2.032 Dynamics

Fall 2004

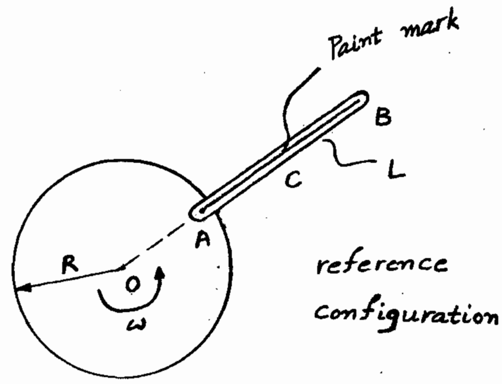
Problem Set No. 3

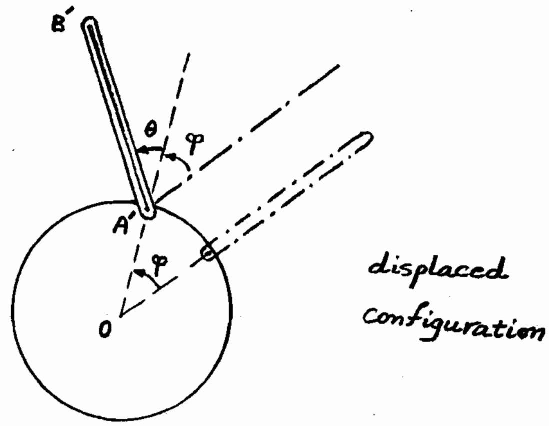
Problem 1

(a)

To find the angular velocity of the rod, compare orientation of AB to A'B':

$$\underline{\omega}_{rod} = (\dot{\varphi} + \dot{\theta})\, \underline{e}_z = (\omega + \dot{\theta})\, \underline{e}_z$$





(b)

$$\underline{v}_C = \underline{v}_A + \underline{\omega}_{rod} \times \underline{AC}$$

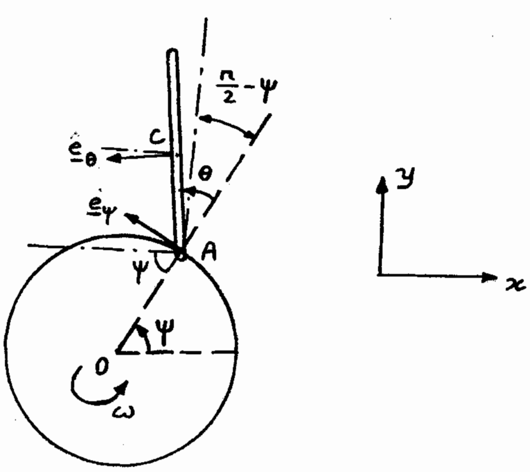
$$\underline{v}_A = \underline{\omega} \times \underline{OA} = \omega R\, \underline{e}_\psi$$

$$\therefore \quad \underline{v}_C = \omega R\, \underline{e}_\psi + (\omega + \dot{\theta}) \frac{L}{2}\, \underline{e}_\theta$$

$$\underline{e}_\psi = -\sin\psi\, \underline{e}_x + \cos\psi\, \underline{e}_y$$

$$\underline{e}_\theta = -\cos\left[\theta - \left(\frac{\pi}{2} - \psi\right)\right] \underline{e}_x - \sin\left[\theta - \left(\frac{\pi}{2} - \psi\right)\right] \underline{e}_y$$

$$\underline{e}_\theta = -\sin(\theta + \psi)\, \underline{e}_x + \cos(\theta + \psi)\, \underline{e}_y$$



