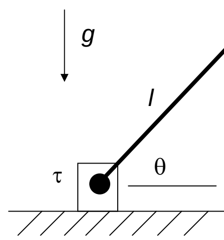
14 Pendulum Dynamics and Linearization

Consider a single-link arm, with length l and all the mass m concentrated at the end. A motor at the fixed pivot point supplies a controllable torque τ. As drawn, a positive torque drives the arm counter-clockwise, so as to drive the arm angle θ positive. The arm is free to swing around the full 360 degrees; gravity pulls the arm downward. There is no damping.



1. Derive and state the equation of motion for this system.

   Solution: It is a basic rotary moment of inertia with a gravity effect and input torque. We will get

$$ml^2\ddot{\theta} = \tau - mgl\cos\theta.$$

2. For each of the linearization angles [-90,0,90] degrees, answer the following:

   (a) What is the static torque needed to support the arm in this configuration?

   Solution: For the three angles given, the static torque is the amount needed to just balance the gravity torque. These values are [0, mgl, 0]; note that the upward and downward configurations do not take any static torque to maintain.

   (b) Assuming that the static torque is continuously applied, but that there is no dynamic torque, what is the equation of motion for small deviations from the static angle? We are looking for a differential equation in the perturbation of θ, say $\tilde{\theta}$.

   Solution: The key here is to write the angles and the torques as being comprised of a static ($\bar{\theta}, t\bar{a}u$) and a dynamic part ($\tilde{\theta}, \tilde{\tau}$):

$$\begin{aligned} ml^2\frac{d}{dt}\left(\bar{\theta}+\tilde{\theta}\right) &= \bar{\tau}+\tilde{\tau}-mgl\cos(\bar{\theta}+\tilde{\theta})+d \\ ml^2\frac{d}{dt}\tilde{\theta} &= \bar{\tau}-mgl\cos\bar{\theta}\cos\tilde{\theta}-mgl\sin\bar{\theta}\sin\tilde{\theta}+d. \end{aligned}$$

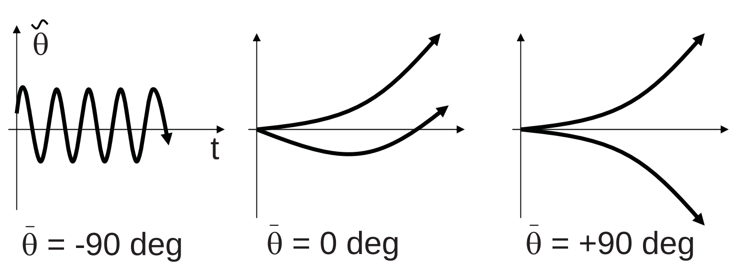
   Along with a trigonometric identity, we used the facts that dynamic torque $\tilde{\tau}$ is zero, and that static angle $\bar{\theta}$ has zero second derivative. We can see that $\cos\tilde{\theta}$

linearizes to simply one, and so the first two terms on the right-hand side in the second line give us precisely the static torques of part (i). The term d represents a torque perturbation. We are left with the following linear equations for the three static angles [-90, 0, 90] degrees, respectively:

$$\begin{aligned} mgl\ddot{\tilde{\theta}} &= -mgl\tilde{\theta} + d \\ mgl\ddot{\tilde{\theta}} &= d \\ mgl\ddot{\tilde{\theta}} &= mgl\tilde{\theta} + d. \end{aligned}$$

(c) In words and in an annotated graph, show what is the response of the equilibrium system to a (small) torque impulse. Be sure to take into account the fact that the system could respond differently to impulses of opposite directions.

Solution: The first differential equation above is that of a second-order, undamped oscillator. A perturbation to the equilibrium will lead to oscillations of constant size. The second equation above is a "double integrator," like a mass on a sliding contact; an impulsive d will set the arm spinning at constant speed, with no damping or restoring force. Of course, as the arm travels our linearizing assumptions (namely that $\tilde{\theta}$ is small) fail, and we see some other interesting behavior. If the perturbation is upward, the static torque will be larger than the gravity torque, and the arm will accelerate in the positive direction, up and over the top! If the perturbation is negative, we can again expect an acceleration in the positive direction for the same reasons, again going up and over. It doesn't get stuck at zero degrees because there is no damping and inertia will carry it through. Finally, the upward static configuration is just plain unstable because it is as if we have a negative spring term - any perturbation will cause the arm to fall over.



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