2.032 Dynamics

Fall 2004

Problem Set No. 6

## Problem 1

(a)

generalized coordinates for the rolling disk:

$$q_1 = x_p\,, \qquad q_2 = y_p\,, \qquad q_3 = \alpha\,, \qquad q_4 = \varphi$$

rolling constraints:

$$(1)\ \dot{x}_p - R\dot{\varphi}\sin\alpha = 0 \longrightarrow a_{11}=1,\ a_{12}=0,\ a_{13}=0,\ a_{14}=-R\sin\alpha$$

$$(2)\ \dot{y}_p - R\dot{\varphi}\cos\alpha = 0 \longrightarrow a_{21}=0,\ a_{22}=1,\ a_{23}=0,\ a_{24}=-R\cos\alpha$$

$$\left.\begin{aligned}\frac{\partial a_{14}}{\partial q_3} &= -R\cos\alpha \neq \frac{\partial a_{13}}{\partial q_4} = 0\\ \frac{\partial a_{24}}{\partial q_3} &= R\sin\alpha \neq \frac{\partial a_{23}}{\partial q_4} = 0\end{aligned}\right\} \Longrightarrow \text{(1) \& (2) are not integrable.}$$

$\longrightarrow$ Constraints are nonholonomic.

(b) Assume that there are integrating factors: $c_1(q_i,t)$, $c_2(q_i,t)$, $i=1,\dots,4$

$$\Longrightarrow \quad \tilde{a}_{11}=c_1\,, \quad \tilde{a}_{12}=0\,, \quad \tilde{a}_{13}=0\,, \quad \tilde{a}_{14}=-Rc_1\sin\alpha$$

$$\tilde{a}_{21}=0\,, \quad \tilde{a}_{22}=c_2\,, \quad \tilde{a}_{23}=0\,, \quad \tilde{a}_{24}=-Rc_2\cos\alpha$$

necessary conditions for the constraints to be integrable:

$$\frac{\partial \tilde{a}_{12}}{\partial q_i}=0 \longrightarrow \frac{\partial \tilde{a}_{1i}}{\partial q_2}=0 \longrightarrow \frac{\partial c_1}{\partial y_p}=0$$

$$\frac{\partial \tilde{a}_{13}}{\partial q_i}=0 \longrightarrow \frac{\partial \tilde{a}_{1i}}{\partial q_3}=0 \longrightarrow \frac{\partial c_1}{\partial \alpha}=0 \ \ \&\ \ -R\sin\alpha\frac{\partial c_1}{\partial \alpha} - Rc_1\cos\alpha = 0 \longrightarrow \underline{c_1 = 0}$$

## Problem 1

$$\frac{\partial \tilde{a}_{23}}{\partial q_i} = 0 \longrightarrow \frac{\partial \tilde{a}_{2i}}{\partial q_3} = 0 \longrightarrow \frac{\partial c_2}{\partial \alpha} = 0 \quad \& \quad -R\cos\alpha \frac{\partial c_2}{\partial \alpha} + Rc_2 \sin\alpha = 0$$

$$\longrightarrow c_2 = 0$$

$\Longrightarrow$ no non trivial integrating factors exist.

two rolling constraints cannot be integrated to holonomic constraints.

Problem 2

generalized coordinates assuming vehicle B does not rotate:

$$q_1 = x_A \qquad q_2 = y_A \qquad q_3 = \theta$$
$$q_4 = x_B \qquad q_5 = y_B$$

