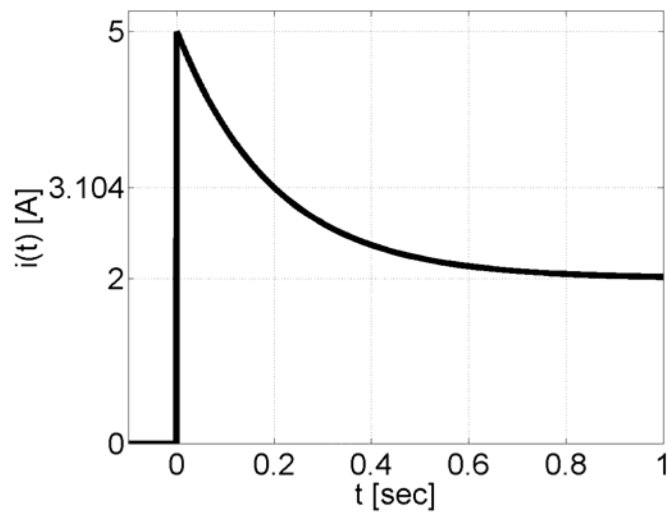
Figure by MIT OpenCourseWare.

Review: poles, zeros, and the forced/natural responses





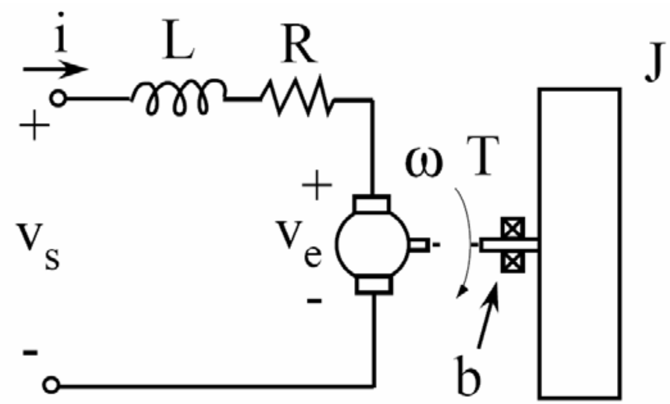




Goals for today

- Second-order systems response
  - types of 2nd-order systems
    - overdamped
    - underdamped
    - undamped
    - critically damped
  - transient behavior of overdamped 2nd-order systems
  - transient behavior of underdamped 2nd-order systems
  - DC motor with non-negligible impedance
- **Next lecture (Friday):**
  - examples of modeling & transient calculations for electro-mechanical 2nd order systems

DC motor system with non-negligible inductance



Recall **combined equations of motion**

$$\left.\begin{aligned} LsI(s) + RI(s) + K_v\Omega(s) &= V_s(s) \\ Js\Omega(s) + b\Omega(s) &= K_m I(s) \end{aligned}\right\} \Rightarrow$$

$$\left\{\begin{aligned} &\left[\frac{LJ}{R}s^2 + \left(\frac{Lb}{R} + J\right)s + \left(b + \frac{K_m K_v}{R}\right)\right]\Omega(s) = \frac{K_m}{R}V_s(s) \\ &(Js + b)\,\Omega(s) = K_m I(s) \end{aligned}\right.$$

Including the DC motor's inductance, we find

$$\left\{\begin{aligned} \frac{\Omega(s)}{V_s(s)} &= \frac{K_m}{LJ}\frac{1}{s^2 + \left(\frac{b}{J} + \frac{R}{L}\right)s + \left(\frac{bR + K_m K_v}{LJ}\right)} \\ \frac{I(s)}{V_s(s)} &= \frac{1}{R}\frac{\left(s + \frac{b}{J}\right)}{s^2 + \left(\frac{b}{J} + \frac{R}{L}\right)s + \left(\frac{bR + K_m K_v}{LJ}\right)} \end{aligned}\right.$$

Quadratic polynomial denominator
Second–order system

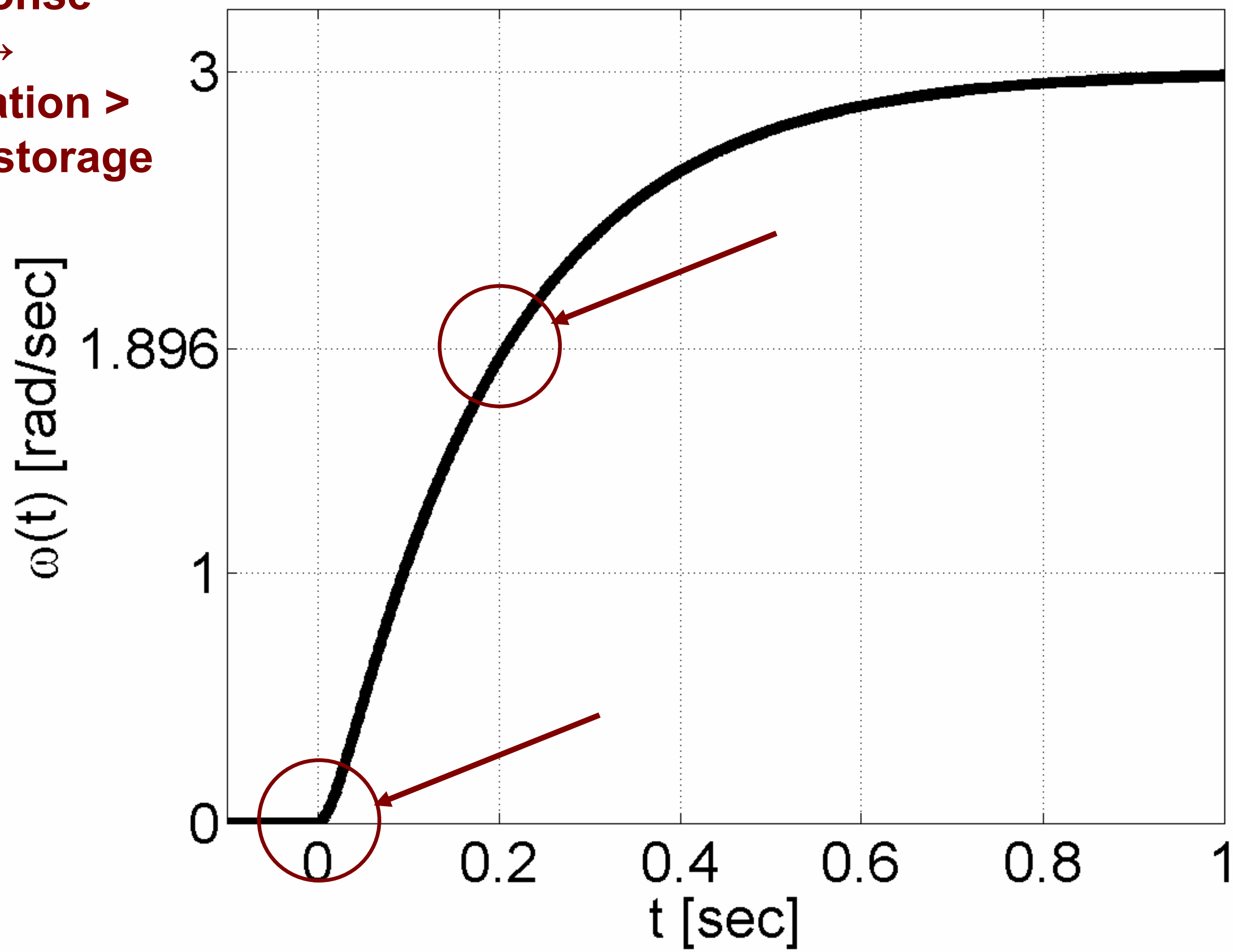
Step response of 2nd order system – large *R*/*L*

