2.032 Dynamics

Fall 2004

Problem Set No. 7

Problem 1

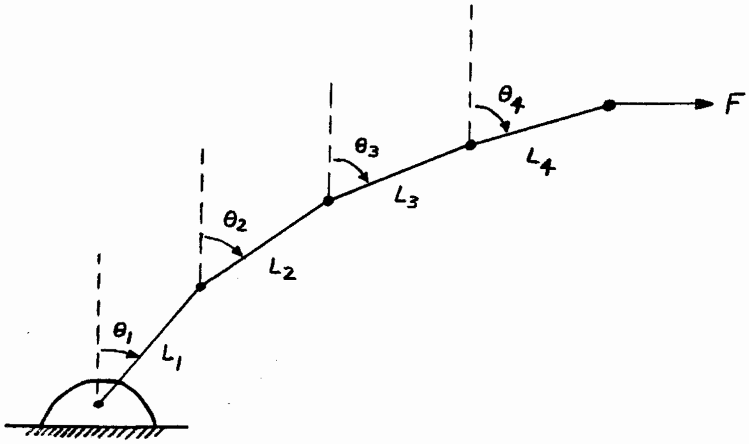
$\xi_1 = \theta_1$

$\xi_2 = \theta_2$

$\xi_3 = \theta_3$

$\xi_4 = \theta_4$

$\Xi_1 = ?$ $\Xi_2 = ?$



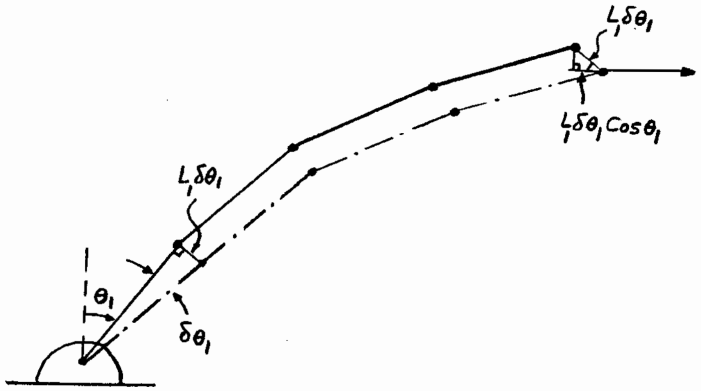
To find Ξ_1,

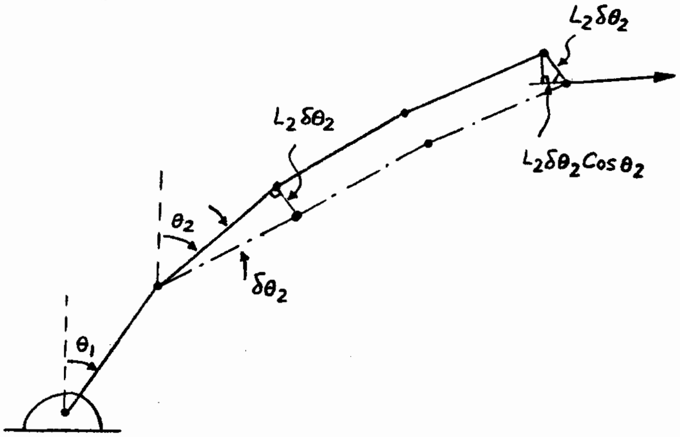
Freeze $\theta_2, \theta_3, \theta_4$, and

vary $\theta_1 \longrightarrow \theta_1 + \delta\theta_1$

$$\delta W_1 = F(L_1\delta\theta_1)\cos\theta_1$$
$$= (FL_1\cos\theta_1)\,\delta\theta_1$$

$$\therefore \quad \Xi_1 = FL_1\cos\theta_1$$





To find Ξ_2,

Freeze $\theta_1, \theta_3, \theta_4$, and

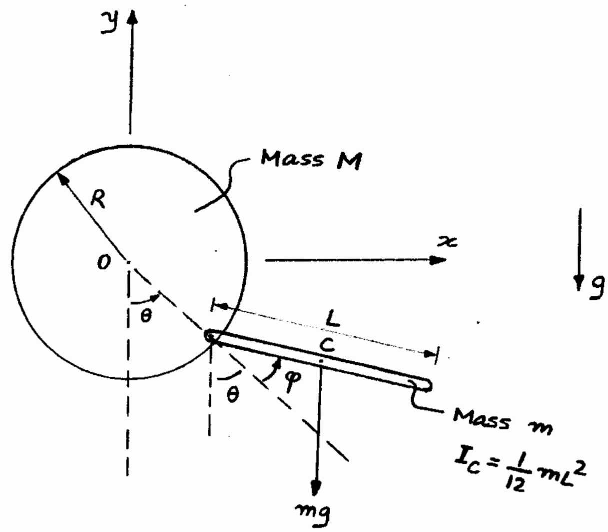
vary $\theta_2 \longrightarrow \theta_2 + \delta\theta_2$

$$\delta W_2 = F(L_2\delta\theta_2)\cos\theta_2 = (FL_2\cos\theta_2)\,\delta\theta_2 \quad \Longrightarrow \quad \Xi_2 = FL_2\cos\theta_2$$

