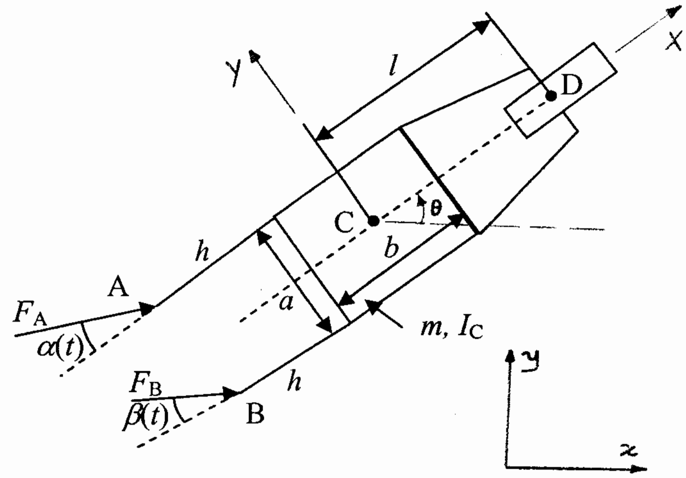
Problem Set No. 9

Problem 1



$\#\ DOF = 3 - 1 = 2$

generalized coordinates:

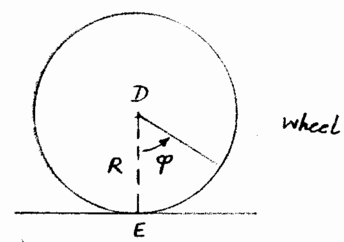
$q_1 = x_C, \quad q_2 = y_C, \quad q_3 = \theta$

$$\underline{v}_D = \underline{v}_C + \underline{\omega} \times \underline{r}_{CD}$$

$$\underline{v}_D = \dot{x}_C \underline{e}_x + \dot{y}_C \underline{e}_y + \dot{\theta}\underline{e}_z \times \ell(\cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y)$$

$$= (\dot{x}_C - \dot{\theta}\ell\sin\theta)\,\underline{e}_x + (\dot{y}_C + \dot{\theta}\ell\cos\theta)\,\underline{e}_y \qquad (1)$$

No slip constraint for the wheel:



$$\underline{0} = \underline{v}_E = \underline{v}_D + \overset{wheel}{\underline{\omega}} \times \underline{r}_{DE}$$

$$= \dot{x}_D \underline{e}_x + \dot{y}_D \underline{e}_y + (\dot{\varphi}\sin\theta\, \underline{e}_x - \dot{\varphi}\cos\theta\, \underline{e}_y + \dot{\theta}\underline{e}_z) \times -R\underline{e}_z$$

$$\longrightarrow \begin{cases} \dot{x}_D = -\dot{\varphi}R\cos\theta \\ \dot{y}_D = -\dot{\varphi}R\sin\theta \end{cases} \Longrightarrow \quad \frac{\dot{y}_D}{\dot{x}_D} = \tan\theta \qquad (2)$$

$$(1), (2) \Longrightarrow \frac{\dot{y}_C + \dot{\theta}\ell\cos\theta}{\dot{x}_C - \dot{\theta}\ell\sin\theta} = \tan\theta \longrightarrow \underline{\dot{x}_C\sin\theta - \dot{y}_C\cos\theta - \dot{\theta}\ell = 0} \qquad \text{constraint}$$

$a_{11} = \sin\theta, \quad a_{12} = -\cos\theta, \quad a_{13} = -\ell$

$$\frac{\partial a_{11}}{\partial q_3} \neq \frac{\partial a_{13}}{\partial q_1} \Longrightarrow \text{constraint is nonholonomic.}$$

One nonholonomic constraint: $m = 1 \longrightarrow$ Lagrange multiplier λ_1

$L = T - V$

$$T = \frac{1}{2}m(\dot{x}_C^2 + \dot{y}_C^2) + \frac{1}{2}I_C\dot{\theta}^2, \qquad V = \text{Const.} = 0$$

Problem 1

generalized forces:

$\underline{F}_A$ and $\underline{F}_B$ are non-potential active forces.