Overdamped response

$\leftrightarrow$

dissipation > energy storage

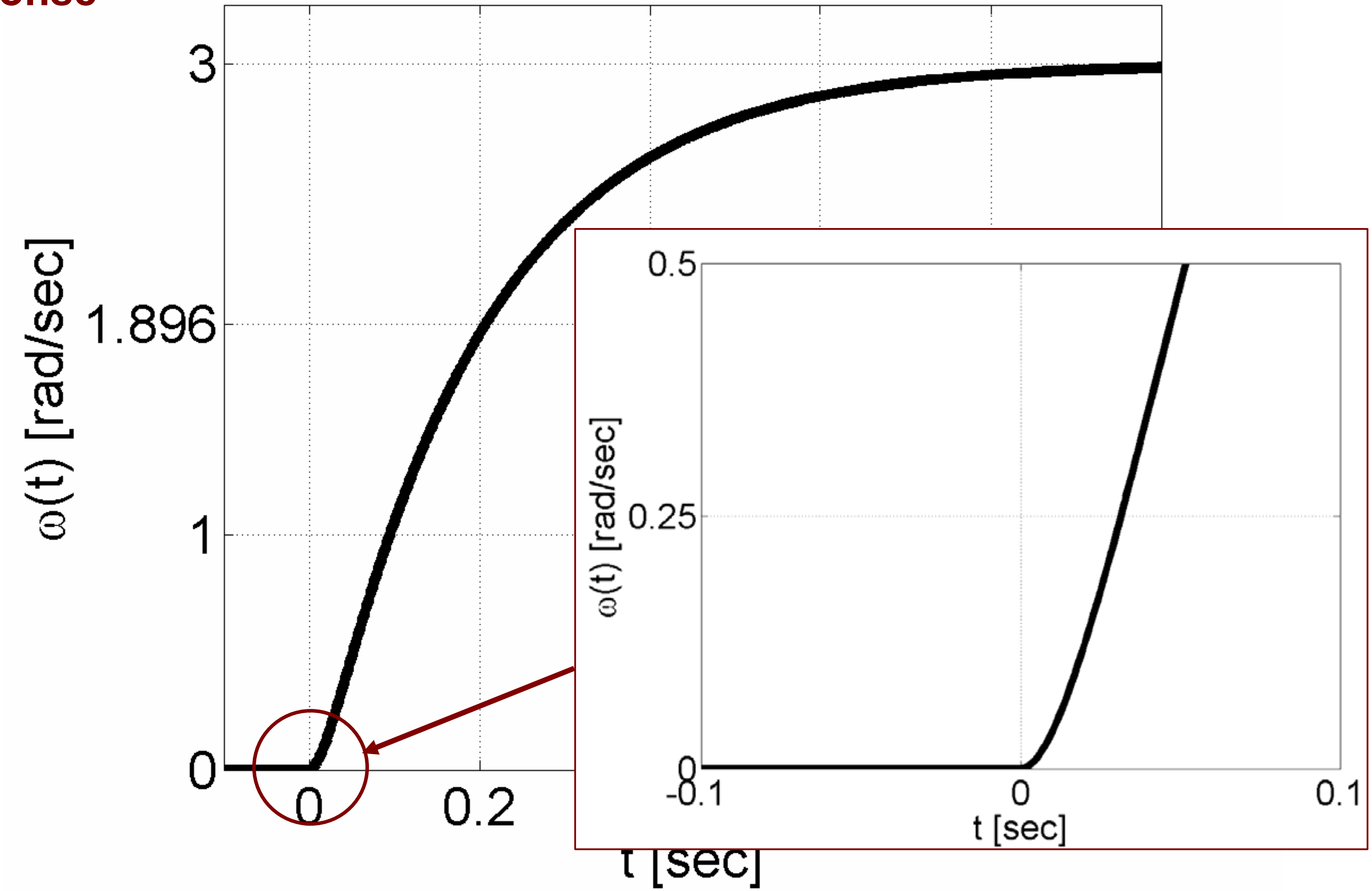
$$\underline{L = 0.1\mathrm{H},}\ K_v = 6\mathrm{V}\cdot\mathrm{sec},\ K_m = 6\mathrm{N}\cdot\mathrm{m/A},\ J = 2\mathrm{kg}\cdot\mathrm{m}^2,$$
$$R = 6\Omega,\ b = 4\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{Hz};\ v_s(t) = 30u(t)\ \mathrm{V}.$$



Step response of 2nd order system – large *R/L*

Overdamped response

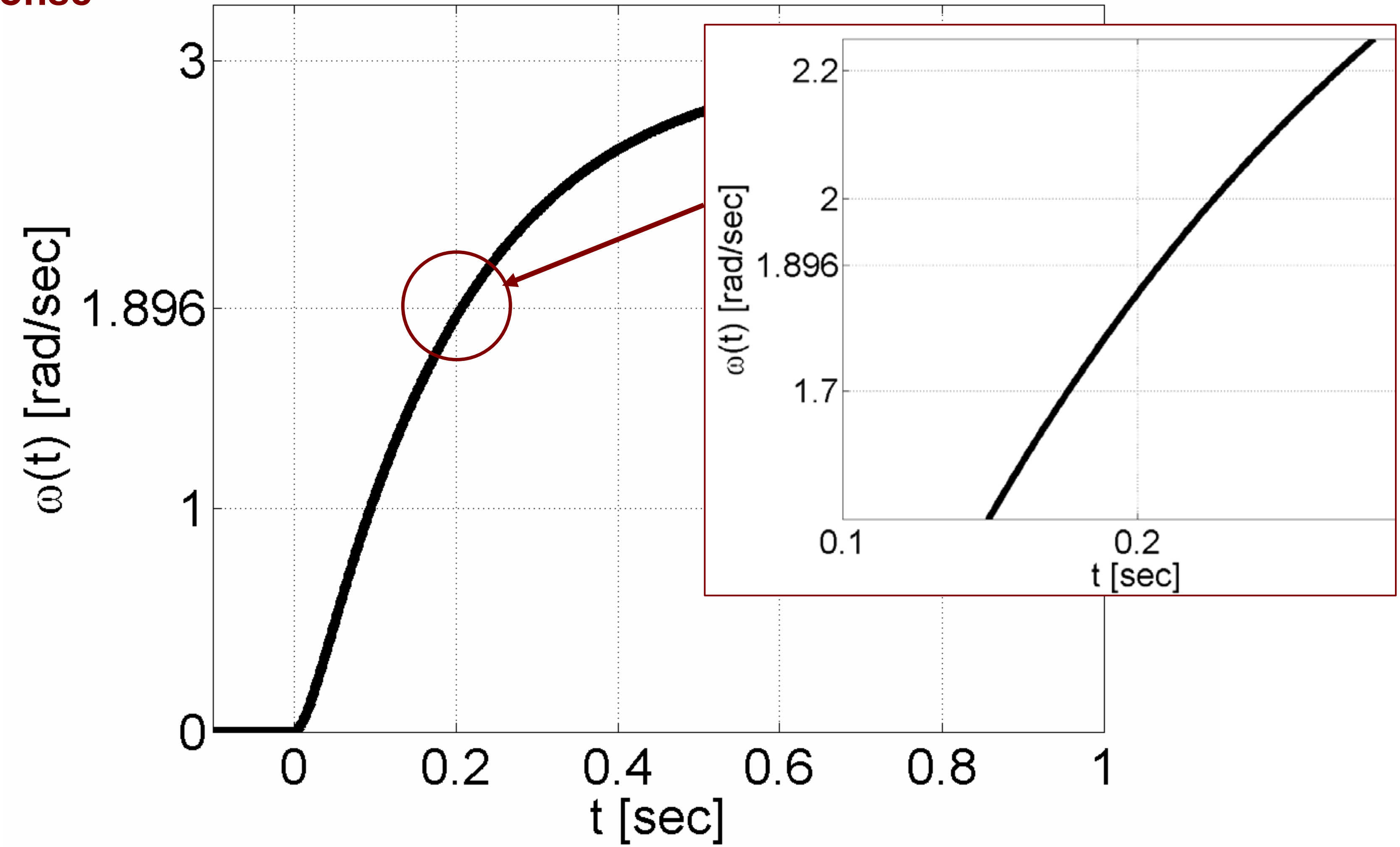
$$L = 0.1\text{H},\ K_v = 6\text{V}\cdot\text{sec},\ K_m = 6\text{N}\cdot\text{m/A},\ J = 2\text{kg}\cdot\text{m}^2,$$
$$R = 6\Omega,\ b = 4\text{kg}\cdot\text{m}^2\cdot\text{Hz};\ v_s(t) = 30u(t)\ \text{V}.$$



