2.032 Dynamics

Fall 2004

Problem Set No. 8

Problem 1

123 system coordinate is fixed to the inner gimbal.

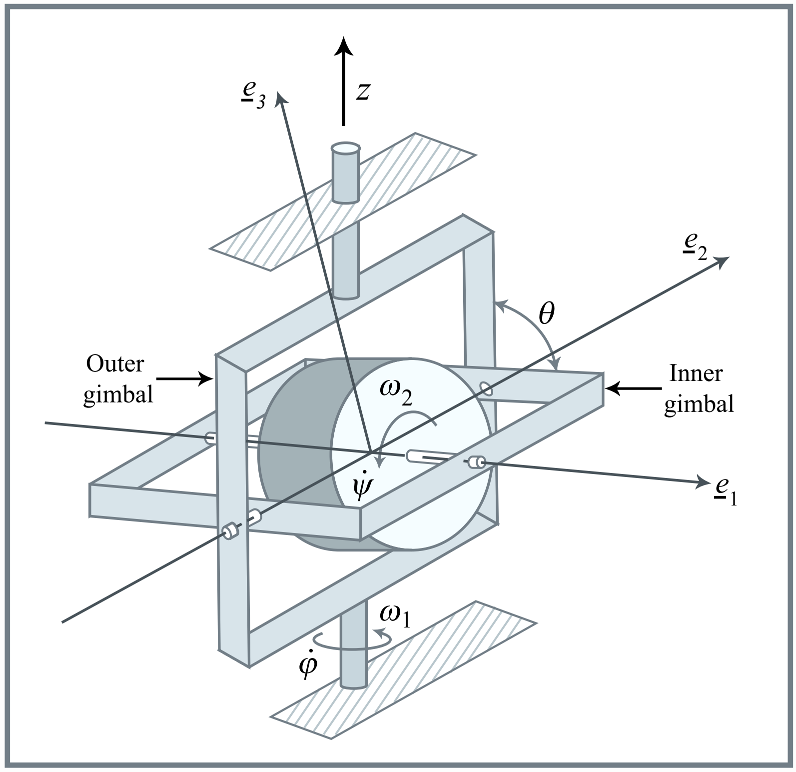


Figure by OCW.

DOF = 6 − 3 = 3

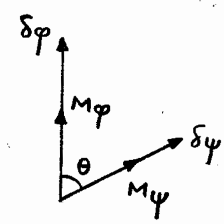
generalized coordinates:

$q_1 = \theta$, $q_2 = \varphi$, $q_3 = \psi$

If we need torque M_ψ about spin axis and torque M_φ about the vertical axis to maintain the motion, the generalized forces can be found as follows:

$$\delta W = M_\psi \delta\psi + M_\varphi (\delta\psi \cos\theta) + M_\varphi \delta\varphi + M_\psi (\delta\varphi \cos\theta)$$

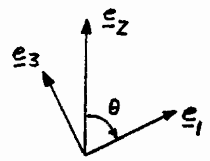
$$= (M_\psi + M_\varphi \cos\theta)\,\delta\psi + (M_\varphi + M_\psi \cos\theta)\,\delta\varphi$$



$$\Longrightarrow \quad Q_\theta = 0$$

$$Q_\varphi = M_\varphi + M_\psi \cos\theta$$

$$Q_\psi = M_\psi + M_\varphi \cos\theta$$



$\mathcal{L} = T - V$

$V = 0$ (center of mass does not move)

$$\underline{e}_z = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_3$$

$$T = \frac{1}{2} \underline{\omega}^T \underline{\underline{I}}_c \underline{\omega}$$

$$\underline{\omega} = \dot{\varphi}\, \underline{e}_z + \dot{\theta}\, \underline{e}_2 + \dot{\psi}\, \underline{e}_1 = (\dot{\psi} + \dot{\varphi}\cos\theta)\underline{e}_1 + \dot{\theta}\, \underline{e}_2 + \dot{\varphi}\sin\theta\, \underline{e}_3$$

$$T = \frac{1}{2} I_1 (\dot{\psi} + \dot{\varphi}\cos\theta)^2 + \frac{1}{2} I_2 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\varphi}^2 \sin^2\theta = L \qquad (V = 0)$$