$$\begin{cases} \underline{v}_A = \underline{v}_C + \underline{\omega} \times \underline{r}_{CA} = \dot{x}_C \underline{e}_x + \dot{y}_C \underline{e}_y + \dot{\theta} \underline{e}_z \times \left[-\left(\frac{b}{2}+h\right) \underline{e}_X + \frac{a}{2} \underline{e}_Y \right] \\ \underline{v}_B = \underline{v}_C + \underline{\omega} \times \underline{r}_{CB} = \dot{x}_C \underline{e}_x + \dot{y}_C \underline{e}_y + \dot{\theta} \underline{e}_z \times \left[-\left(\frac{b}{2}+h\right) \underline{e}_X - \frac{a}{2} \underline{e}_Y \right] \end{cases}$$

$$\begin{cases} \underline{e}_x = \cos\theta \, \underline{e}_X - \sin\theta \, \underline{e}_Y \\ \underline{e}_y = \sin\theta \, \underline{e}_X + \cos\theta \, \underline{e}_Y \end{cases}$$

$\longrightarrow$

$$\begin{cases} \underline{v}_A = \left(\dot{x}_C \cos\theta + \dot{y}_C \sin\theta - \frac{a}{2} \dot{\theta} \right) \underline{e}_X + \left(-\dot{x}_C \sin\theta + \dot{y}_C \cos\theta - \dot{\theta} \left(\frac{b}{2} + h \right) \right) \underline{e}_Y \\ \underline{v}_B = \left(\dot{x}_C \cos\theta + \dot{y}_C \sin\theta + \frac{a}{2} \dot{\theta} \right) \underline{e}_X + \left(-\dot{x}_C \sin\theta + \dot{y}_C \cos\theta - \dot{\theta} \left(\frac{b}{2} + h \right) \right) \underline{e}_Y \end{cases}$$

$\Longrightarrow$

$$\delta \underline{r}_{A(B)} = \left(\delta x_C \cos\theta + \delta y_C \sin\theta - (+) \frac{a}{2} \delta\theta \right) \underline{e}_X + \left(-\delta x_C \sin\theta + \delta y_C \cos\theta - \delta\theta \left(\frac{b}{2} + h \right) \right) \underline{e}_Y$$

$$\begin{cases} \underline{F}_A = F_A \left(\cos\alpha \, \underline{e}_X - \sin\alpha \, \underline{e}_Y \right) \\ \underline{F}_B = F_B \left(\cos\beta \, \underline{e}_X - \sin\beta \, \underline{e}_Y \right) \end{cases}$$

$$\delta W = \underline{F}_A \cdot \delta \underline{r}_A + \underline{F}_B \cdot \delta \underline{r}_B = \left[F_A \cos(\theta - \alpha) + F_B \cos(\theta - \beta) \right] \delta x_C + \left[F_A \sin(\theta - \alpha) + F_B \sin(\theta - \beta) \right] \delta y_C$$
$$+ \left[-F_A \frac{a}{2} \cos\alpha + F_A \sin\alpha \left(\frac{b}{2} + h \right) + F_B \frac{a}{2} \cos\beta + F_B \sin\beta \left(\frac{b}{2} + h \right) \right] \delta\theta$$

$\therefore$

$$\left. \begin{aligned} Q_1 &= F_A \cos(\theta - \alpha) + F_B \cos(\theta - \beta) \\ Q_2 &= F_A \sin(\theta - \alpha) + F_B \sin(\theta - \beta) \\ Q_3 &= -F_A \frac{a}{2} \cos\alpha + F_A \left(\frac{b}{2} + h \right) \sin\alpha + F_B \frac{a}{2} \cos\beta + F_B \left(\frac{b}{2} + h \right) \sin\beta \end{aligned} \right\} \text{generalized forces}$$

Equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_C} \right) - \frac{\partial L}{\partial x_C} = Q_1 + \lambda_1 a_{11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_C} \right) - \frac{\partial L}{\partial y_C} = Q_2 + \lambda_1 a_{12}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_3 + \lambda_1 a_{13}$$

Problem 1

$\therefore$

$$\begin{cases} m\ddot{x}_c = Q_1 + \lambda_1 \sin\theta & (3) \\ m\ddot{y}_c = Q_2 - \lambda_1 \cos\theta & (4) \\ I_c\ddot{\theta} = Q_3 - \ell\lambda_1 & (5) \end{cases}$$

Constraint : $\dot{x}_c \sin\theta - \dot{y}_c \cos\theta - \dot{\theta}\ell = 0$ (6)

use (5) to eliminate λ_1 from the remaining equations:

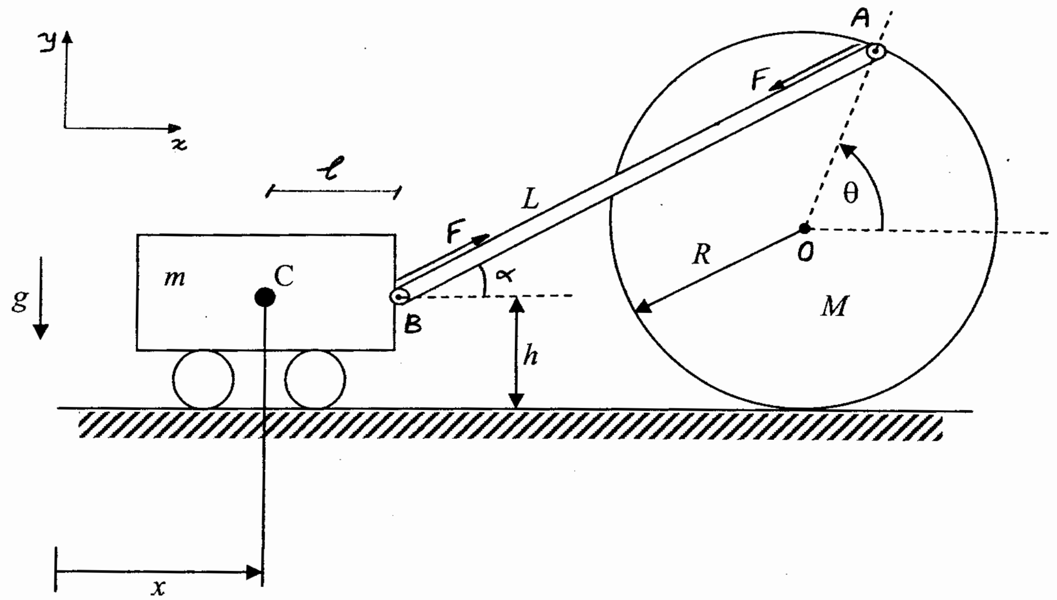
$$\lambda_1 = \frac{Q_3}{\ell} - \frac{I_c}{\ell}\ddot{\theta}$$

$\Longrightarrow$

$$\begin{cases} m\ddot{x}_c = Q_1 + (Q_3 - I_c\ddot{\theta})\dfrac{\sin\theta}{\ell} \\ m\ddot{y}_c = Q_2 - (Q_3 - I_c\ddot{\theta})\dfrac{\cos\theta}{\ell} \\ \dot{x}_c \sin\theta - \dot{y}_c \cos\theta - \dot{\theta}\ell = 0 \end{cases}$$

3 ODE for the three coordinates.

Problem 2



$\# \text{ DOF} = 2 \times 3 - 2 - 2 - 1 = 1$ $\qquad q_1 = x$

link is massless so the force in the link is along the link.

To find the constraint force F :

Introduce $q_2 = \theta$ $\qquad n = 2$

constraint : $\quad AB = L$

$$(x_A - x_B)^2 + (y_A - y_B)^2 = L^2$$

$m = 1 \longrightarrow$ one Lagrange multiplier λ_1

$x_0 = D - R\theta \qquad \left(D = x_0 \big|_{\theta = 0} \right)$

$x_C = x$

$x_A - x_B = R\cos\theta + D - R\theta - x - \ell$

$y_A - y_B = R\sin\theta + R - h$

$$\longrightarrow \quad \left(R\cos\theta + D - R\theta - x - \ell \right)^2 + \left(R\sin\theta + R - h \right)^2 = L^2$$

$$\longrightarrow \quad \left(R\cos\theta - R\theta - x + D - \ell \right)\left(-R\sin\theta\, \dot\theta - R\dot\theta - \dot x \right) + \left(R\sin\theta + R - h \right)\left(R\cos\theta\, \dot\theta \right) = 0$$

$$\longrightarrow \quad \dot x \left(R\cos\theta - R\theta - x + D - \ell \right) + \dot\theta \left[R(1 + \sin\theta)(D - R\theta - x - \ell) + hR\cos\theta \right] = 0$$