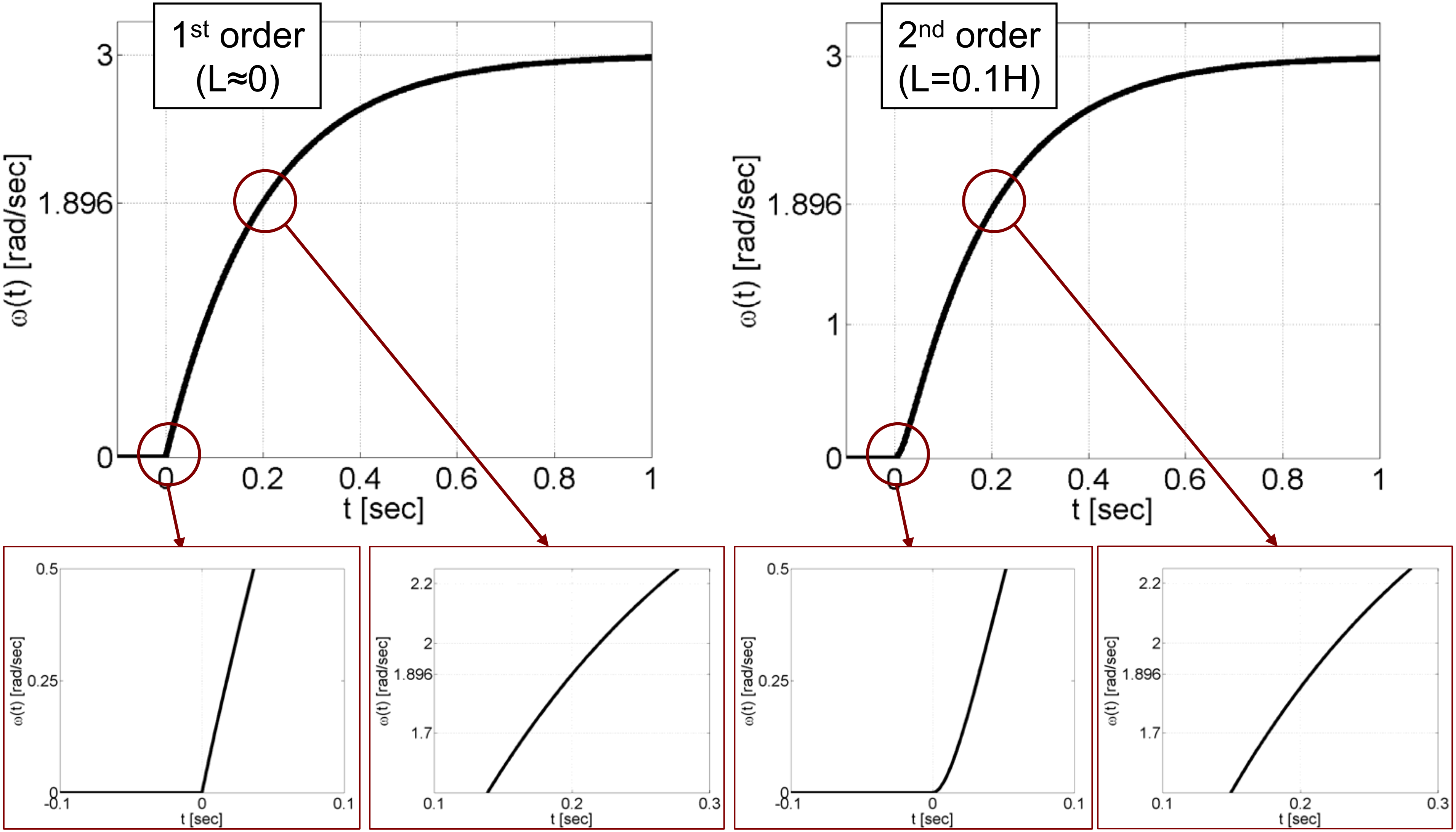
Step response of 2nd order system – large *R*/*L*

Overdamped response

$$L = 0.1\text{H},\ K_v = 6\text{V}\cdot\text{sec},\ K_m = 6\text{N}\cdot\text{m/A},\ J = 2\text{kg}\cdot\text{m}^2,$$
$$R = 6\Omega,\ b = 4\text{kg}\cdot\text{m}^2\cdot\text{Hz};\ v_s(t) = 30u(t)\ \text{V}.$$



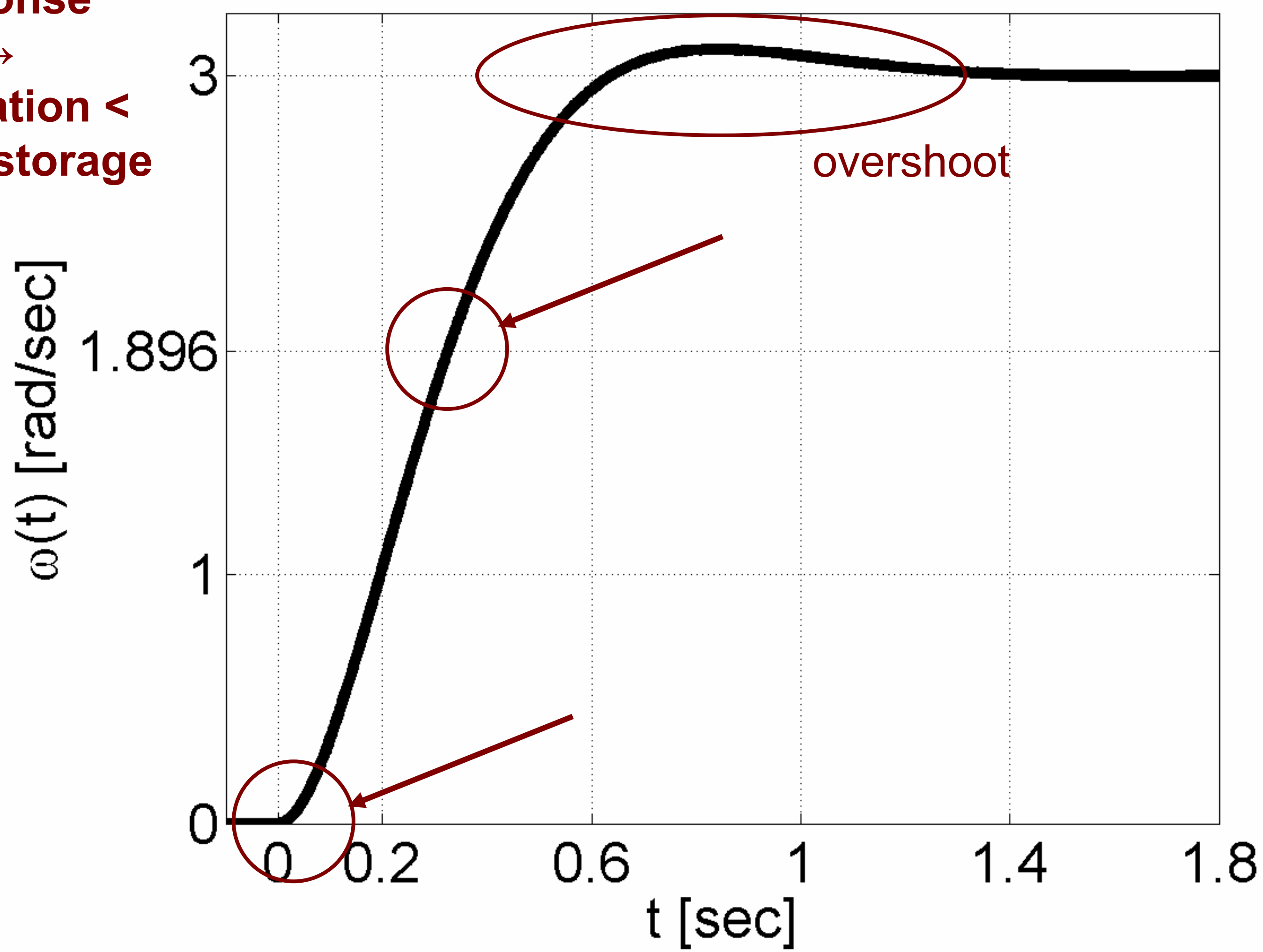
Comparison of 1st order and 2nd order overdamped