Problem 1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = Q_\theta \longrightarrow I_2\ddot\theta + I_1(\dot\psi + \dot\varphi \cos\theta)\dot\varphi \sin\theta - I_2\dot\varphi^2 \sin\theta\cos\theta = 0$$

Note that $\dot\varphi = \omega_1$ and $\dot\psi = \omega_2$ are known so the system has one degree of freedom θ and the equation of motion would be:

$$\underline{I_2\ddot\theta + (I_1 - I_2)\omega_1^2 \sin\theta\cos\theta + I_1\omega_1\omega_2\sin\theta = 0}$$

To find the generalized forces Q_φ and Q_ψ :

$$Q_\varphi = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left[I_1(\dot\psi + \dot\varphi\cos\theta)\cos\theta + I_2\dot\varphi\sin^2\theta\right]$$

$$\underline{Q_\varphi = I_1\dot\omega_1\cos^2\theta - I_1\dot\theta\omega_2\sin\theta + I_2\dot\omega_1\sin^2\theta + (I_2 - I_1)\dot\theta\omega_1\sin 2\theta}$$

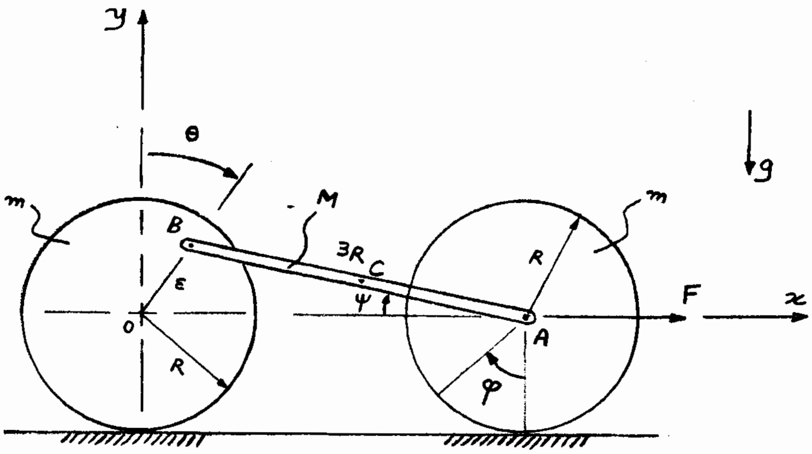
$$Q_\psi = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\psi}\right) - \frac{\partial L}{\partial \psi} = I_1(\ddot\psi + \ddot\varphi\cos\theta) - I_1\dot\varphi\dot\theta\sin\theta =$$

$$\underline{Q_\psi = I_1\dot\omega_1\cos\theta - I_1\omega_1\dot\theta\sin\theta}$$

M_φ and M_ψ which, respectively, correspond to M_z and M_1' (found in the Quiz problem) can be determined using the following equations:

$$\begin{cases} M_\varphi = \dfrac{Q_\varphi - \cos\theta\, Q_\psi}{\sin^2\theta} \\[2ex] M_\psi = \dfrac{Q_\psi - \cos\theta\, Q_\varphi}{\sin^2\theta} \end{cases}$$

Problem 2

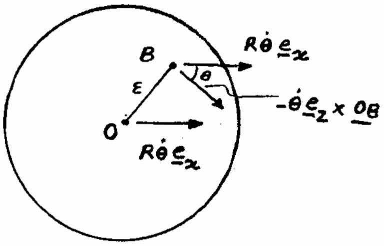


$$\underline{\omega}\big|_{\text{left cyl.}} = -\dot{\theta}\,\underline{e}_z$$

$$\underline{\omega}\big|_{\text{right cyl.}} = -\dot{\varphi}\,\underline{e}_z$$

