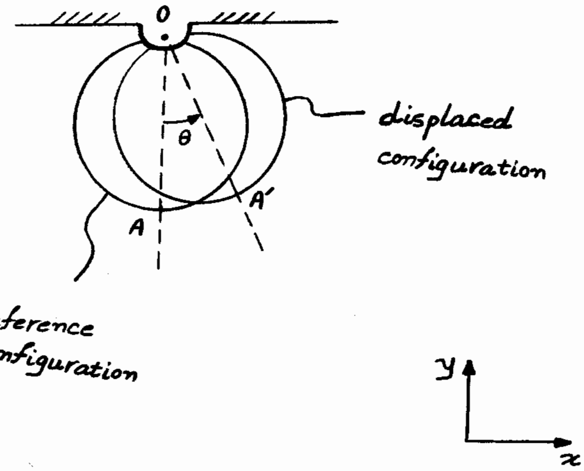
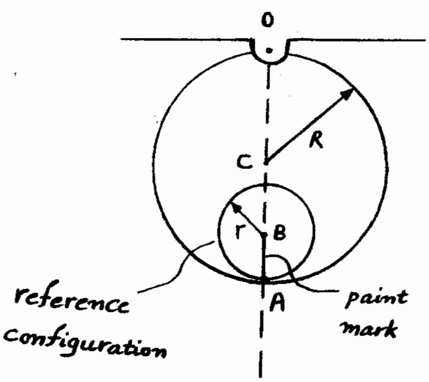
Problem 1

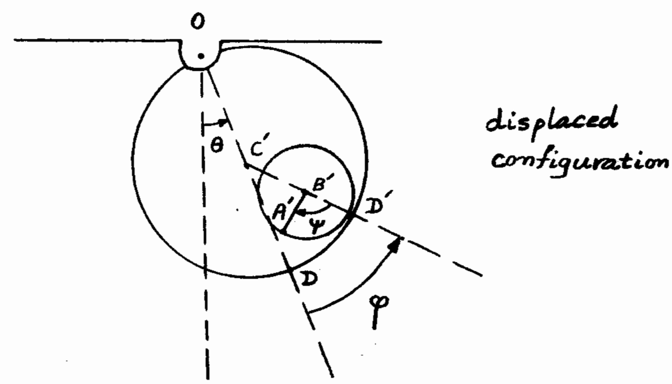
Practice Problems No. 1

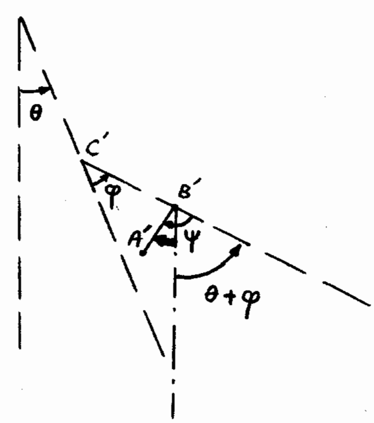


Comparing the orientation of OA to OA′, the angular velocity of the ring would be:

$$\underset{\sim}{\omega}_{ring} = \dot{\theta}\, \hat{\underset{\sim}{e}}_z$$







Comparing BA to B′A′,

$$\underset{\sim}{\omega}_{disk} = -\left[\dot{\psi} - (\dot{\theta}+\dot{\varphi})\right] \hat{\underset{\sim}{e}}_z$$