Problem 2

a) Constraints are holonomic. $\# DOF = 2\times 3 - 2 - 2 = 2$

$q_1 = \theta$ and $q_2 = \varphi$ are complete and independent set of generalized coordinates.



b) all active forces are potential $\Longrightarrow$ $Q_j = 0$

$$\underline{\omega}_M = \underline{\omega}\big|_{flywheel} = \dot\theta\, \underline{e}_z$$

$$\underline{\omega}_m = \underline{\omega}\big|_{rod} = (\dot\theta + \dot\varphi)\, \underline{e}_z$$

$$\begin{cases} x_c = R\sin\theta + \frac{L}{2}\sin(\theta+\varphi) \\ y_c = -R\cos\theta - \frac{L}{2}\cos(\theta+\varphi) \end{cases}$$

$$\Longrightarrow \quad \underline{v}_c = \left[R\cos\theta\,\dot\theta + \frac{L}{2}\cos(\theta+\varphi)\,(\dot\theta+\dot\varphi)\right]\underline{e}_x + \left[R\sin\theta\,\dot\theta + \frac{L}{2}\sin(\theta+\varphi)\,(\dot\theta+\dot\varphi)\right]\underline{e}_y$$

Construct Lagrangian:

$$\mathcal{L} = T - V$$

$$T = T_M + T_m$$

$$T_M = \frac{1}{2} I_0\, \underline{\omega}_M\cdot\underline{\omega}_M = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot\theta^2 = \frac{1}{4}MR^2\dot\theta^2$$

$$T_m = \frac{1}{2} m\, \underline{v}_c\cdot\underline{v}_c + \frac{1}{2} I_c\, \underline{\omega}_m\cdot\underline{\omega}_m = \frac{1}{2}m\left[R^2\dot\theta^2 + \frac{L^2}{4}(\dot\theta+\dot\varphi)^2 + LR\dot\theta\,(\dot\theta+\dot\varphi)\cos\varphi\right] + \frac{1}{2}\left(\frac{1}{12}mL^2\right)(\dot\theta+\dot\varphi)^2$$

$$V = mg\,y_c = -mg\left[R\cos\theta + \frac{L}{2}\cos(\theta+\varphi)\right] \qquad (y_0 = 0)$$

$$\therefore \quad \mathcal{L} = \left(\frac{M}{4}+\frac{m}{2}\right)R^2\dot\theta^2 + \frac{1}{6}mL^2(\dot\theta+\dot\varphi)^2 + \frac{1}{2}mLR\dot\theta\,(\dot\theta+\dot\varphi)\cos\varphi + mgR\cos\theta + mg\frac{L}{2}\cos(\theta+\varphi)$$

Problem 2

$$\left|\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot\theta} - \dfrac{\partial \mathcal{L}}{\partial \theta} = 0 \\ \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot\varphi} - \dfrac{\partial \mathcal{L}}{\partial \varphi} = 0 \end{array}\right.$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = \left(\frac{M}{2}+m\right)R^2\dot\theta + \frac{1}{3}mL^2(\dot\theta+\dot\varphi) + \frac{1}{2}mLR(2\dot\theta+\dot\varphi)\cos\varphi$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgR\sin\theta - mg\frac{L}{2}\sin(\theta+\varphi)$$