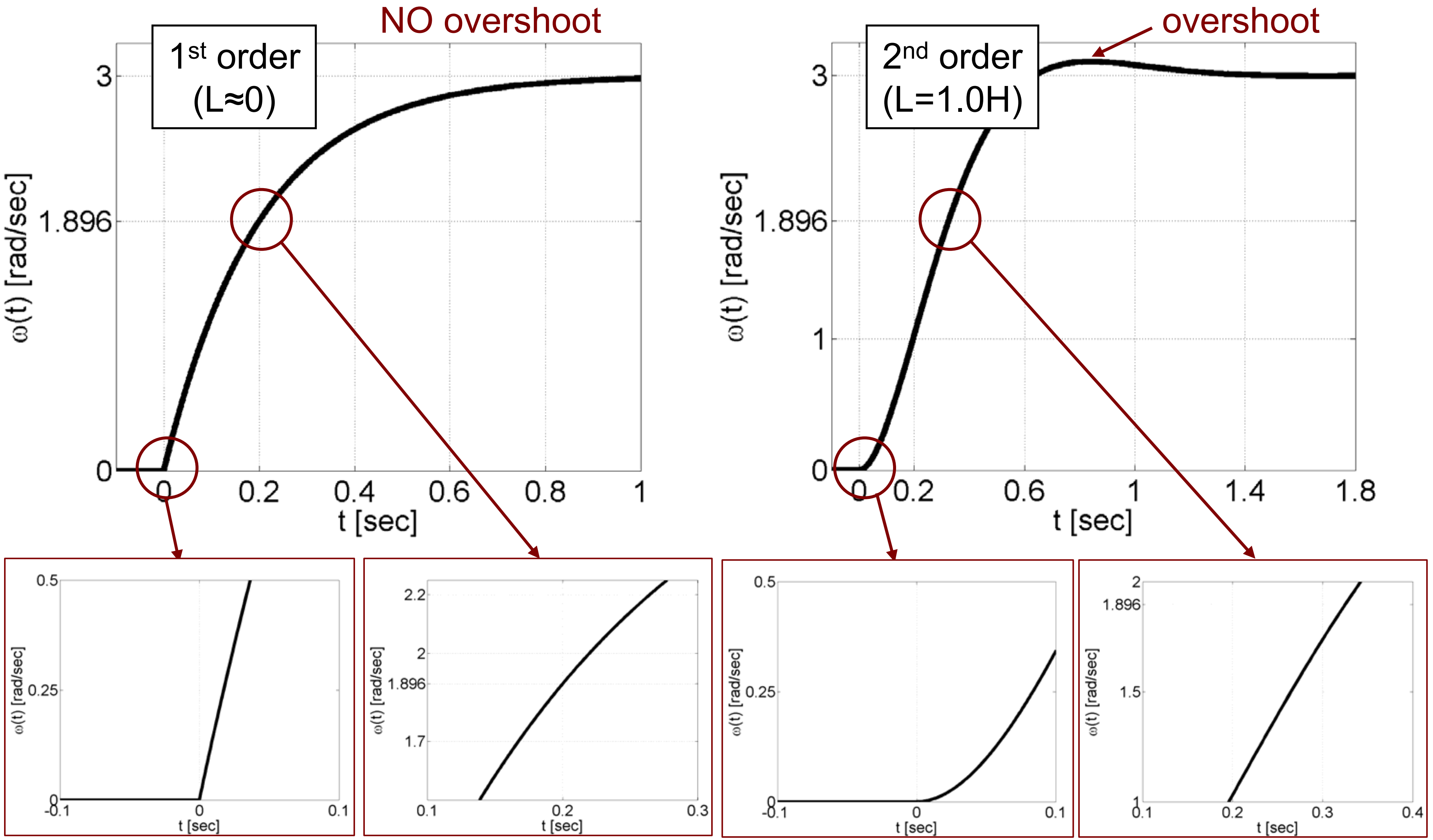
Step response of 2nd order system – small *R*/*L*

Underdamped response
↔
dissipation < energy storage

$$L = 1.0\text{H},\ K_v = 6\text{V}\cdot\text{sec},\ K_m = 6\text{N}\cdot\text{m/A},\ J = 2\text{kg}\cdot\text{m}^2,$$
$$R = 6\Omega,\ b = 4\text{kg}\cdot\text{m}^2\cdot\text{Hz};\ v_s(t) = 30u(t)\ \text{V}.$$



Comparison of 1st order and 2nd order underdamped



Overdamped DC motor: derivation of the step response

Using the numerical values $L = 0.1\text{H}$, $K_v = 6\text{V}\cdot\text{sec}$, $K_m = 6\text{N}\cdot\text{m/A}$, $J = 2\text{kg}\cdot\text{m}^2$, $R = 6\Omega$, $b = 4\text{kg}\cdot\text{m}^2\cdot\text{Hz}$ we find

$$\frac{K_m}{LJ} = 30\frac{\text{rad}}{\text{sec}\cdot\text{V}}; \qquad \frac{b}{J} + \frac{R}{L} = 62\text{rad/sec}; \qquad \frac{bR + K_m K_v}{LJ} = 300\,(\text{rad/sec})^2 .$$

Therefore, the transfer function for the angular velocity is

$$\frac{\Omega(s)}{V_s(s)} = \frac{30}{s^2 + 62s + 300}.$$

A 2^{nd}–order system is **overdamped** if the transfer function denominator has **two real roots**.

We find that the denominator has **two real roots**,

$$s_1 = -5.290\text{Hz}, \qquad s_2 = -56.71\text{Hz} \quad \Rightarrow \quad \frac{\Omega(s)}{V_s(s)} = \frac{30}{(s+5.290)(s+56.71)}.$$

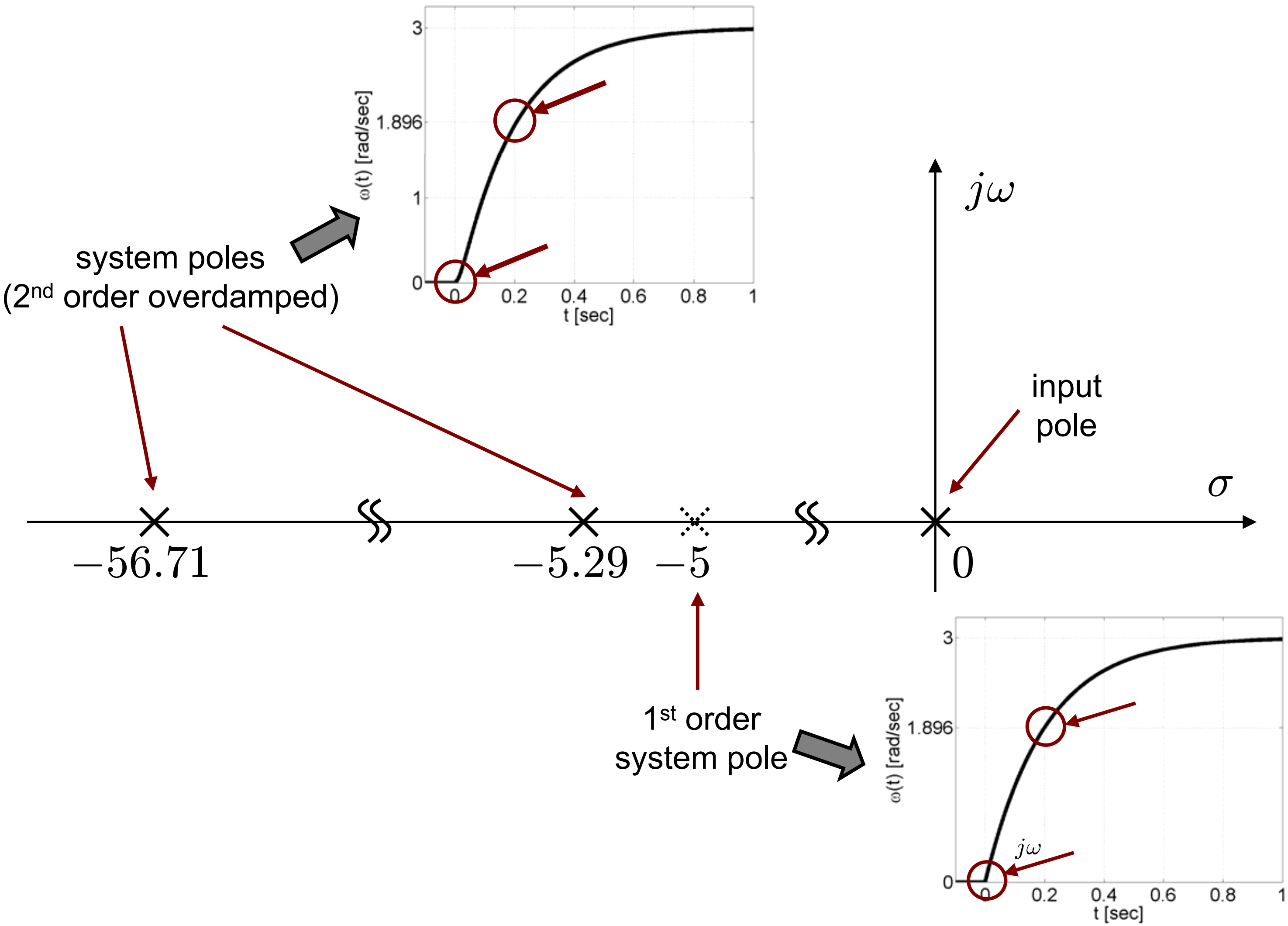
To compute the step response we substitute the Laplace transform of the voltage source $V_s(s) = 30/s$ and carry out the partial fraction expansion:

$$\Omega(s) = \frac{900}{s(s+5.290)(s+56.71)} = \frac{3}{s} - \frac{3.3}{s+5.290} + \frac{0.3}{s+56.71} \Rightarrow$$

$$\omega(t) = \left[3 - 3.3\mathrm{e}^{-5.29t} + 0.3\mathrm{e}^{-56.71t}\right] u(t).$$

This is the function whose plot we analyzed in slides #5–8.

Overdamped DC motor in the *s*-domain



Undamped DC motor: *no dissipation*

Consider the opposite extreme where the dissipation due to both the resistor and bearings friction is negligible, *i.e.* $R = 0$ and $b = 0$. Using the same remaining numerical values $L = 0.1\text{H}$, $K_v = 6\text{V}\cdot\text{sec}$, $K_m = 6\text{N}\cdot\text{m/A}$, $J = 2\text{kg}\cdot\text{m}^2$, we find

$$\frac{K_m}{LJ} = 30\frac{\text{rad}}{\text{sec}\cdot\text{V}}; \qquad \frac{b}{J} + \frac{R}{L} = 0; \qquad \frac{bR + K_m K_v}{LJ} = 180\,(\text{rad/sec})^2\,.$$

Therefore, the transfer function for the angular velocity is

$$\frac{\Omega(s)}{V_s(s)} = \frac{30}{s^2 + 180}.$$

A 2^{nd}–order system is **undamped** if the transfer function denominator has a **conjugate pair of two imaginary roots**.

The denominator has **a conjugate pair of two imaginary roots**,

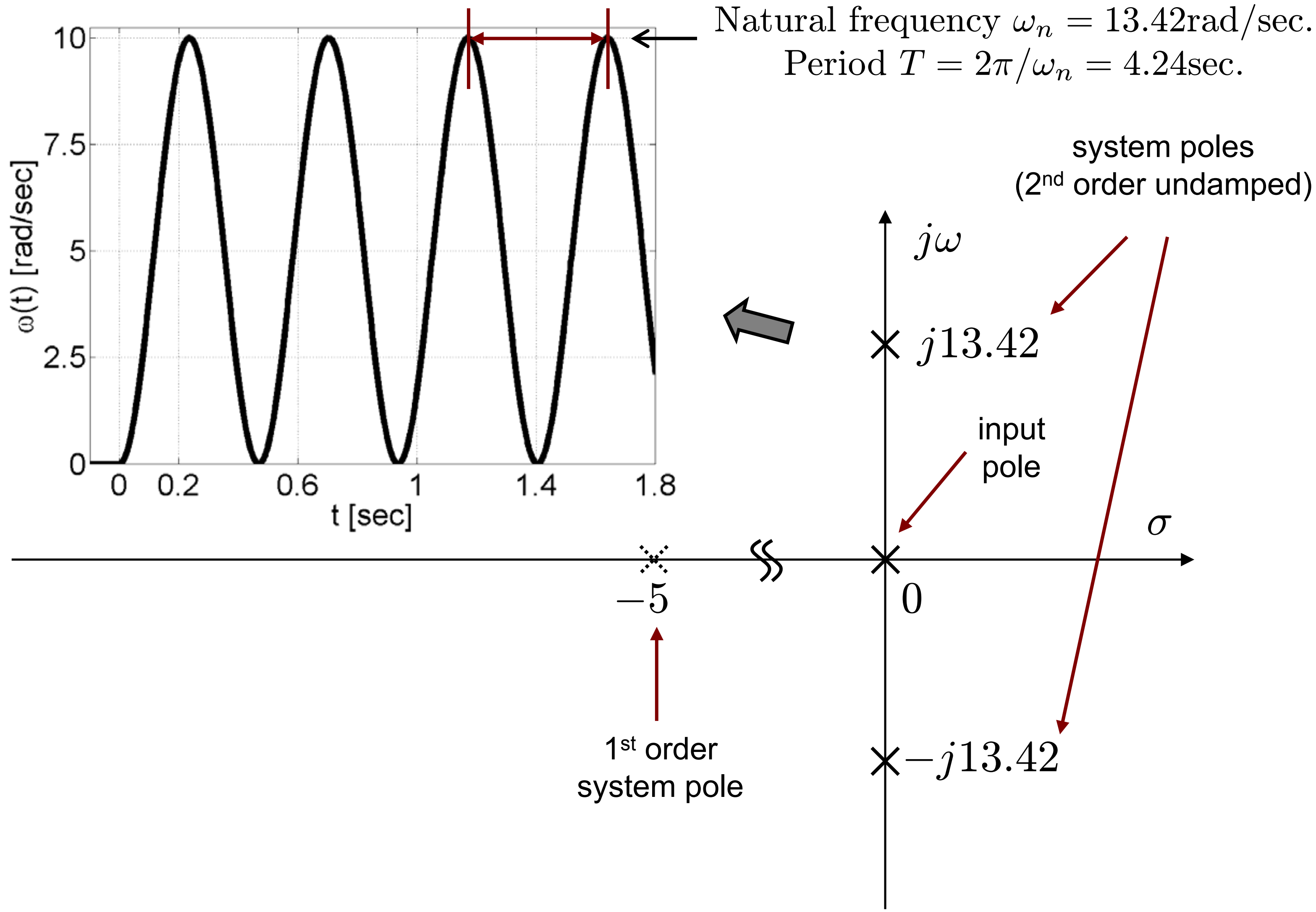
$$s_{1,2} = \pm j13.42\text{Hz} \quad \Rightarrow \quad \frac{\Omega(s)}{V_s(s)} = \frac{30}{(s + j13.42)(s - j13.42)}.$$

Again, the step response is found by partial fraction expansion:

$$\Omega(s) = \frac{900}{s(s + j13.42)(s - j13.42)} = \frac{5}{s} - \frac{5s}{s^2 + (13.42)^2} \Rightarrow$$

$$\omega(t) = [5 - 5\cos(13.42t)]\,u(t).$$

Undamped DC motor in the *s*-domain



Underdamped DC motor: *small dissipation*

Finally, let us return to what we previously labelled as "underdamped" case, *i.e.* $L = 1.0\text{H}$, $K_v = 6\text{V}\cdot\text{sec}$, $K_m = 6\text{N}\cdot\text{m/A}$, $J = 2\text{kg}\cdot\text{m}^2$, $R = 6\Omega$, $b = 4\text{kg}\cdot\text{m}^2\cdot\text{Hz}$. The values of L, R are such that the dissipation in the system is negligible compared to the energy storage capacity. We then find

$$\frac{K_m}{LJ} = 3\frac{\text{rad}}{\text{sec}\cdot\text{V}}; \qquad \frac{b}{J} + \frac{R}{L} = 8\text{rad/sec}; \qquad \frac{bR + K_m K_v}{LJ} = 30\,(\text{rad/sec})^2\,.$$

Therefore, the transfer function for the angular velocity is

$$\frac{\Omega(s)}{V_s(s)} = \frac{30}{s^2 + 8s + 30}.$$

A 2^{nd}–order system is **underdamped** if the transfer function denominator has a **conjugate pair of two complex roots**.