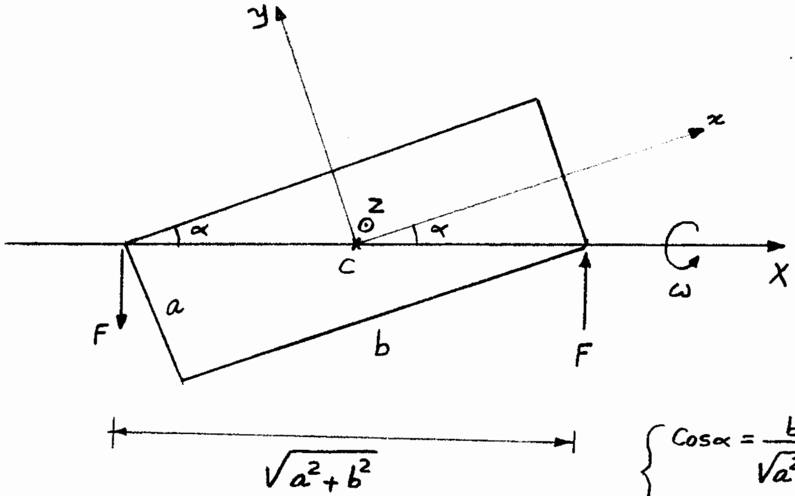
$$= (\dot{\theta}+\dot{\varphi}-\dot{\psi})\, \hat{\underset{\sim}{e}}_z$$

No slip $\Longrightarrow$ $\begin{cases} \underset{\sim}{v}_{D'}\big|_{disk} = \underset{\sim}{v}_{D'}\big|_{ring} \\ A'D' = DD' \end{cases}$ $\Longrightarrow$ $r\psi = R\varphi$ $\Longrightarrow$ $\psi = \frac{R}{r}\varphi$

$$\therefore \quad \underset{\sim}{\omega}_{disk} = \left[\dot{\theta} + \left(1-\frac{R}{r}\right)\dot{\varphi}\right] \hat{\underset{\sim}{e}}_z$$

Problem 2

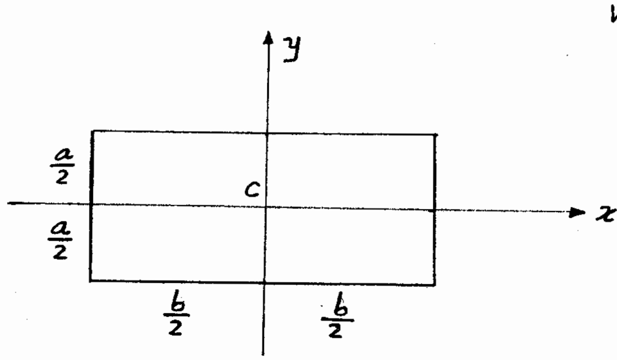
Introducing xyz coordinate system which is fixed to the plate and rotates with ω about X axis:



$$\begin{cases} \cos\alpha = \dfrac{b}{\sqrt{a^2+b^2}} \\ \sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}} \end{cases}$$

(a)

Since $\underset{\sim}{v}_C = 0$, forces on the bearings are equal and in opposite directions.



$$I_x = \int (\rho dv)(y^2+z^2) \qquad z^2 \approx 0 \quad \text{(thin plate)}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{b/2} \rho\, dx\, dy\, (y^2) = \rho b \frac{a^3}{12} = \underline{\frac{1}{12} M a^2}$$

$$\underline{I_y = \frac{1}{12} M b^2} \qquad \underline{I_z = \frac{1}{12} M (a^2+b^2)}, \qquad \underline{I_{xy} = I_{yz} = I_{zx} = 0}$$

$$\underset{\sim}{\omega}_{plate} = \omega\, \hat{\underset{\sim}{e}}_X = \omega \cos\alpha\, \hat{\underset{\sim}{e}}_x - \omega \sin\alpha\, \hat{\underset{\sim}{e}}_y \longrightarrow \begin{cases} \omega_x = \omega \cos\alpha \\ \omega_y = -\omega \sin\alpha \end{cases} \qquad \omega_z = 0$$

$$\underset{\sim}{H}_c = [I]_c\, \underset{\sim}{\omega} \quad \Longrightarrow \quad \underset{\sim}{H}_c = I_x \omega_x \hat{\underset{\sim}{e}}_x + I_y \omega_y \hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{\tau}_C = \frac{d \underset{\sim}{H}_c}{dt}$$

