3 Fourier Series

Compute the Fourier series coefficients A_0, A_n, and B_n for the following signals on the interval $t = [0, 2\pi]$:

1. $f(t) = 4\sin(t + \pi/3) + \cos(3t)$
   First, write this in a fully expanded form: $y(t) = 4\sin(t)\cos(\pi/3) + 4\cos(t)\sin(\pi/3) + \cos(3t)$. Then it is obvious that

$$\begin{aligned} A_0 &= 0 \text{ (the mean)} \\ A_1 &= 4\sin(\pi/3) \\ B_1 &= 4\cos(\pi/3) \\ A_3 &= 1, \end{aligned}$$

   and all other terms are zero, due to orthogonality.

2. $f(t) = \begin{cases} t, t < T/2 \\ t - T/2, t \geq T/2 \end{cases}$ (biased sawtooth)
   $A_0 = \pi/2$, the mean value of the function. Let's next do the A_n's:

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} \cos(nt) f(t) dt \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(nt) t dt + \frac{1}{\pi}\int_{\pi}^{2\pi} \cos(nt)(t - \pi) dt \\ &= \frac{1}{\pi}\int_0^{2\pi} \cos(nt) t dt - \int_{\pi}^{2\pi} \cos(nt) dt \\ &= \frac{1}{\pi}\left(\frac{\cos(nt)}{n^2} + \frac{t\sin(nt)}{n}\right)\Bigg|_0^{2\pi} - 0 \\ &= 0 \end{aligned}$$

   This makes sense intuitively because the cosines are symmetric functions around zero (even), whereas $f(t)$ is not. The signal's information is carried in the sine terms:

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_0^{2\pi} \sin(nt) f(t) dt \\ &= \frac{1}{\pi}\int_0^{\pi} \sin(nt) t dt + \frac{1}{\pi}\int_{\pi}^{2\pi} \sin(nt)(t - \pi) dt \\ &= \frac{1}{\pi}\int_0^{2\pi} \sin(nt) t dt - \int_{\pi}^{2\pi} \sin(nt) dt \end{aligned}$$

   Now the second integral is $-2/n$ for n odd, and zero otherwise. Let's call it $q(n)$. Then continuing we see

$$\begin{aligned} B_n &= \frac{1}{\pi}\left(\frac{\sin(nt)}{n^2} - \frac{t\cos(nt)}{n}\right)\Bigg|_0^{2\pi} - q(n) \\ &= -2/n - q(n). \end{aligned}$$

   Hence $B_n = -2/n$ for even n, and zero otherwise. Try it out by making a plot!