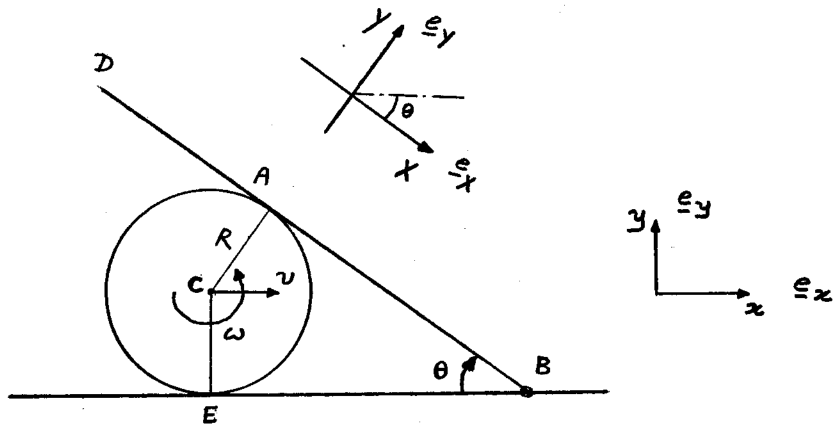
If $\psi(t=0) = \psi_0 \implies \psi(t) = \psi_0 + \omega t$

So, $$\underline{v}_C = \left[-\omega R \sin\psi - (\omega + \dot{\theta}) \frac{L}{2} \sin(\theta + \psi)\right] \underline{e}_x + \left[\omega R \cos\psi + (\omega + \dot{\theta}) \frac{L}{2} \cos(\theta + \psi)\right] \underline{e}_y$$

Problem 2

(a)

$$\underline{v}_C = v\,\underline{e}_x = v\cos\theta\,\underline{e}_X + v\sin\theta\,\underline{e}_Y$$



No slip between cylinder and the bar :

$$\underline{v}_A\big|_{bar} = \underline{v}_A\big|_{Cylinder}$$

$$\underline{v}_A\big|_{bar} = \underline{\omega}_{bar} \times \underline{BA} \quad\Longrightarrow\quad \underline{v}_A\big|_{bar} = -AB\,\dot{\theta}\,\underline{e}_Y$$

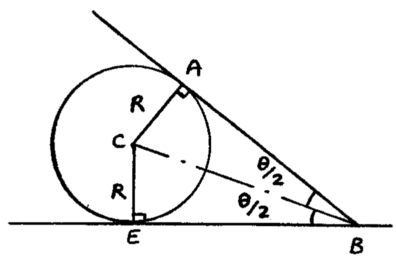
$$\underline{v}_A\big|_{cylinder} = \underline{v}_C + \underline{\omega}_{cylinder} \times \underline{CA}$$

$$= \left(v\cos\theta\,\underline{e}_X + v\sin\theta\,\underline{e}_Y\right) - R\,\omega_{cylinder}\,\underline{e}_X$$

$$\therefore \quad AB\,\dot{\theta}\,\underline{e}_Y = \left(v\cos\theta\,\underline{e}_X + v\sin\theta\,\underline{e}_Y\right) - R\,\omega_{cyl.}\,\underline{e}_X$$

$$\Longrightarrow \quad \left(v\cos\theta - R\,\omega_{cyl.}\right)\underline{e}_X + \left(v\sin\theta - AB\,\dot{\theta}\right)\underline{e}_Y = \underline{0}$$

$$\Longrightarrow \quad \omega_{cylinder} = \frac{v\cos\theta}{R} \quad \& \quad \dot{\theta} = \frac{v\sin\theta}{AB}$$



$$\frac{CA}{AB} = \tan\frac{\theta}{2} \quad\Longrightarrow\quad AB = R\cot\frac{\theta}{2}$$

So,
$$\underline{\omega}_{bar} = -\dot{\theta}\,\underline{e}_Z = -\frac{v\sin\theta}{AB}\,\underline{e}_Z = -\frac{v}{R}\sin\theta\tan\frac{\theta}{2}\,\underline{e}_Z = \boxed{-\frac{2v}{R}\sin^2\frac{\theta}{2}\,\underline{e}_Z}$$

Angular velocity of the bar BD

(b)
$$\underline{v}_E\big|_{cylinder} = \underline{v}_C + \underline{\omega}_{cylinder} \times \underline{CE} = \left(v + \frac{v\cos\theta}{R}R\right)\underline{e}_x = \boxed{v\left(1+\cos\theta\right)\underline{e}_x}$$

Velocity of the cylinder at the point where it contacts the ground.