$$\underset{\sim}{\omega} \times \hat{\underset{\sim}{e}}_x = \omega \sin\alpha\, \hat{\underset{\sim}{e}}_z \qquad \underset{\sim}{\omega} \times \hat{\underset{\sim}{e}}_y = \omega \cos\alpha\, \hat{\underset{\sim}{e}}_z$$

$$\frac{d\underset{\sim}{H}_c}{dt} = I_x \omega_x \frac{d\hat{\underset{\sim}{e}}_x}{dt} + I_y \omega_y \frac{d\hat{\underset{\sim}{e}}_y}{dt}$$

$$= \left(\frac{1}{12} M a^2 \omega^2 \cos\alpha \sin\alpha - \frac{1}{12} M b^2 \omega^2 \sin\alpha \cos\alpha\right) \hat{\underset{\sim}{e}}_z = \frac{1}{12} M \omega^2 \sin\alpha \cos\alpha\, (a^2 - b^2)\, \hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\tau}_c = F \sqrt{a^2+b^2}\; \hat{\underset{\sim}{e}}_z$$

Problem 2

$$\therefore \quad F\sqrt{a^2+b^2} = \frac{1}{12} M\omega^2 \frac{ab}{a^2+b^2}(a^2-b^2)$$

$$\Longrightarrow \quad \boxed{F = \frac{1}{12} M\omega^2 ab \frac{a^2-b^2}{(a^2+b^2)^{3/2}}}$$

Note that force F rotates about X axis and is always in xy plane.

(b)

$$K.E. = \frac{1}{2}\{\omega\}^t [I]_c \{\omega\} + \cancel{\frac{1}{2} M \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c} \xrightarrow{} 0$$

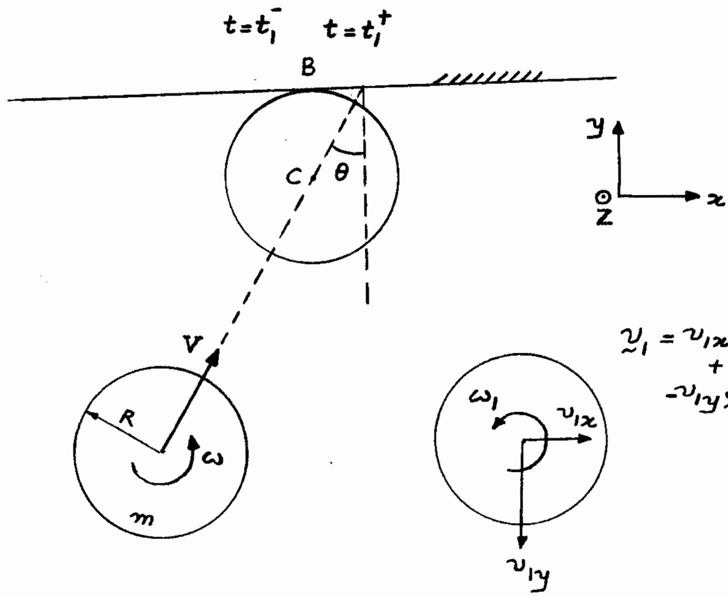
$$= \frac{1}{2}\left(I_x \omega_x^2 + I_y \omega_y^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{12} M a^2 \omega^2 \cos^2\alpha + \frac{1}{12} M b^2 \omega^2 \sin^2\alpha\right)$$

$$= \boxed{\frac{1}{12} M\omega^2 \frac{a^2 b^2}{a^2+b^2}}$$

kinetic energy of the rotating plate

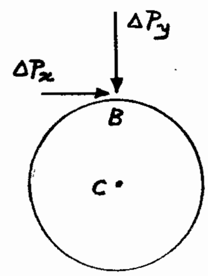
Problem 3



$\underset{\sim}{v}_1 = v_{1x}\hat{\underset{\sim}{e}}_x + -v_{1y}\hat{\underset{\sim}{e}}_y$

Assume the disk collides with the wall at point B at $t=t_1$.