$$\underline{\omega}\big|_{\text{rod}} = -\dot{\psi}\,\underline{e}_z$$

$$\underline{v}_A = R\dot{\varphi}\,\underline{e}_x$$

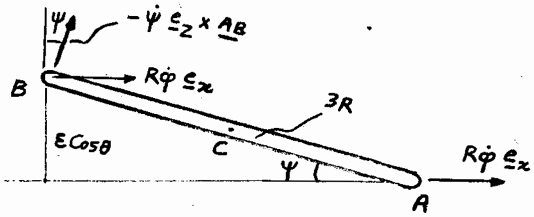


$$\underline{v}_B = \underline{v}_0 + (-\dot{\theta}\,\underline{e}_z) \times \underline{OB}$$

$$= R\dot{\theta}\,\underline{e}_x + \dot{\theta}\,\varepsilon\left(\cos\theta\,\underline{e}_x - \sin\theta\,\underline{e}_y\right)$$

$$= (R + \varepsilon\cos\theta)\,\dot{\theta}\,\underline{e}_x - \varepsilon\sin\theta\,\dot{\theta}\,\underline{e}_y$$

(point B on the left cylinder)



$$\underline{v}_B\big|_{\text{rod}} = \underline{v}_A + \overbrace{\underline{\omega}\big|_{\text{rod}}}^{-\dot{\psi}\,\underline{e}_z} \times \underline{AB}$$

$$= R\dot{\varphi}\,\underline{e}_x + 3R\dot{\psi}\left(\sin\psi\,\underline{e}_x + \cos\psi\,\underline{e}_y\right)$$

$$= (R\dot{\varphi} + 3R\dot{\psi}\sin\psi)\,\underline{e}_x + 3R\dot{\psi}\cos\psi\,\underline{e}_y$$

$$\underline{v}_B\big|_{\text{left cyl.}} = \underline{v}_B\big|_{\text{rod}} \quad \Longrightarrow \quad \begin{cases} (R + \varepsilon\cos\theta)\,\dot{\theta} = R\dot{\varphi} + 3R\dot{\psi}\,\overbrace{\sin\psi}^{\frac{\varepsilon\cos\theta}{3R}} \\ -\varepsilon\sin\theta\,\dot{\theta} = 3R\dot{\psi}\,\underbrace{\cos\psi}_{\frac{\sqrt{9R^2 - \varepsilon^2\cos^2\theta}}{3R}} \end{cases}$$

$$\Longrightarrow \begin{cases} \dot{\psi} = \dfrac{-\varepsilon\sin\theta\,\dot{\theta}}{\sqrt{9R^2 - \varepsilon^2\cos^2\theta}} \\[2ex] \dot{\varphi} = \dfrac{(R + \varepsilon\cos\theta)\,\dot{\theta}}{R} + \dfrac{\varepsilon^2\sin\theta\cos\theta\,\dot{\theta}}{R\sqrt{9R^2 - \varepsilon^2\cos^2\theta}} \end{cases} \qquad (*)$$

Problem 2

$$\underline{v}_C = \underline{v}_A + \underline{\omega}|_{rod} \times \underline{AC} = R\dot{\varphi}\,\underline{e}_x + \dot{\psi}\,\frac{3R}{2}\left(\sin\psi\,\underline{e}_x + \cos\psi\,\underline{e}_y\right)$$

$$= \left(R\dot{\varphi} + \frac{3R}{2}\dot{\psi}\sin\psi\right)\underline{e}_x + \frac{3R}{2}\dot{\psi}\cos\psi\,\underline{e}_y$$

$$\underline{v}_C = \left[(R+\varepsilon\cos\theta)\,\dot{\theta} + \frac{\varepsilon^2\sin\theta\cos\theta\,\dot{\theta}}{2\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right]\underline{e}_x - \frac{\varepsilon\sin\theta\,\dot{\theta}}{2}\,\underline{e}_y = G(\theta)\,\dot{\theta}\,\underline{e}_x - \frac{\varepsilon\sin\theta\,\dot{\theta}}{2}\,\underline{e}_y$$