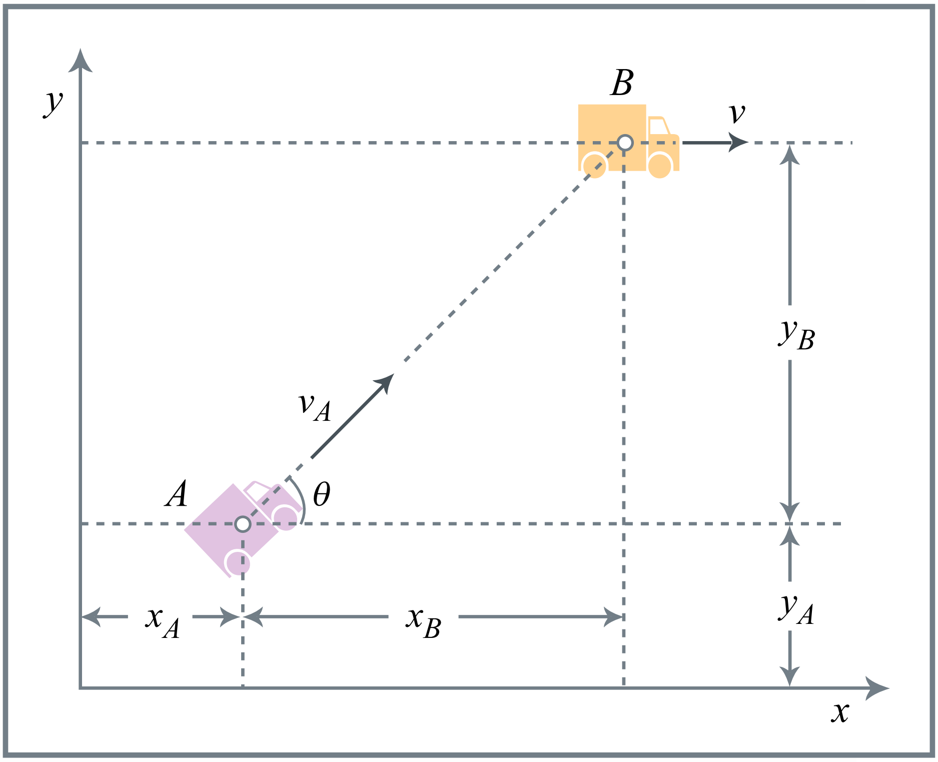



Figure by OCW. After 4.3/4 in Baruh, H. *Analytical Dynamics*. Boston MA: McGraw-Hill, 1999.

Constraint: The orientation of vehicle A must alway be toward vehicle B:

$$\underline{v}_A \times \underline{r}_{AB} = \underline{0}$$

$$(\dot{x}_A \underline{i} + \dot{y}_A \underline{j}) \times (x_B \underline{i} + y_B \underline{j}) = \underline{0}$$

$$\dot{x}_A y_B - \dot{y}_A x_B = 0$$ nonholonomic scleronomic constraint

Check the itegrability:

$$a_{11} = y_B, \quad a_{12} = -x_B, \quad a_{13} = a_{14} = a_{15} = 0$$

$$\frac{\partial a_{11}}{\partial q_5} = 1 \neq \frac{\partial a_{15}}{\partial q_1} = 0 \longrightarrow$$ constraint is not integrable.

If we allow for an integrating factor: $c(q_i, t)$, $i = 1, \ldots, 5$

$$\longrightarrow \quad \tilde{a}_{11} = c y_B, \quad \tilde{a}_{12} = -c x_B, \quad \tilde{a}_{13} = \tilde{a}_{14} = \tilde{a}_{15} = 0$$

$$\frac{\partial \tilde{a}_{14}}{\partial q_i} = 0 \longrightarrow \frac{\partial \tilde{a}_{1i}}{\partial q_4} = 0 \longrightarrow \frac{\partial c}{\partial x_B} = 0 \;\; \& \;\; -\frac{\partial c}{\partial x_B} x_B - c = 0 \longrightarrow \underline{c = 0}$$

$\Longrightarrow$ constraint is truely nonholonomic.

# Problem 3

For point P:

$$\begin{cases} x = (L_1 + L_2 \sin\theta)\cos\varphi \\ y = (L_1 + L_2 \sin\theta)\sin\varphi \\ z = -L_2 \cos\theta \end{cases}$$

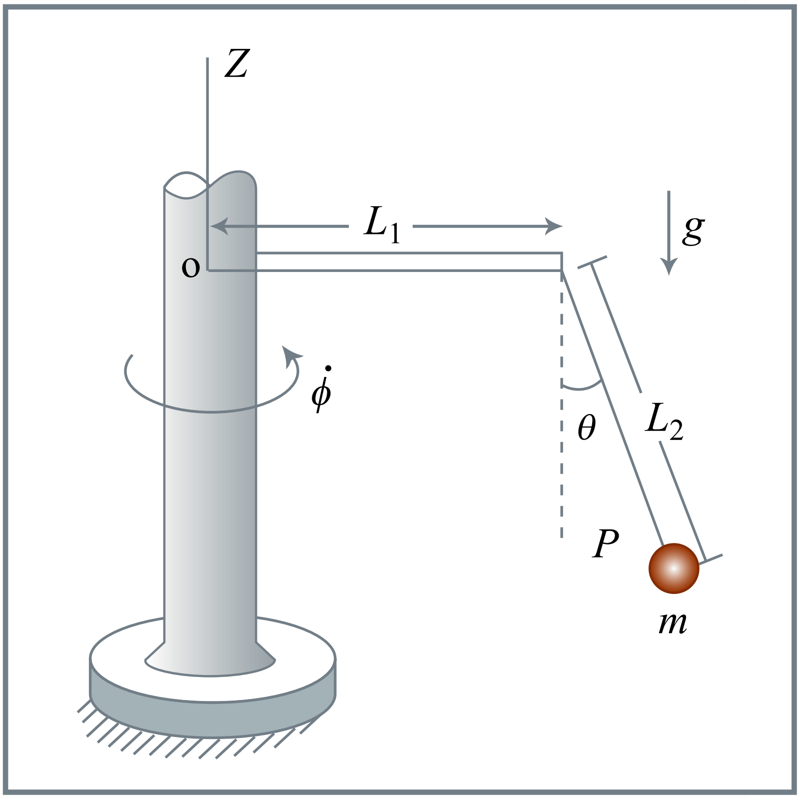



Figure by OCW.
After 4.4/7 in Baruh, H. *Analytical Dynamics*. Boston MA: McGraw-Hill, 1999.