An impulse acts on the disk at $t=t_1$ $(\Delta P_x, \Delta P_y)$.



F.B.D. at $t=t_1$

Collision in the normal direction (y) is elastic so the magnitude of the velocity in the normal direction is conserved:

$$v_{1y}\Big|_{t=t_1^+} = v_y\Big|_{t=t_1^-} \Rightarrow v_{1y}\Big|_{t=t_1^+} = V\cos\theta$$

$$\therefore \quad \underline{v_{1y} = V\cos\theta}$$

No slip occurs at the wall. $\Rightarrow$ $v_{Bx}\Big|_{t=t_1^+} = 0$ $\Rightarrow$ $\Big[v_{Cx} + \underbrace{(\underset{\sim}{\omega}\times \underset{\sim}{CB})_x}_{x\text{ component}}\Big]_{t=t_1^+} = 0$

$$\Rightarrow v_{1x} - \omega_1 R = 0 \Rightarrow \underline{v_{1x} = R\omega_1}$$

Angular momentum about point B : $\underset{\sim}{\tau}_B = \frac{d}{dt}\underset{\sim}{H}_B + \cancel{\underset{\sim}{v}_B}^{\,0} \times \underset{\sim}{P}$

$$\underset{\sim}{\tau}_B = 0 \longrightarrow \frac{d}{dt}\underset{\sim}{H}_B = 0 \longrightarrow \underset{\sim}{H}_B\Big|_{t=t_1^-} = \underset{\sim}{H}_B\Big|_{t=t_1^+} \qquad (\underset{\sim}{H}_B = \underset{\sim}{H}_C + \underset{\sim}{BC}\times\underset{\sim}{P})$$

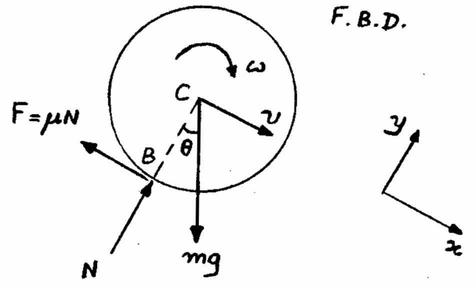
$$\longrightarrow \frac{1}{2}mR^2\omega + mVR\sin\theta = \frac{1}{2}mR^2\omega_1 + mRv_{1x} \xRightarrow{\omega_1 = v_{1x}/R} \underline{v_{1x} = \frac{2}{3}V\sin\theta + \frac{1}{3}R\omega}$$

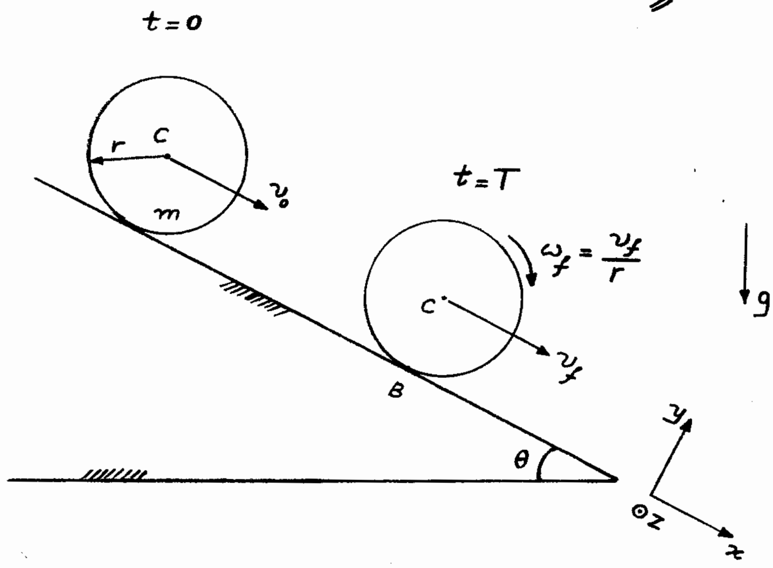
$$v_1 = \sqrt{v_{1x}^2 + v_{1y}^2} = \underline{\sqrt{\left(\tfrac{2}{3}V\sin\theta + \tfrac{1}{3}R\omega\right)^2 + (V\cos\theta)^2}}$$