Problem 3

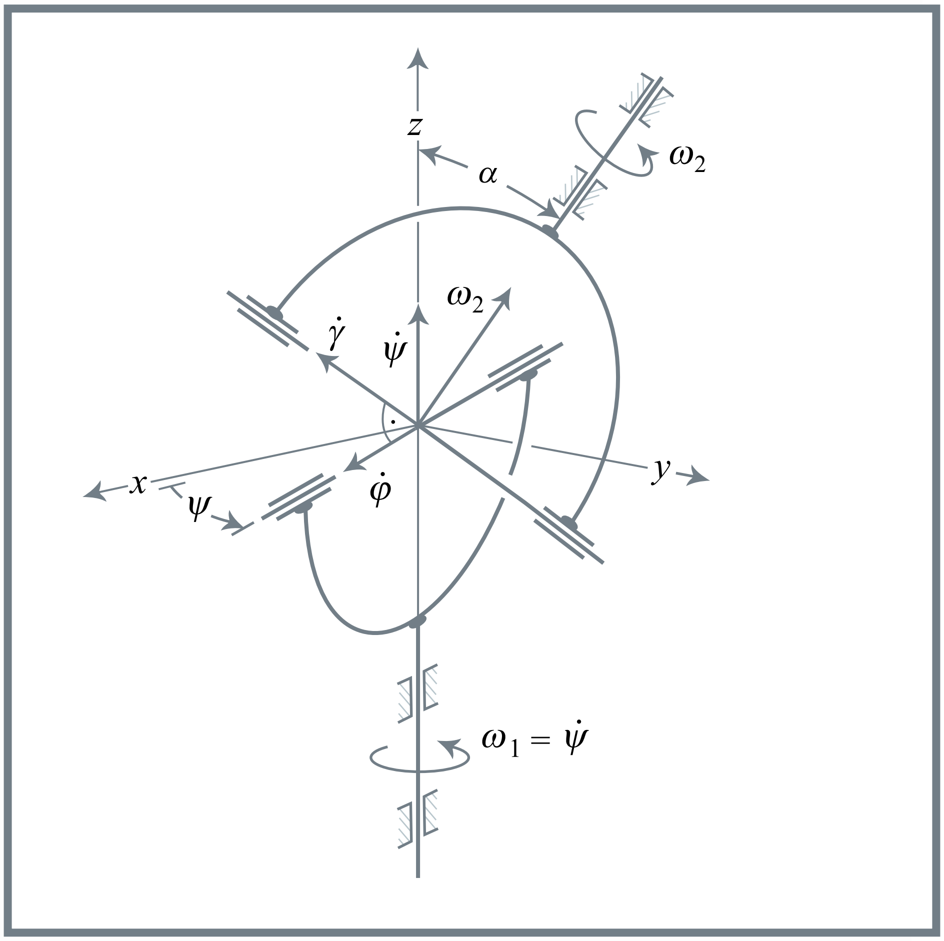


Figure by OCW.

After Problem 4-14 in Crandall, S. H., et al. *Dynamics of Mechanical and Electromechanical Systems*. Malabar, FL: Krieger, 1982.