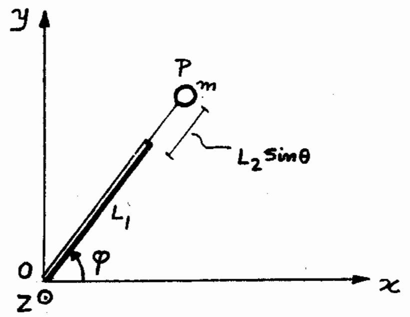
Virtual displacement:

$$\begin{cases} \delta x = L_2 \cos\theta \cos\varphi\, \delta\theta - (L_1 + L_2 \sin\theta)\sin\varphi\, \delta\varphi \\ \delta y = L_2 \cos\theta \sin\varphi\, \delta\theta + (L_1 + L_2 \sin\theta)\cos\varphi\, \delta\varphi \\ \delta z = L_2 \sin\theta\, \delta\theta \end{cases}$$

In the case of constant $\dot{\varphi}$ $(\delta\varphi = 0)$:

$$x^2 + y^2 + z^2 = OP^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(90+\theta)$$

$$= L_1^2 + L_2^2 + 2L_1L_2\sin\theta$$




Top view

Virtual displacement:

$$2x\,\delta x + 2y\,\delta y + 2z\,\delta z = 2L_1L_2\cos\theta\,\delta\theta$$

# Problem 4



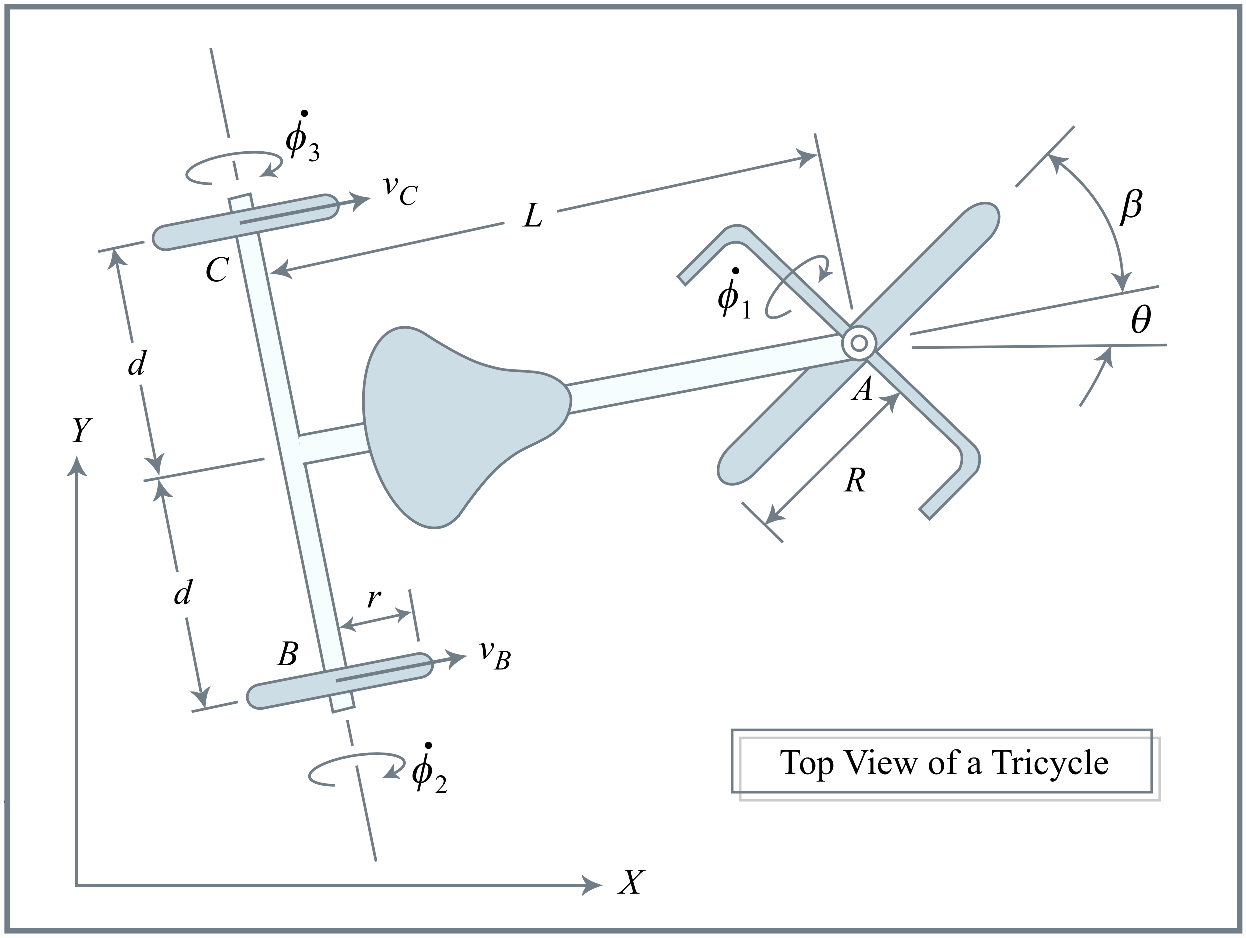


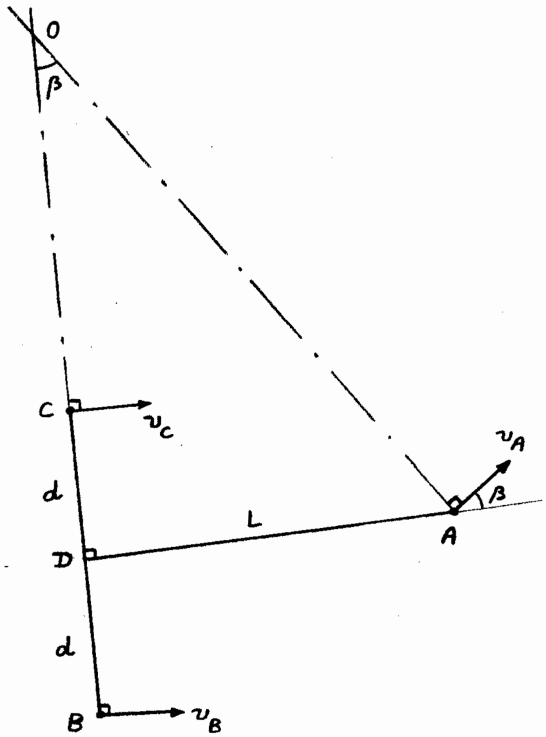

Figure by OCW. After 6.8 in Ginsberg, J. H. *Advanced Engineering Dynamics.* 2nd ed. New York: Cambridge University Press, 1998.

Wheels do not slip :

$$v_A = R\dot{\varphi}_1$$

$$v_B = r\dot{\varphi}_2$$

$$v_C = r\dot{\varphi}_3$$




Points A, B, and C are on the frame as well.
One can find the instantaneous axis of rotation for the frame :

$$\omega_{frame} = \dot{\theta}\,\underline{k}$$

$$OA = \frac{L}{\sin\beta}, \qquad OC = \frac{L}{\tan\beta} - d, \qquad OB = \frac{L}{\tan\beta} + d$$