velocity of the center of the disk after collision

Problem 4

$0 < t < T$





End of the period of slipping $t=T$.

Linear mom. : $\quad m \dfrac{d\underset{\sim}{v}}{dt} = \underset{\sim}{F}$

in x direction : $\quad m \dfrac{dv}{dt} = mg\sin\theta - F \qquad$ ($F = \mu N$ during the period of slipping)

in y direction : $\quad 0 = mg\cos\theta - N \longrightarrow N = mg\cos\theta$

$$\therefore \quad m\frac{dv}{dt} = mg\,(\sin\theta - \mu\cos\theta) \longrightarrow \int_0^T dv = \int_0^T g\,(\sin\theta - \mu\cos\theta)\,dt$$

$$\Longrightarrow \quad v_f - v_o = gT\,(\sin\theta - \mu\cos\theta)$$

Ang. mom. about point C : $\quad \underset{\sim}{\tau}_C = \dfrac{d}{dt}\underset{\sim}{H}_C$

$$\left.\begin{aligned} \underset{\sim}{\tau}_C &= -\mu N r\,\hat{\underset{\sim}{e}}_z = -\mu mg\cos\theta\, r\,\hat{\underset{\sim}{e}}_z \\ \frac{d}{dt}\underset{\sim}{H}_C &= -I_C\frac{d\omega}{dt}\hat{\underset{\sim}{e}}_z = -\frac{2}{5}mr^2\frac{d\omega}{dt}\hat{\underset{\sim}{e}}_z \end{aligned}\right\} \Longrightarrow \mu g\cos\theta = \frac{2}{5}r\frac{d\omega}{dt}$$

$$\Longrightarrow \int_0^T d\omega = \int_0^T \frac{5}{2}\frac{\mu}{r}g\cos\theta\,dt \longrightarrow \omega_f - 0 = \frac{5}{2}\mu g\cos\theta\,\frac{T}{r}$$