DOF $= 3\times3 - 2-2-2-2 = 1$ $\qquad q_1 = \theta$ $\qquad$ F in the only active non-potential force.

$$\circledast \longrightarrow \qquad \delta\varphi = \left(\frac{R+\varepsilon\cos\theta}{R} + \frac{\varepsilon^2\sin\theta\cos\theta}{R\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right)\delta\theta$$

To find Q_θ, $\qquad \delta W_\theta = F(R\,\delta\varphi) = F\left(R+\varepsilon\cos\theta + \frac{\varepsilon^2\sin\theta\cos\theta}{\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right)\delta\theta$

$$\Longrightarrow \qquad Q_\theta = F\left(R+\varepsilon\cos\theta + \varepsilon^2\frac{\sin\theta\cos\theta}{\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right)$$

Construct Lagrangian: $\qquad \mathcal{L} = T - V$

$$T = \frac{1}{2}m\,\underline{v}_0\cdot\underline{v}_0 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{\theta}^2 + \frac{1}{2}m\,\underline{v}_A\cdot\underline{v}_A + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{\varphi}^2 + \frac{1}{2}M\,\underline{v}_C\cdot\underline{v}_C + \frac{1}{2}\left(\frac{1}{12}M(3R)^2\right)\dot{\psi}^2$$

$$T = \frac{1}{2}m(R\dot{\theta})^2 + \frac{1}{4}mR^2\dot{\theta}^2 + \frac{1}{2}m(R\dot{\varphi})^2 + \frac{1}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}M\left[G^2(\theta) + \frac{\varepsilon^2\sin^2\theta}{4}\right]\dot{\theta}^2 + \frac{3}{8}MR^2\frac{\varepsilon^2\sin^2\theta}{9R^2-\varepsilon^2\cos^2\theta}\dot{\theta}^2$$

$$T = \frac{3}{4}mR^2\dot{\theta}^2 + \frac{3}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}M\left[G^2(\theta) + \frac{\varepsilon^2\sin^2\theta}{4}\right]\dot{\theta}^2 + \frac{3}{8}MR^2\frac{\varepsilon^2\sin^2\theta}{9R^2-\varepsilon^2\cos^2\theta}\dot{\theta}^2$$

$$V = Mg\,y_C = Mg\left(\frac{3R}{2}\sin\psi\right) = Mg\,\frac{\varepsilon\cos\theta}{2} \qquad (y_0 = 0,\; y_A = 0)$$

Problem 2

$$\therefore \quad \mathcal{L} = \frac{3}{4} m R^2 \dot{\theta}^2 + \frac{3}{4} m \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2 \dot{\theta}^2$$

$$+ \frac{1}{2} M \left\{ \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{2\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2 + \frac{\varepsilon^2 \sin^2\theta}{4} \right\} \dot{\theta}^2 + \frac{3}{8} M R^2 \frac{\varepsilon^2 \sin^2\theta}{9R^2 - \varepsilon^2 \cos^2\theta} \dot{\theta}^2$$

$$- Mg \frac{\varepsilon}{2} \cos\theta = \underline{H(\theta)\, \dot{\theta}^2 - Mg \frac{\varepsilon}{2} \cos\theta}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt} \left[2H(\theta)\, \dot{\theta} \right] - \left[\frac{dH}{d\theta} \dot{\theta}^2 + Mg \frac{\varepsilon}{2} \sin\theta \right] = Q_\theta$$

$$\boxed{2H(\theta)\, \ddot{\theta} + \frac{dH}{d\theta} \dot{\theta}^2 - Mg \frac{\varepsilon}{2} \sin\theta = Q_\theta}$$

governing equation of motion

where

$$H(\theta) = \frac{3}{4} m R^2 + \frac{3}{4} m \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2$$

$$+ \frac{1}{2} M \left\{ \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{2\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2 + \frac{\varepsilon^2 \sin^2\theta}{4} \right\} + \frac{3}{8} M R^2 \frac{\varepsilon^2 \sin^2\theta}{9R^2 - \varepsilon^2 \cos^2\theta}$$

and

$$Q_\theta = F \left(R + \varepsilon \cos\theta + \varepsilon^2 \frac{\sin\theta \cos\theta}{\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right)$$