constraint

Problem 2

$$a_{11} = R\cos\theta - R\theta - x + D - \ell$$

$$a_{12} = R(1+\sin\theta)(D - R\theta - x - \ell) + hR\cos\theta$$

There are no active nonpotential forces (unrelated to the constraint) $\longrightarrow$ $Q_1 = Q_2 = 0$

$$\mathcal{L} = T - V$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(R^2\dot{\theta}^2) + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2 = \frac{1}{2}m\dot{x}^2 + \frac{3}{4}MR^2\dot{\theta}^2$$

$$V = \text{Const.}$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot{x}} - \dfrac{\partial\mathcal{L}}{\partial x} = \overbrace{\lambda_1 a_{11}}^{K_1} & \longrightarrow \quad m\ddot{x} = K_1 \\ \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot{\theta}} - \dfrac{\partial\mathcal{L}}{\partial\theta} = \underbrace{\lambda_1 a_{12}}_{K_2} & \longrightarrow \quad \frac{3}{2}MR^2\ddot{\theta} = K_2 \end{cases}$$

λ_1 can be eliminated so we have 2 ODE (one constraint equation) for the two coordinates.

To find F:

$$\delta w = K_1\,\delta x + K_2\,\delta\theta = F(\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j})\cdot\delta\underline{r}_B + F(-\cos\alpha\,\underline{i} - \sin\alpha\,\underline{j})\cdot\delta\underline{r}_A$$

$$\underline{v}_A = \underline{v}_0 + \underline{\omega}\times\underline{r}_{0A} = -R\dot{\theta}\,\underline{i} + \dot{\theta}\,\underline{k}\times R(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) = -(R\dot{\theta} + R\dot{\theta}\sin\theta)\,\underline{i} + R\dot{\theta}\cos\theta\,\underline{j}$$

$$\longrightarrow \quad \delta\underline{r}_A = \left[-R(1+\sin\theta)\,\underline{i} + R\cos\theta\,\underline{j}\right]\delta\theta$$

$$\delta\underline{r}_B = \delta x\,\underline{i}$$

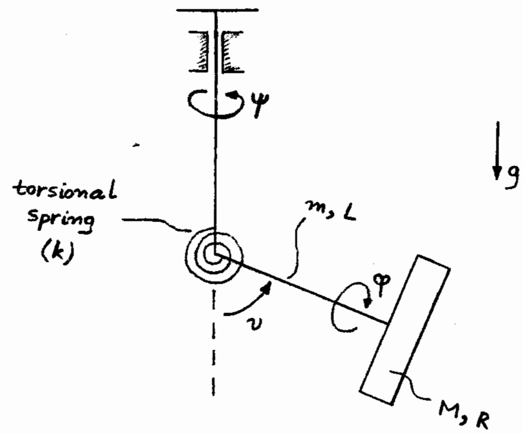
$$\therefore \quad \delta w = \overbrace{(F\cos\alpha)}^{K_1 = m\ddot{x}}\,\delta x + \overbrace{\left[FR\cos\alpha\,(1+\sin\theta) - FR\cos\theta\sin\alpha\right]}^{K_2 = \frac{3}{2}MR^2\ddot{\theta}}\,\delta\theta$$

$$\Longrightarrow \quad \underline{F = \frac{m\ddot{x}}{\cos\alpha}} \quad \text{or} \quad \underline{F = \frac{3/2\,MR\,\ddot{\theta}}{\cos\alpha + \sin(\theta - \alpha)}}$$

where $\underline{\alpha = \sin^{-1}\left(\dfrac{R\sin\theta + R - h}{L}\right)}$

Problem 3

(a)



$\#$ DOF = 3

$$T = T^{beam} + T^{disk}$$
$$= \frac{1}{6} m L^2 \left(\dot\psi^2 \sin^2\upsilon + \dot\upsilon^2 \right) + \frac{M}{2} \dot\psi^2 \sin^2\upsilon \left(L^2 + \frac{R^2}{4} \right)$$
$$+ \frac{1}{2} M \dot\upsilon^2 \left(L^2 + \frac{R^2}{4} \right) + \frac{1}{4} M R^2 \left(\dot\varphi + \dot\psi \cos\upsilon \right)^2$$