No slip at $t=T \Longrightarrow \quad r\omega_f = v_f$

Problem 4

$$\therefore \quad v_f = \frac{5}{2}\mu g \cos\theta \, T$$

$$v_f - v_0 = gT(\sin\theta - \mu\cos\theta)$$

$$\Rightarrow \quad \boxed{T = \frac{2v_0}{g(7\mu\cos\theta - 2\sin\theta)}}$$

time duration of slipping

$$\boxed{v_f = \frac{5\mu\cos\theta \, v_0}{7\mu\cos\theta - 2\sin\theta}}$$

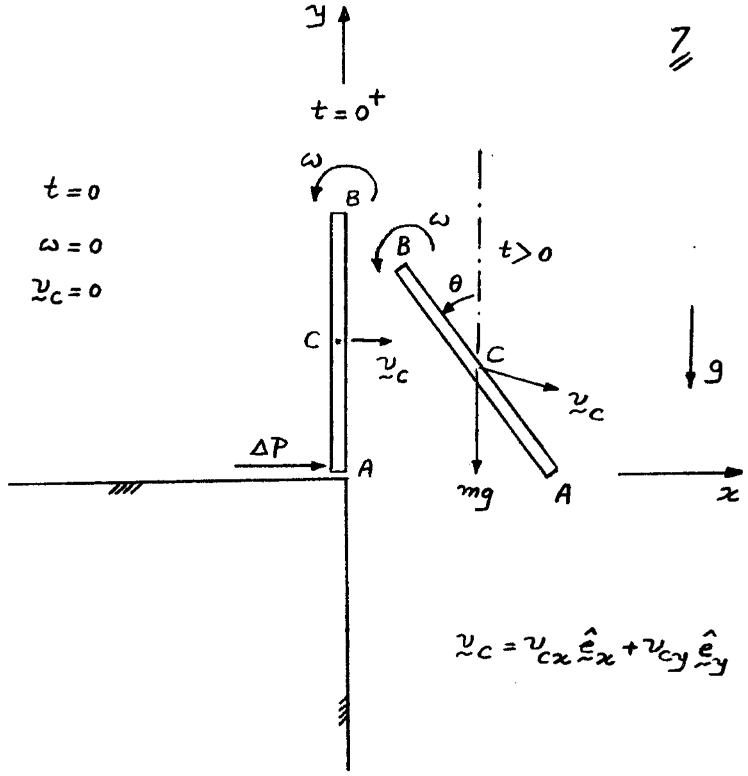
velocity of the center of mass C at the end of the period of slipping

Note that $7\mu\cos\theta - 2\sin\theta$ has to be positive :

$$\tan\theta < 3.5\mu$$

Problem 5

$AB = 2a$



Horizontal impulse ΔP at $t=0$.

During the impulse period, other forces do not have enough time to act :

Linear momentum: $\quad m \dfrac{d\underset{\sim}{v}_c}{dt} = \underset{\sim}{F}$

$$m\, d\underset{\sim}{v}_c = \underset{\sim}{F}\, dt$$

$$m\left(\underset{\sim}{v}_c\big|_{t=0^+} - 0\right) = \int_0^{0^+} \underset{\sim}{F}\, dt = \Delta \underset{\sim}{P} = \Delta P\, \hat{\underset{\sim}{e}}_x$$

$$\Longrightarrow \quad \underset{\sim}{v}_c\big|_{t=0^+} = \frac{\Delta P}{m}\hat{\underset{\sim}{e}}_x \quad \longrightarrow \quad \begin{cases} v_{cx} = \dfrac{\Delta P}{m} \\ v_{cy} = 0 \end{cases}, \quad \text{at } t = 0^+$$

Ang. mom. about C : $\quad \underset{\sim}{\tau}_c = \dfrac{d}{dt}\underset{\sim}{H}_c = I_c \dfrac{d\omega}{dt}\hat{\underset{\sim}{e}}_z$

$$\underset{\sim}{\tau}_c\, dt = I_c\, d\omega\, \hat{\underset{\sim}{e}}_z \quad \Longrightarrow \quad \int_0^{0^+} \underset{\sim}{\tau}_c\, dt = I_c\left(\omega\big|_{t=0^+} - 0\right)\hat{\underset{\sim}{e}}_z$$

$$\Longrightarrow \quad \Delta P a = \frac{1}{12} m (2a)^2 \left[\omega\big|_{t=0^+} - 0\right] \quad \longrightarrow \quad \omega\big|_{t=0^+} = \frac{3\Delta P}{ma}$$

$\underset{\sim}{v}_c\big|_{t=0^+}$ and $\omega\big|_{t=0^+}$ are the initial conditions for the next stage of the motion which is free fall :

$$\underset{\sim}{\tau}_c = \frac{d}{dt}\underset{\sim}{H}_c, \quad \underset{\sim}{\tau}_c = 0 \quad \Longrightarrow \quad I_c\,\omega = \text{const.} \quad \longrightarrow \quad \omega = \text{Const.} = \omega\big|_{t=0^+} = \frac{3\Delta P}{ma}$$

$$m \frac{d\underset{\sim}{v}_c}{dt} = \underset{\sim}{F} = -mg\, \hat{\underset{\sim}{e}}_y$$