$\underline{\omega}_2$ is alway in yz plane

and $\dot{\underline{\varphi}}$ in xy plane:

$$\underline{\omega}_2 = \omega_2 \begin{Bmatrix} 0 \\ \sin\alpha \\ \cos\alpha \end{Bmatrix}$$

$$\dot{\underline{\varphi}} = \dot\varphi \begin{Bmatrix} \cos\psi \\ \sin\psi \\ 0 \end{Bmatrix}$$

$$\underline{\omega}_1 = \omega_1 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$\dot{\underline{\gamma}}$ is perpendicular to both $\underline{\omega}_2$ and $\dot{\underline{\varphi}}$ so the direction of $\dot{\underline{\gamma}}$ can be found:

$$\underline{e}_{\dot\gamma} = \frac{\underline{e}_{\dot\varphi} \times \underline{e}_{\omega_2}}{|\underline{e}_{\dot\varphi} \times \underline{e}_{\omega_2}|} = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \cos\psi & \sin\psi & 0 \\ 0 & \sin\alpha & \cos\alpha \end{vmatrix} \frac{1}{|\underline{e}_{\dot\varphi} \times \underline{e}_{\omega_2}|}$$

$$\underline{e}_{\dot\gamma} = \begin{Bmatrix} \sin\psi\cos\alpha \\ -\cos\psi\cos\alpha \\ \cos\psi\sin\alpha \end{Bmatrix} \frac{1}{\sqrt{\cos^2\alpha + \sin^2\alpha\cos^2\psi}} \quad \Longrightarrow \quad \dot{\underline{\gamma}} = \frac{\dot\gamma}{\sqrt{\cos^2\alpha + \sin^2\alpha\cos^2\psi}} \begin{Bmatrix} \sin\psi\cos\alpha \\ -\cos\psi\cos\alpha \\ \cos\psi\sin\alpha \end{Bmatrix}$$

The angular velocity of the cross can be determined in two ways:

$$\underline{\omega}_{cross} = \underline{\omega}_1 + \dot{\underline{\varphi}}$$

$$\underline{\omega}_{cross} = \underline{\omega}_2 + \dot{\underline{\gamma}}$$

Since the cross is a rigid body, it has a unique angular velocity:

$$\therefore \quad \underline{\omega}_1 + \dot{\underline{\varphi}} = \underline{\omega}_2 + \dot{\underline{\gamma}} \qquad (*)$$

Problem 3

$$(*) \implies \begin{Bmatrix} 0 \\ 0 \\ \omega_1 \end{Bmatrix} + \begin{Bmatrix} \dot{\varphi} \cos\psi \\ \dot{\varphi} \sin\psi \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \omega_2 \sin\alpha \\ \omega_2 \cos\alpha \end{Bmatrix} + \frac{\dot{\gamma}}{\sqrt{\cos^2\alpha + \sin^2\alpha \cos^2\psi}} \begin{Bmatrix} \sin\psi \cos\alpha \\ -\cos\psi \cos\alpha \\ \cos\psi \sin\alpha \end{Bmatrix}$$

Let $$\dot{\gamma}_1 = \frac{\dot{\gamma}}{\sqrt{\cos^2\alpha + \sin^2\alpha \cos^2\psi}}$$

So we get three equations in three unknowns ω_2, $\dot{\gamma}_1$, and $\dot{\varphi}$:

$$\begin{cases} \dot{\varphi} \cos\psi = \dot{\gamma}_1 \sin\psi \cos\alpha \\ \dot{\varphi} \sin\psi = \omega_2 \sin\alpha - \dot{\gamma}_1 \cos\psi \cos\alpha \\ \omega_1 = \omega_2 \cos\alpha + \dot{\gamma}_1 \cos\psi \sin\alpha \end{cases}$$

$$\implies \quad \dot{\gamma}_1 = \omega_1 \frac{\sin\alpha \cos\psi}{\cos^2\alpha + \sin^2\alpha \cos^2\psi} \quad \longrightarrow \quad \dot{\gamma} = \omega_1 \frac{\sin\alpha \cos\psi}{\sqrt{\cos^2\alpha + \sin^2\alpha \cos^2\psi}}$$

$$\dot{\varphi} = \omega_1 \frac{\sin\alpha \cos\alpha \sin\psi}{\cos^2\alpha + \sin^2\alpha \cos^2\psi}$$

$$\omega_2 = \omega_1 \frac{\cos\alpha}{\cos^2\alpha + \sin^2\alpha \cos^2\psi}$$