Problem 3

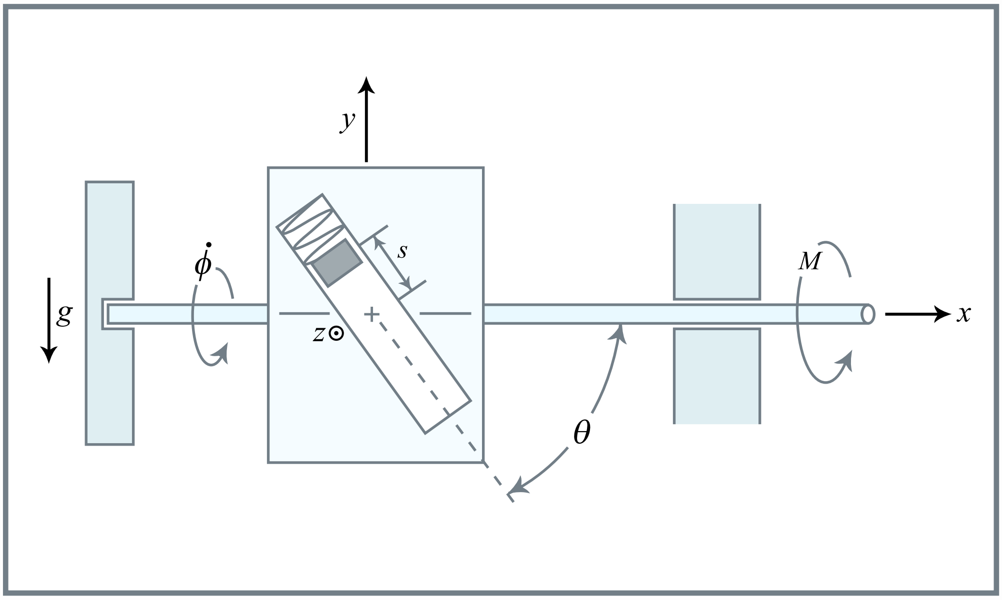


Figure by OCW. After problem 6.41 in Ginsberg, J. H. *Advanced Engineering Dynamics.* 2nd ed. New York: Cambridge University Press, 1998.

DOF = 2

generalized coordinates :

$q_1 = s \,, \quad q_2 = \varphi$

generalized forces :

$Q_s = 0 \,, \quad Q_\varphi = M$

$(\delta W = M\,\delta\varphi)$

$L = T - V$

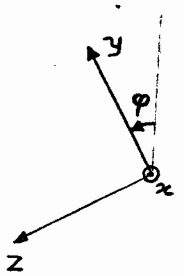
$V_{slider} = mg\, s \sin\theta \cos\varphi$

$V_{spring} = \frac{1}{2} k s^2$ (k spring stiffness)

$V_{housing} = 0$

$T_{housing} = \frac{1}{2} I \dot\varphi^2$

$T_{slider} = \frac{1}{2} m v_s^2$



xyz coordinate system rotates with $\dot\varphi$ about x:

$$\underline{r}_s = -s\cos\theta\, \underline{e}_x + s\sin\theta\, \underline{e}_y$$

$$\underline{v}_s = \overset{\circ}{\underline{r}}_s + \underbrace{\underline{\omega}}_{\dot\varphi \underline{e}_x} \times \underline{r}_s = -\dot s\cos\theta\, \underline{e}_x + \dot s \sin\theta\, \underline{e}_y + \underbrace{\dot\varphi\, \underline{e}_x \times \underline{r}_s}_{\dot\varphi s \sin\theta\, \underline{e}_z}$$

$$T_{slider} = \frac{1}{2} m \left(\dot s^2 + \dot\varphi^2 s^2 \sin^2\theta\right)$$