Problem 5

$$\therefore \begin{cases} m\dfrac{dv_{cx}}{dt} = 0 \longrightarrow v_{cx} = \text{const.} = v_{cx}\Big|_{t=0^+} = \dfrac{\Delta P}{m} \\[2ex] m\dfrac{dv_{cy}}{dt} = -mg \longrightarrow v_{cy} = -gt + \overset{\nearrow 0}{v_{cy}\Big|_{t=0^+}} = -gt \end{cases}$$

$$\begin{cases} v_{cx} = \dfrac{dx_c}{dt} = \dfrac{\Delta P}{m} \longrightarrow x_c = \dfrac{\Delta P}{m}t + \overset{\nearrow 0}{x_c\Big|_{t=0}} = \dfrac{\Delta P}{m}t \\[2ex] v_{cy} = \dfrac{dy_c}{dt} = -gt \longrightarrow y_c = -g\dfrac{t^2}{2} + \overset{\nearrow a}{y_c\Big|_{t=0}} = -g\dfrac{t^2}{2} + a \end{cases}$$

$$\omega = \frac{d\theta}{dt} = \frac{3\Delta P}{ma} \longrightarrow \theta = \frac{3\Delta P}{ma}t + \overset{\nearrow 0}{\theta\Big|_{t=0}} = \frac{3\Delta P}{ma}t$$

$$\begin{cases} x_B = x_c - a\sin\theta = \dfrac{\Delta P}{m}t - a\sin\left(\dfrac{3\Delta P}{ma}t\right) \\[2ex] y_B = y_c + a\cos\theta = -g\dfrac{t^2}{2} + a\left[1 + \cos\left(\dfrac{3\Delta P}{ma}t\right)\right] \end{cases}$$

Point B clips the edge of the table $(x = 0,\ y = 0)$:

$$\begin{cases} x_B = 0 = \dfrac{\Delta P}{m}t - a\sin\left(\dfrac{3\Delta P}{ma}t\right) = 0 \quad \xRightarrow{X = \frac{\Delta P t}{ma}} \quad X - \sin(3X) = 0 \longrightarrow X = 0.76 = \dfrac{\Delta P t}{ma} \\[2ex] y_B = 0 = -g\dfrac{t^2}{2} + a\left[1 + \cos\left(\dfrac{3\Delta P}{ma}t\right)\right] = 0 \end{cases}$$

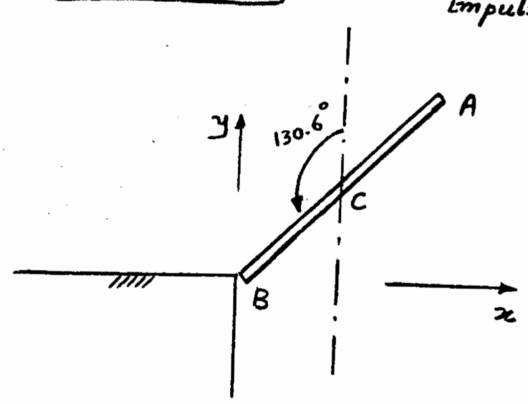
$$-g\frac{t^2}{2} + a\left[1 + \cos(3X)\right] = 0 \longrightarrow g\frac{t^2}{2} = a(1 - 0.65) \longrightarrow \underline{t = 0.84\sqrt{\frac{a}{g}}}$$

$$\Delta P t = 0.76\, ma \longrightarrow \Delta P = \frac{0.76\, ma}{0.84\sqrt{\frac{a}{g}}} = \boxed{0.91\, m\sqrt{ga}}$$

value of horizontal impulse

At this instant :

$$\begin{cases} \theta = \dfrac{3\Delta P}{ma}t = 3(0.76) = \underline{2.28 = 130.6^\circ} \\[2ex] x_B = y_B = 0 \end{cases}$$



Position of the pencil at this instant.