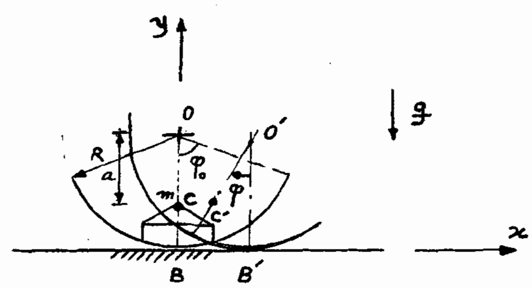
Problem 4

equation of motion :

$$\left(R^2 + a^2 - 2aR\cos\varphi\right)\ddot{\varphi} + aR\sin\varphi\,\dot{\varphi}^2 + ag\sin\varphi = 0\,,$$

$$-\varphi_0 < \varphi < \varphi_0$$

(range of validity of this governing eq.)



system is conservative. $\longrightarrow$ $T + V = \text{Const.}$

$$\underline{r}_c = (R\varphi - a\sin\varphi)\,\underline{e}_x + (R - a\cos\varphi)\,\underline{e}_y$$

$$\underline{v}_c = \dot{\underline{r}}_c = (R - a\cos\varphi)\,\dot{\varphi}\,\underline{e}_x + a\sin\varphi\,\dot{\varphi}\,\underline{e}_y$$

$$V = mg\left(R - a\cos\varphi\right)$$

$$T = \frac{1}{2} m\dot{\varphi}^2 \left\{ \left(R - a\cos\varphi\right)^2 + a^2\sin^2\varphi \right\}$$

$$T + V = \text{const.} = E_0 \quad \longrightarrow \quad \dot{\varphi} = \pm\sqrt{2\,\frac{E_0 - mg(R - a\cos\varphi)}{m\underbrace{\left(R^2 + a^2 - 2aR\cos\varphi\right)}_{>0}}}$$

a real solution for $\dot{\varphi}$ exists if $E_0 \geqslant mg\left(R - a\cos\varphi\right)$.

For $E_0^{(n)} \geqslant mg(R-a)$, if $E_0^{(n)} = mg\left(R - a\cos\varphi^{*(n)}\right)$ then one can examine the behavior of $\dot{\varphi}$ in the neighborhood of $\varphi^{*(n)}$, $\varphi^{*(n)} + \varphi$ (φ small) :