This denominator has **a conjugate pair of two complex roots**,

$$s_{1,2} = -4 \pm j3.74 \text{ rad/sec} \quad \Rightarrow \quad \frac{\Omega(s)}{V_s(s)} = \frac{30}{(s+4+j3.74)(s+4-j3.74)}.$$

We will now develop the partial fraction expansion method for this case, aiming to find the step response:

$$\Omega(s) = \frac{90}{s(s+4+j3.74)(s+4-j3.74)} = \frac{90}{s(s^2+8s+30)} = 90\left(\frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+8s+30}\right).$$

Underdamped DC motor: *small dissipation*

Using the familiar partial fraction method, we can find $K_1 = 1/30$, $K_2 = -1/30$, $K_3 = -8/30$, therefore

$$\Omega(s) = 3\left(\frac{1}{s} - \frac{s+8}{s^2+8s+30}\right).$$

To find the inverse Laplace transform, we rewrite the denominator as a complete square plus a constant, and break down the numerator into the sum of *the same factor* that appeared in the denominator's complete square plus another constant:

$$\Omega(s) = 3\left(\frac{1}{s} - \frac{(s+4)+4}{(s+4)^2+14}\right).$$

If the complete square instead of $(s+4)^2$ were of the form s^2, the inverse Laplace transform would have followed easily from Nise Table 2.1:

$$\mathcal{L}^{-1}\left[\frac{s+4}{s^2+14}\right] = \mathcal{L}^{-1}\left[\frac{s+4}{s^2+(3.74)^2}\right] = \cos(3.74t) + 4\sin(3.74t).$$

To take the extra factor of 4 into account, we must use yet another property of Laplace transforms, which we have not seen until now:

$$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a). \qquad \text{(Nise Table 2.2, \#4).}$$

Underdamped DC motor: *small dissipation*

We apply this "frequency shift" property as follows:

$$\mathcal{L}^{-1}\left[\frac{s+4}{s^2+(3.74)^2}\right]=\cos(3.74t)+4\sin(3.74t)\Rightarrow$$

$$\Rightarrow\mathcal{L}^{-1}\left[\frac{(s+4)+4}{(s+4)^2+(3.74)^2}\right]=\mathrm{e}^{-4t}\left[\cos(3.74t)+4\sin(3.74t)\right].$$

Combining all of the above results, we can finally compute the step response for the angular velocity of the DC motor as

$$\omega(t)=\left\{3-3\left(\mathrm{e}^{-4t}\left[\cos(3.74t)+4\sin(3.74t)\right]\right)\right\}u(t).$$

With a little bit of trigonometry, which we leave to you to do as exercise, we can rewrite the step response as

$$\omega(t)=\left\{3-4.39\mathrm{e}^{-4t}\cos(3.74t-0.82)\right\}u(t).$$

So the step response of the 2^{nd}–order **under**damped system is characterized by a *phase–shifted sinusoid enveloped by an exponential decay.*

This step response was analyzed in slides #9–10 of today's notes.

What the real and imaginary parts of the poles do

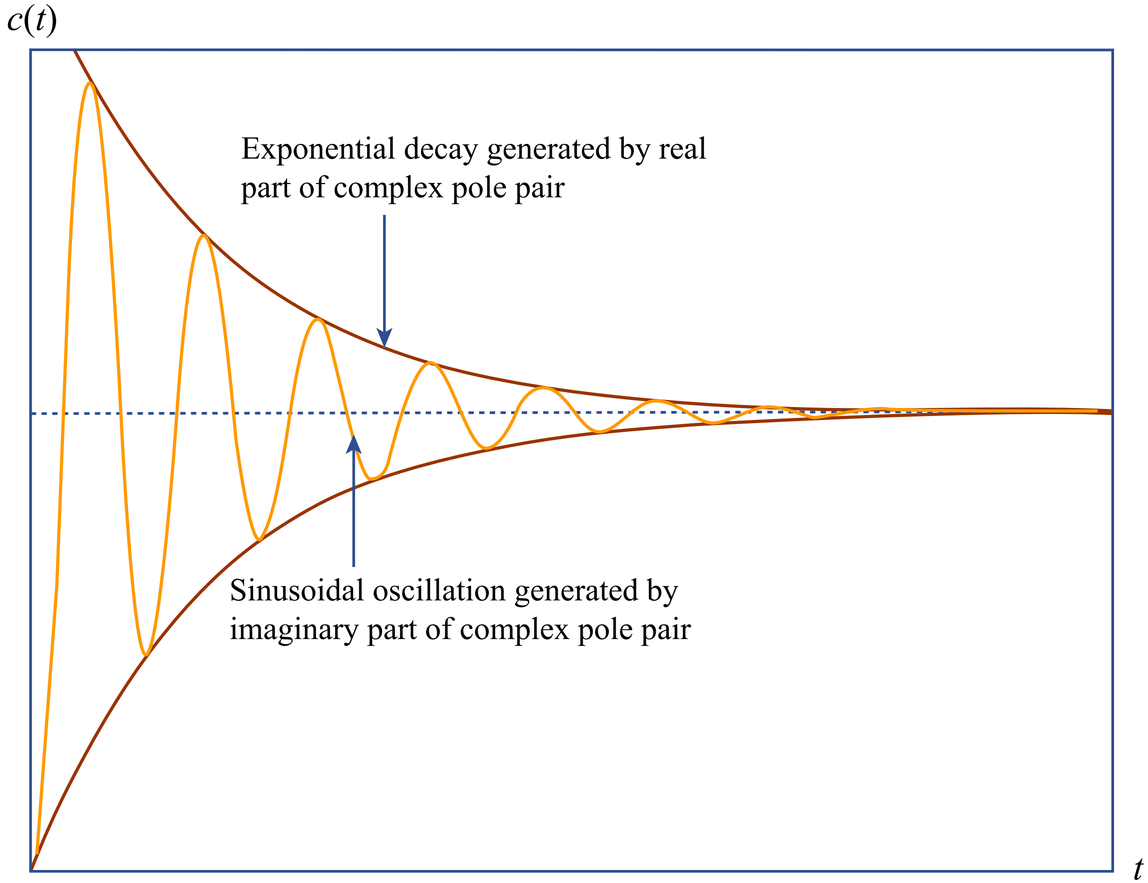


Figure by MIT OpenCourseWare.