$$V = -mg\frac{L}{2}\cos\upsilon - MgL\cos\upsilon + \frac{1}{2}k\upsilon^2$$

$$L = T - V$$

Equations of motion,

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\psi}\right) - \cancelto{0}{\dfrac{\partial L}{\partial \psi}} = 0 & \Longrightarrow \quad \dfrac{\partial L}{\partial \dot\psi} = \text{Const.} = P_\psi \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\upsilon}\right) - \dfrac{\partial L}{\partial \upsilon} = 0 & \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\varphi}\right) - \cancelto{0}{\dfrac{\partial L}{\partial \varphi}} = 0 & \Longrightarrow \quad \dfrac{\partial L}{\partial \dot\varphi} = \text{Const.} = P_\varphi \end{cases}$$

$$L = \left[\frac{mL^2}{6} + \frac{M}{2}\left(L^2 + \frac{R^2}{4}\right) \right] \left(\dot\psi^2 \sin^2\upsilon + \dot\upsilon^2 \right) + \frac{1}{4} M R^2 \left(\dot\varphi + \dot\psi\cos\upsilon \right)^2 + mg\frac{L}{2}\cos\upsilon + MgL\cos\upsilon - \frac{1}{2}k\upsilon^2$$

$$\frac{\partial L}{\partial \dot\psi} = \left[\frac{mL^2}{3} + M\left(L^2 + \frac{R^2}{4}\right) \right] \dot\psi \sin^2\upsilon + \underbrace{\frac{1}{2} M R^2 \left(\dot\varphi + \dot\psi \cos\upsilon \right)}_{P_\varphi} \cos\upsilon = P_\psi$$

$$\frac{\partial L}{\partial \dot\varphi} = \frac{1}{2} M R^2 \left(\dot\varphi + \dot\psi \cos\upsilon \right) = P_\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\upsilon}\right) - \frac{\partial L}{\partial \upsilon} = \left[\frac{mL^2}{3} + M\left(L^2 + \frac{R^2}{4}\right) \right] \ddot\upsilon - 2\sin\upsilon\cos\upsilon\,\dot\psi^2 \left[\frac{mL^2}{6} + \frac{M}{2}\left(L^2 + \frac{R^2}{4}\right) \right] + \overbrace{\frac{MR^2}{2}\left(\dot\varphi + \dot\psi\cos\upsilon\right)}^{P_\varphi} \dot\psi \sin\upsilon$$
$$+ mg\frac{L}{2}\sin\upsilon + MgL\sin\upsilon + k\upsilon = 0$$

Let $\quad M_1 = \dfrac{mL^2}{6} + \dfrac{M}{2}\left(L^2 + \dfrac{R^2}{4}\right)$

$$\longrightarrow \qquad \dot\psi = \frac{P_\psi - P_\varphi \cos\upsilon}{2 M_1 \sin^2\upsilon}$$

Problem 3

$$\Longrightarrow \quad 2M_1\ddot{v} - \frac{\cos v}{\sin^3 v}\frac{(P_\psi - P_\varphi \cos v)^2}{2M_1} + \frac{P_\varphi(P_\psi - P_\varphi\cos v)}{2M_1 \sin v} + mg\frac{L}{2}\sin v + MgL\sin v + kv = 0$$

Single equation of motion for v

(b)

$P_\varphi = 0$

$$\Longrightarrow \quad \ddot{v} - \frac{P_\psi^2}{4M_1^2}\frac{\cos v}{\sin^3 v} + \frac{1}{2M_1}\left[gL\left(\frac{m}{2}+M\right)\sin v + kv\right] = 0$$