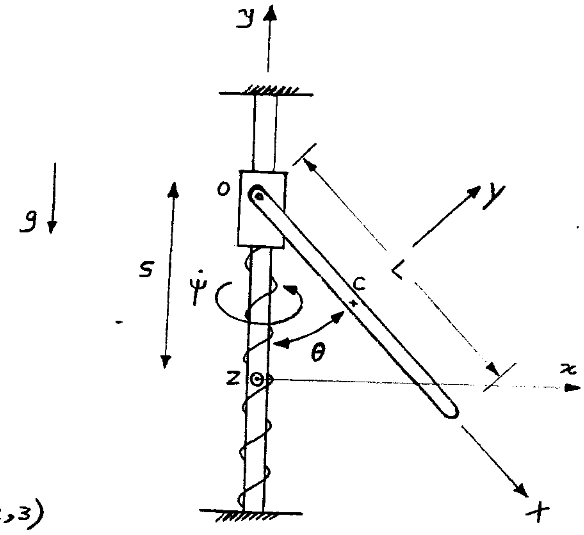
$$\therefore \quad L = \frac{1}{2} I \dot\varphi^2 + \frac{1}{2} m \left(\dot s^2 + \dot\varphi^2 s^2 \sin^2\theta\right) - mg\, s \sin\theta \cos\varphi - \frac{1}{2} k s^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = M \longrightarrow \left(I + m s^2 \sin^2\theta\right)\ddot\varphi + 2 m s \dot s \dot\varphi \sin^2\theta - mg\, s \sin\theta \sin\varphi = M$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot s}\right) - \frac{\partial L}{\partial s} = 0 \longrightarrow m\ddot s - m s \dot\varphi^2 \sin^2\theta + mg \sin\theta \cos\varphi + ks = 0$$

equations of motion

Problem 4



DOF = 3

$q_1 = \theta$, $q_2 = \psi$, $q_3 = s$

(spring is unstretched when $s = 0$ and $\psi = 0$)

All active forces are potential. $\Longrightarrow$ $Q_j = 0$ $(j = 1, 2, 3)$

$L = T - V$

$$T = \frac{1}{2} m v_C^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{I}}_C \underline{\omega}$$

xyz rotates with $\dot\psi$ about vertical axis so that the bar is always in xy plane. XYZ is fixed to the bar.

Introducing additional generalized coordinates and using Lagrange multipliers, one can find the constraint forces as well.

The constraint force at point O has three components. The vertical component is simply $-k_e s$ since the collar is massless.

To find the horizonal components of the constraint force:

select $q_4 = x$, $q_5 = z$ (x and z are the coordinates of the upper end of the rod,

$\longrightarrow$ $n = 5$

Constraints: $\begin{cases} x = 0 & \text{or} \quad q_4 = 0 \\ z = 0 & \text{or} \quad q_5 = 0 \end{cases}$ $\longrightarrow$ $m = 2$ $\longrightarrow$ two Lagrange multipliers λ_1, λ_2

$q_4 = 0 \longrightarrow a_{11} = a_{12} = a_{13} = a_{15} = 0$, $a_{14} = 1$

$q_5 = 0 \longrightarrow a_{21} = a_{22} = a_{23} = a_{24} = 0$, $a_{25} = 1$

Problem 4

$$\underline{r}_c = \left(\frac{L}{2}\sin\theta + x\right)\underline{e}_x + \left(s - \frac{L}{2}\cos\theta\right)\underline{e}_y + z\,\underline{e}_z$$

$$\underline{v}_c = \dot{\underline{r}}_c = \overset{\circ}{\underline{r}}_c + \underbrace{\underline{\omega}_{xyz}}_{\dot\psi\,\underline{e}_y} \times \underline{r}_c = \left(\dot x + \dot\psi z + \frac{L}{2}\cos\theta\,\dot\theta\right)\underline{e}_x + \left(\dot s + \frac{L}{2}\sin\theta\,\dot\theta\right)\underline{e}_y + \left(\dot z - \dot\psi x - \dot\psi\frac{L}{2}\sin\theta\right)\underline{e}_z$$