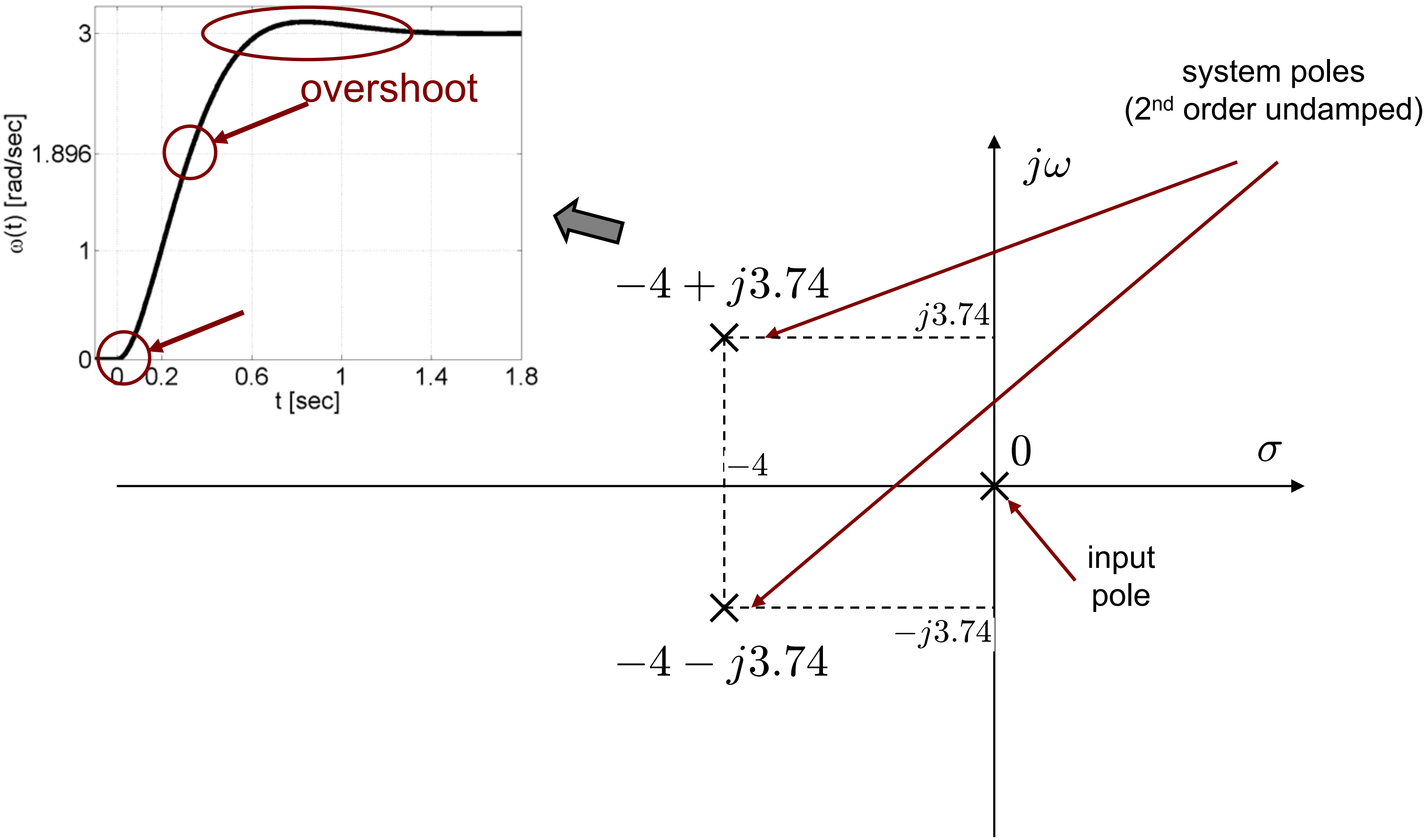
Damping ratio

$$\zeta \equiv \frac{1}{2\pi} \frac{\text{Undamped (``natural'') period}}{\text{Time constant of exponential decay}}.$$

Figure 4.8

Note: the underdamped oscillation frequency is ***<u>not</u>*** the same as the natural frequency!

Underdamped DC motor in the *s*-domain



The general 2nd order system

We can write the transfer function of the general 2^{nd}–order system with *unit* steady state response as follows:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \text{where}$$

- ω_n is the system's *natural frequency*, and
- ζ is the system's *damping ratio*.

The natural frequency indicates the oscillation frequency of the undamped ("natural") system, *i.e.* the system with energy storage elements only and without any dissipative elements. The damping ratio denotes the relative contribution to the system dynamics by energy storage elements and dissipative elements. Recall,

$$\zeta \equiv \frac{1}{2\pi} \frac{\text{Undamped (``natural'') period}}{\text{Time constant of exponential decay}}.$$

Depending on the damping ratio ζ, the system response is

- *undamped* if $\zeta = 0$;
- *underdamped* if $0 < \zeta < 1$;
- *critically damped* if $\zeta = 1$;
- *overdamped* if $\zeta > 1$.

The general 2nd order system

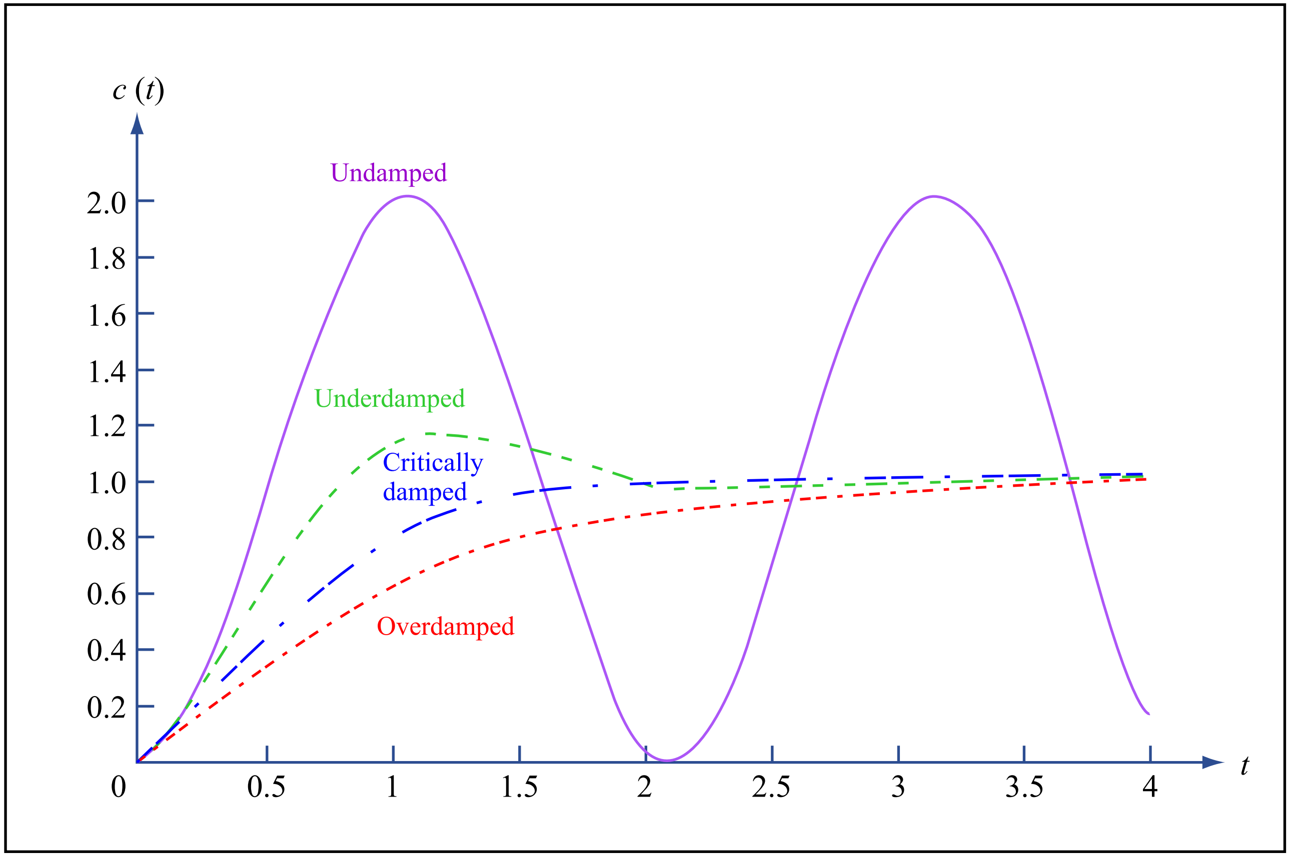


Figure by MIT OpenCourseWare.

The general 2nd order system

Images removed due to copyright restrictions.