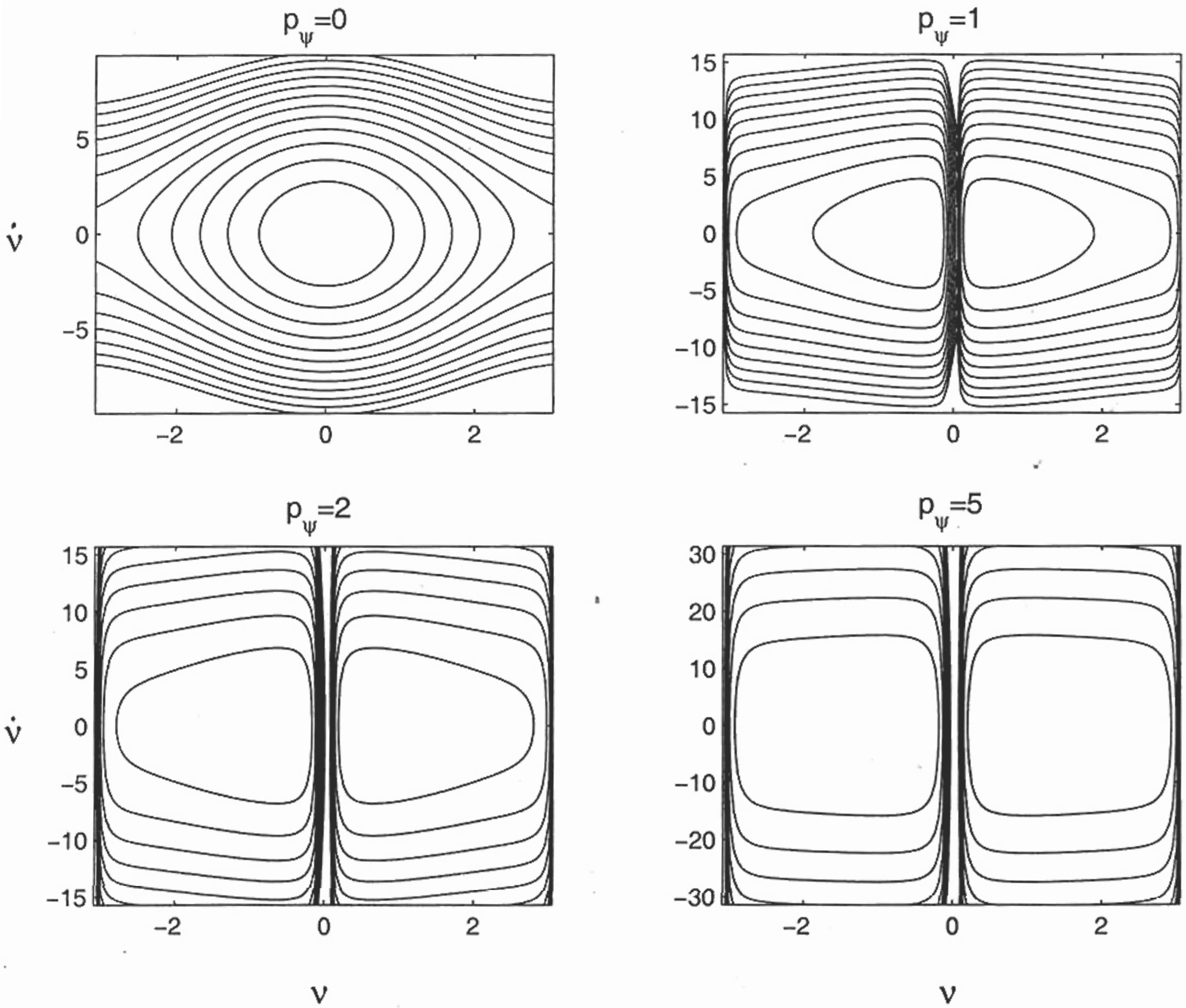
$$\longrightarrow \quad \dot{v}\ddot{v} - \frac{P_\psi^2}{4M_1^2}\frac{\cos v}{\sin^3 v}\dot{v} + \frac{1}{2M_1}\left[gL\left(\frac{m}{2}+M\right)\sin v\,\dot{v} + kv\dot{v}\right] = 0$$

$$\longrightarrow \quad \frac{1}{2}\dot{v}^2 + \frac{P_\psi^2}{8M_1^2}\frac{1}{\sin^2 v} - \frac{1}{2M_1}\left[gL\left(\frac{m}{2}+M\right)\cos v - k\frac{v^2}{2}\right] = \text{Const.}$$

$m = M = L = R = 1 \quad \longrightarrow \quad M_1 = \frac{19}{24}, \qquad g = 9.8 \qquad k = 1$

$$\longrightarrow \quad \dot{v}^2 + \frac{P_\psi^2}{4M_1^2}\frac{1}{\sin^2 v} - \frac{1}{M_1}\left[14.7\cos v - \frac{v^2}{2}\right] = E_0$$

See the following page for the trajectories on the $(v,\dot{v})$ phase plane.



$p_\psi=0$
$p_\psi=1$
$p_\psi=2$
$p_\psi=5$
$\dot{\nu}$
ν