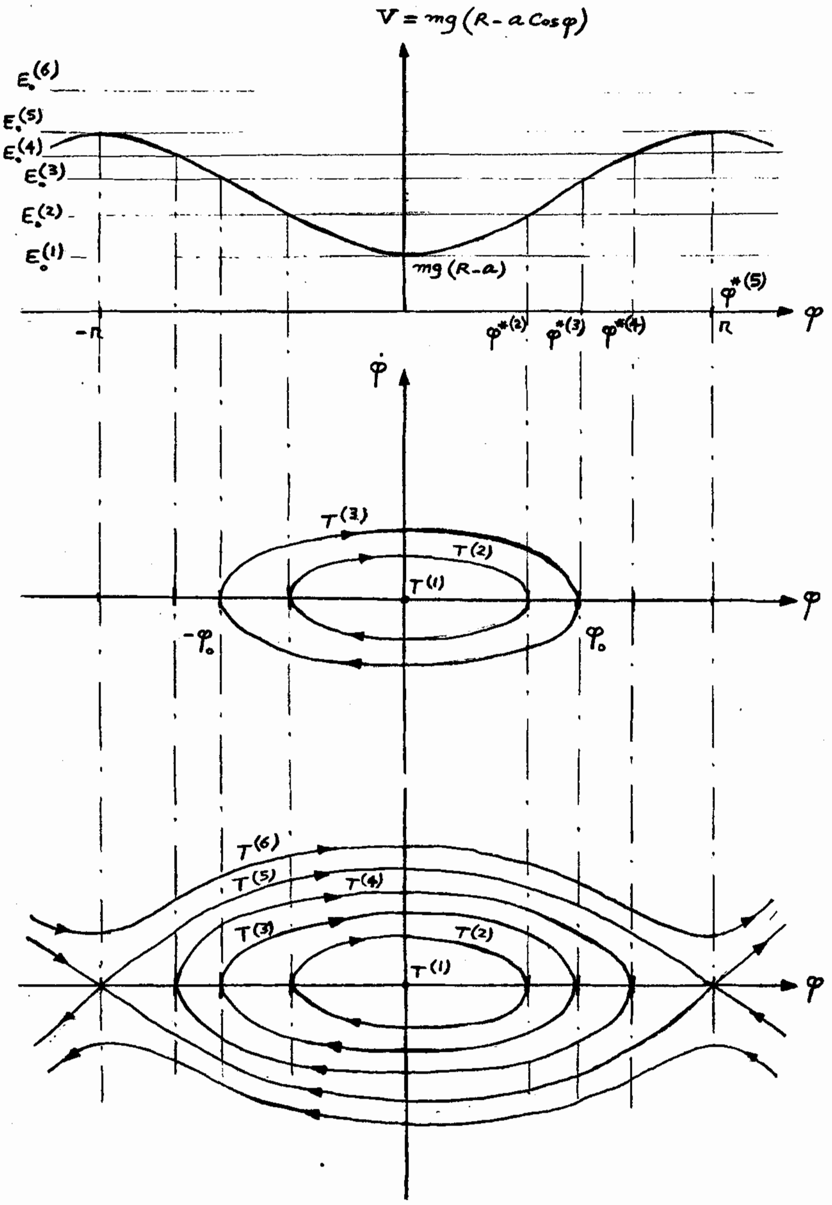
$$\dot{\varphi} = \pm\sqrt{2\,\frac{E_0^{(n)} - mg\left(R - a\cos(\varphi^{*(n)} + \varphi)\right)}{m\left(R^2 + a^2 - 2aR\cos(\varphi^{*(n)} + \varphi)\right)}} = \pm\sqrt{2\,\frac{\cancel{E_0^{(n)} - mg\left(R - a\cos\varphi^{*(n)}\right)}^{\,0} - mga\sin\varphi^{*(n)}\cdot\varphi - mg\frac{a}{2}\cos\varphi^{*(n)}\cdot\varphi^2 + \cdots}{m\left(R^2 + a^2 - 2aR\cos\varphi^{*(n)}\right) + 2maR\sin\varphi^{*(n)}\cdot\varphi + \cdots}}$$

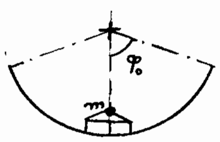
$$\therefore \quad \dot{\varphi} \sim \sqrt{|\varphi|} \qquad (\varphi \text{ close to } \varphi^{*(n)})$$

if $\sin\varphi^{*(n)} = 0$ $\left(\varphi^{*(n)} = \pm n\right)$, $\quad \dot{\varphi} \sim \pm\varphi$

Problem 4

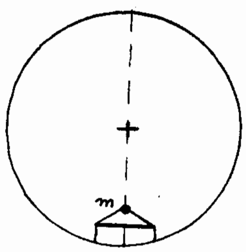
Qualitative plots of trajectories in the phase plane :





$(-\varphi_0 < \varphi < \varphi_0)$

For the case of $\varphi_0 = \pi$,



Note that $\varphi = 0$ is a stable equilibrium point and $\varphi = \pm\pi$ (in the case of $\varphi_0 = \pi$) is an unstable equilibrium point.