$$\underline{\omega}_{rod} = \dot\psi\,\underline{e}_y + \dot\theta\,\underline{e}_z = \dot\psi\left(-\cos\theta\,\underline{e}_X + \sin\theta\,\underline{e}_Y\right) + \dot\theta\,\underline{e}_Z \qquad \underline{\underline{I}}_c = \frac{1}{12}mL^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow$

$$T = \frac{1}{2}m\left[\left(\dot x + \dot\psi z + \frac{L}{2}\cos\theta\,\dot\theta\right)^2 + \left(\dot s + \frac{L}{2}\sin\theta\,\dot\theta\right)^2 + \left(\dot z - \dot\psi x - \dot\psi\frac{L}{2}\sin\theta\right)^2\right]$$
$$+ \frac{1}{2}\left(\frac{1}{12}mL^2\right)\left(\dot\theta^2 + \dot\psi^2\sin^2\theta\right)$$

$$V = mg\,y_c + \frac{1}{2}k_e s^2 + \frac{1}{2}k_t\psi^2 = mg\left(s - \frac{L}{2}\cos\theta\right) + \frac{1}{2}k_e s^2 + \frac{1}{2}k_t\psi^2$$

$$Q_j = 0 \qquad (j = 1, \cdots, 5)$$

$\therefore$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = \lambda_1 a_{11} + \lambda_2 a_{21} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\psi} - \frac{\partial L}{\partial\psi} = \lambda_1 a_{12} + \lambda_2 a_{22} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot s} - \frac{\partial L}{\partial s} = \lambda_1 a_{13} + \lambda_2 a_{23} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot x} - \frac{\partial L}{\partial x} = \lambda_1 a_{14} + \lambda_2 a_{24} = \overset{K_4}{\lambda_1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot z} - \frac{\partial L}{\partial z} = \lambda_1 a_{15} + \lambda_2 a_{25} = \overset{K_5}{\lambda_2}$$

$$\delta W^{\text{non-potential}} = \overset{K_4}{F_x}\,\delta x + \overset{K_5}{F_z}\,\delta z \qquad (F_x, F_z)\ \text{horizontal components of the contraint force}$$

Problem 4

Applying the constraints $x = 0$ & $z = 0$, the first three equations yield the governing equations:

$$\begin{cases} \frac{1}{3} m L^2 \ddot{\theta} + \frac{1}{2} m L \sin\theta \, \ddot{s} - \frac{1}{3} m L^2 \sin\theta \cos\theta \, \dot{\psi}^2 + \frac{1}{2} m g L \sin\theta = 0 \\ \frac{1}{3} m L^2 \sin^2\theta \cdot \ddot{\psi} + \frac{2}{3} m L^2 \dot{\psi} \dot{\theta} \sin\theta \cos\theta + k_t \psi = 0 \\ m \ddot{s} + \frac{1}{2} m L \sin\theta \cdot \ddot{\theta} + \frac{1}{2} m L \dot{\theta}^2 \cos\theta + k_e s + m g = 0 \end{cases}$$

Constraint forces F_x and F_z:

$$F_x = K_4 = \lambda_1 = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right) \Bigg|_{\substack{x = \dot{x} = 0 \\ z = \dot{z} = 0}} = \underline{m \frac{L}{2} \ddot{\theta} \cos\theta - \frac{1}{2} m L \dot{\theta}^2 \sin\theta - \frac{1}{2} m L \dot{\psi}^2 \sin\theta}$$

$$F_z = K_5 = \lambda_2 = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} \right) \Bigg|_{\substack{x = \dot{x} = 0 \\ z = \dot{z} = 0}} = \underline{- m \frac{L}{2} \ddot{\psi} \sin\theta - m L \dot{\psi} \dot{\theta} \cos\theta}$$

Note that the constraint torque about y axis is $-k_t \psi$ and the constraint torque about x axis can be found by introducing a rotation about this axis.

Center of mass of the collar is stationary in x & z so the horizontal constraint force between the collar and the vertical rod is equal to (F_x, F_z).