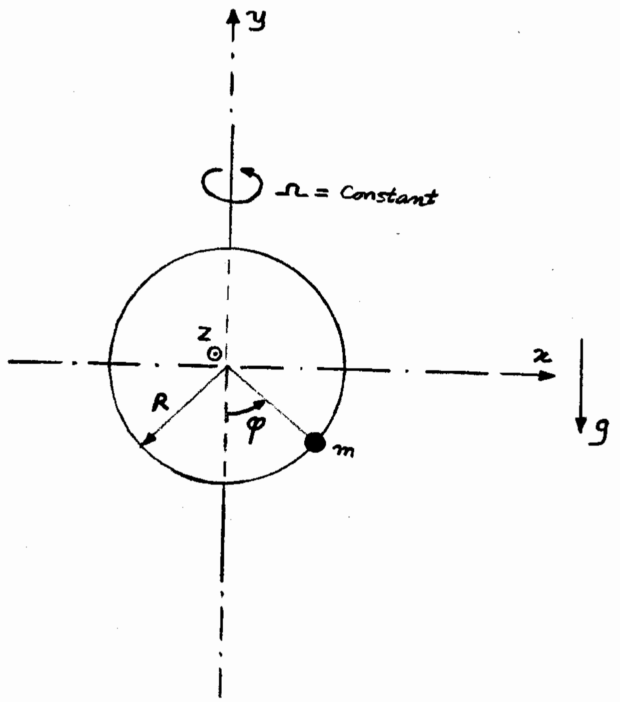
$$\frac{\partial \mathcal{L}}{\partial \dot\varphi} = \frac{1}{3}mL^2(\dot\theta+\dot\varphi) + \frac{1}{2}mLR\dot\theta\cos\varphi$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{1}{2}mLR\dot\theta(\dot\theta+\dot\varphi)\sin\varphi - mg\frac{L}{2}\sin(\theta+\varphi)$$

$$\therefore \left|\begin{array}{l} \left[\left(\frac{M}{2}+m\right)R^2 + \frac{1}{3}mL^2 + mLR\cos\varphi\right]\ddot\theta + mL\left(\frac{L}{3}+\frac{R}{2}\cos\varphi\right)\ddot\varphi - \frac{1}{2}mLR(2\dot\theta+\dot\varphi)\dot\varphi\sin\varphi \\ \quad + mgR\sin\theta + mg\frac{L}{2}\sin(\theta+\varphi) = 0 \\ \\ mL\left(\frac{L}{3}+\frac{R}{2}\cos\varphi\right)\ddot\theta + \frac{1}{3}mL^2\ddot\varphi + \frac{1}{2}mLR\dot\theta^2\sin\varphi + mg\frac{L}{2}\sin(\theta+\varphi) = 0 \end{array}\right.$$

equations of motion

Problem 3



Constraints are holonomic.

→ Constraints are ideal.

→ D'Alembert's principle applies.

$\# \text{ DOF} = 3 - 2 = 1 \implies \varphi$ generalized coordinate

xyz is fixed to the ring so rotates with Ω about y.

$$\left| \begin{array}{l} x = r\sin\varphi \\ y = -r\cos\varphi \end{array} \right.$$

$$\underline{r}_m = r\sin\varphi \, \underline{e}_x - r\cos\varphi \, \underline{e}_y$$

$$\dot{\underline{r}}_m = r\cos\varphi \, \dot\varphi \, \underline{e}_x + r\sin\varphi \, \dot\varphi \, \underline{e}_y + \underbrace{\Omega \underline{e}_y \times \underline{r}_m}_{-r\Omega\sin\varphi \, \underline{e}_z}$$

$$\underline{F} = -mg \, \underline{e}_y \qquad \text{(active force)}$$

$$\dot{\underline{P}} = m\underline{v}_m = m\ddot{\underline{r}}_m = m\left(-r\sin\varphi \, \dot\varphi^2 + r\cos\varphi \, \ddot\varphi\right)\underline{e}_x + m\left(r\cos\varphi \, \dot\varphi^2 + r\sin\varphi \, \ddot\varphi\right)\underline{e}_y - mr\Omega\cos\varphi \, \dot\varphi \, \underline{e}_z$$
$$+ \underbrace{m\Omega \underline{e}_y \times \dot{\underline{r}}_m}_{-m\Omega r\cos\varphi \, \dot\varphi \, \underline{e}_z - mr\Omega^2\sin\varphi \, \underline{e}_x}$$

D'Alembert's principle: $(\underline{F} - \dot{\underline{P}}) \cdot \delta\underline{r} = 0$

$$\rightarrow m\left[\left(r\sin\varphi \, \dot\varphi^2 - r\cos\varphi \, \ddot\varphi + r\Omega^2\sin\varphi\right)\underline{e}_x - \left(g + r\cos\varphi \, \dot\varphi^2 + r\sin\varphi \, \ddot\varphi\right)\underline{e}_y + \left(2r\Omega\cos\varphi \, \dot\varphi\right)\underline{e}_z\right]$$
$$\cdot \left[r\cos\varphi \, \underline{e}_x + r\sin\varphi \, \underline{e}_y\right]\delta\varphi = 0$$

$$\implies \boxed{\ddot\varphi - \Omega^2\sin\varphi\cos\varphi + \frac{g}{r}\sin\varphi = 0} \qquad \text{equation of motion}$$