$$v_A = OA \cdot \dot{\theta} \qquad v_C = OC \cdot \dot{\theta} \qquad v_B = OB \cdot \dot{\theta}$$

## Problem 4

$$v_A = R\dot{\varphi}_1 = \frac{L}{\sin\beta}\dot{\theta} \longrightarrow \underline{L\dot{\theta} = R\dot{\varphi}_1 \sin\beta}$$

$$\left.\begin{aligned} v_B &= r\dot{\varphi}_2 = \left(\frac{L}{\tan\beta} + d\right)\dot{\theta} \\ v_C &= r\dot{\varphi}_3 = \left(\frac{L}{\tan\beta} - d\right)\dot{\theta} \end{aligned}\right\} \Longrightarrow \begin{aligned} &\underline{r(\dot{\varphi}_2 - \dot{\varphi}_3) = 2d\dot{\theta}} \\ &r(\dot{\varphi}_2 + \dot{\varphi}_3) = \frac{2L\dot{\theta}}{\tan\beta} = \frac{2R\dot{\varphi}_1 \sin\beta}{\tan\beta} = 2R\dot{\varphi}_1 \cos\beta \end{aligned}$$

$$\longrightarrow \underline{r(\dot{\varphi}_2 + \dot{\varphi}_3) = 2R\dot{\varphi}_1 \cos\beta}$$

Velocity constraints

To find the number of degrees of freedom:

Each wheel is like a vertical disk rolling without slip on a 2-D plane so it has 2 degrees of freedom. We have the following constraints:

1- $AB$ = Const.

2- $AC$ = Const.

3- $BC$ = Const.

4- wheel $B$ has to be perpendicular to the shaft $BC$. Satisfying the no slip constraint forces the wheel $C$ to be perpendicular to the shaft $BC$ as well.

$\therefore$ $\# DOF = 3 \times 2 - 4 = \boxed{2}$

Note that we need 4 generalized coordinate to define the system.