Please see: Fig. 4.11 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Nise Figure 4.11

The underdamped 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \qquad 0 < \zeta < 1$$

The step response's Laplace transform is

$$\frac{1}{s} \times \frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

We find

$$K_1 = \frac{1}{\omega_n^2}, \qquad K_2 = -\frac{1}{\omega_n^2}, \qquad K_3 = \frac{2\zeta}{\omega_n}$$

Substituting and applying the same method of completing squares that we did in the numerical example of the DC motor's angular velocity response, we can rewrite the laplace transform of the step response as

$$\frac{1}{s} - \frac{(s + \zeta\omega_n) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2\left(1 - \zeta^2\right)}.$$

Using the frequency shifting property of Laplace transforms we finally obtain the step response in the time domain as

$$1 - \mathrm{e}^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right].$$

The underdamped 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \qquad 0 < \zeta < 1$$

Finally, using some additional trigonometry and the definitions

$$\sigma_d = \zeta\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}, \qquad \tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

we can rewrite the step response as

$$1 \quad - \quad \frac{1}{\sqrt{1-\zeta^2}} \quad \times \quad e^{-\sigma_d t} \quad \times \quad \cos\left(\omega_d t - \phi\right)$$

The definitions above can be re–written

$$\zeta = \frac{\sigma_d}{\omega_n},$$

$$\sqrt{1-\zeta^2} = \frac{\omega_d}{\omega_n},$$

$$\Rightarrow \tan\theta = \frac{\omega_d}{\sigma_d} = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

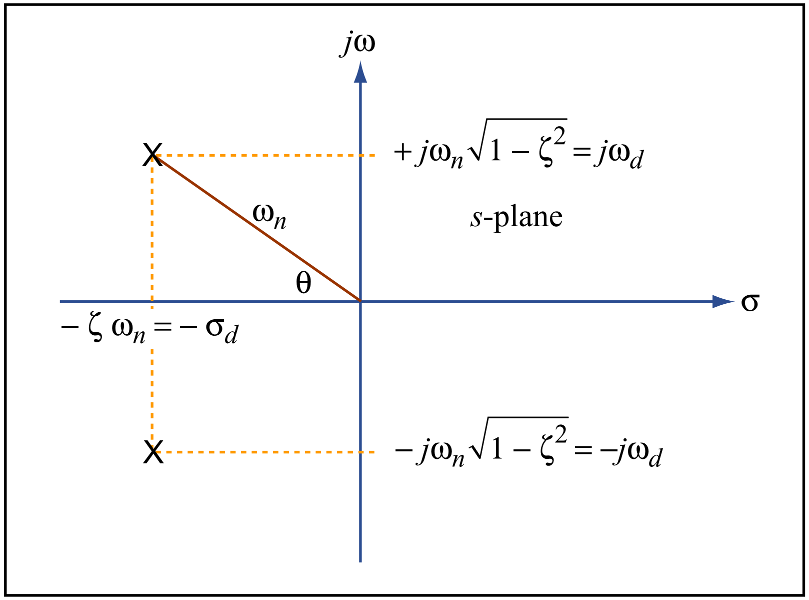


Figure 4.10

Figure by MIT OpenCourseWare.

The underdamped 2nd order system

$$\frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \qquad 0 < \zeta < 1$$

Finally, using some additional trigonometry and the definitions

$$\sigma_d = \zeta\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}, \qquad \tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

we can rewrite the step response as

$$1 \quad - \quad \frac{1}{\sqrt{1-\zeta^2}} \quad \times \quad e^{-\sigma_d t} \quad \times \quad \cos(\omega_d t - \phi)$$

forced response, sets steady state

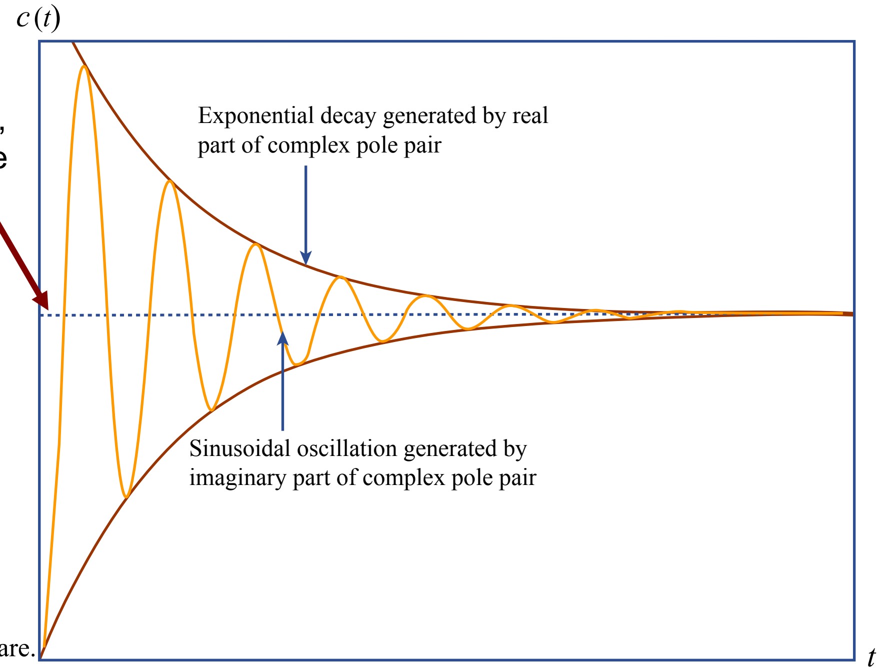


Figure 4.10

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Transients in the underdamped 2nd order system

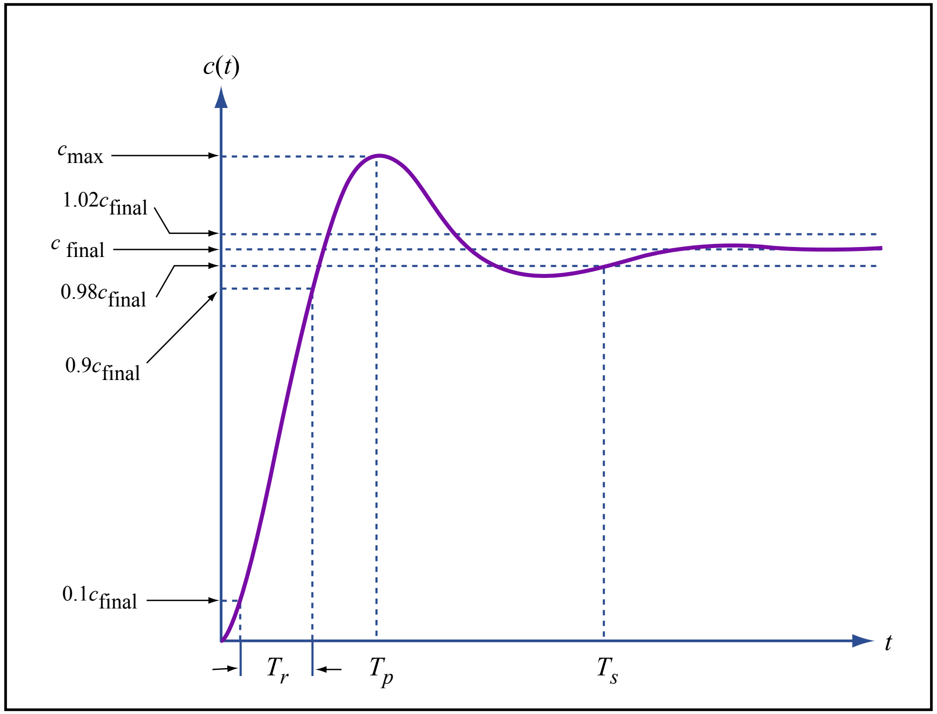

Figure by MIT OpenCourseWare.

Peak time

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}.$$

Percent overshoot (%OS)

$$\%OS = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100$$

$$\Leftrightarrow \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Settling time
(to within $\pm 2\%$ of steady state)

$$T_s = -\frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}.$$

(approximation valid for
$0 < \zeta < 0.9$.)

Figure 4.14

Transient qualities from pole location in the *s*-plane

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Please see: Fig. 4.19 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Recall

$$\zeta = \frac{\sigma_d}{\omega_n},$$

$$\sqrt{1-\zeta^2} = \frac{\omega_d}{\omega_n},$$

$$\Rightarrow \tan\theta = \frac{\omega_d}{\sigma_d} = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

Nise Figure 4.19