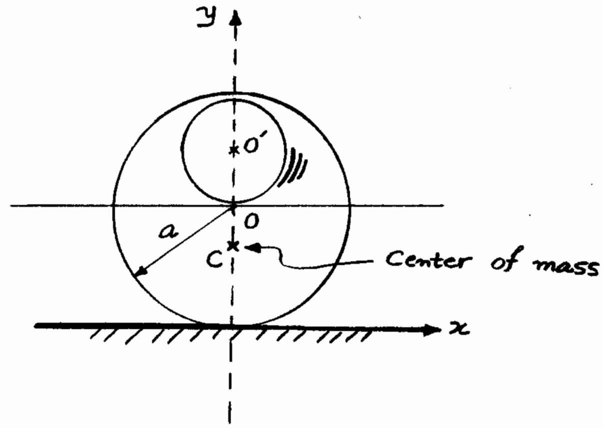
2.032 Dynamics

Fall 2004

Problem Set No. 4

Problem 1

Assume



Length of the cylinder $= h$

Density $= \rho$

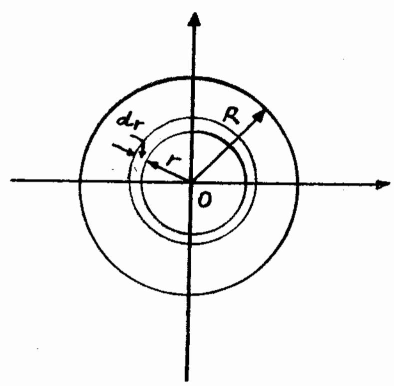
So, $M = \rho h \left[\pi a^2 - \pi\left(\frac{a}{2}\right)^2\right] = \frac{3}{4}\pi a^2 \rho h$

(i)

$x_c = 0$ because of symmetry

$$y_c = \frac{\sum m_i y_i}{\sum m_i} = \frac{\pi a^2 (a) - \pi\left(\frac{a}{2}\right)^2\left(a + \frac{a}{2}\right)}{\pi a^2 - \pi\left(\frac{a}{2}\right)^2} = \frac{5}{6}a$$

$$\Rightarrow \quad co = \frac{a}{6} \quad \& \quad co' = \frac{a}{6} + \frac{a}{2} = \frac{2}{3}a$$



$$I_0 = \sum_i m_i r_i^2 = \rho h \int_0^R 2\pi r\, dr\, (r^2) = \frac{\pi}{2} R^4 \rho h$$

Finding moment of inertia using Parallel-axes theorem:

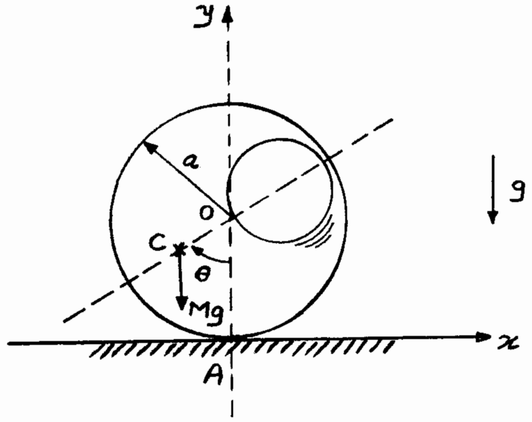
$$I_c = \rho h \left\{ \frac{\pi}{2} a^4 + (\pi a^2)\, \overset{\left(\frac{a}{6}\right)^2}{co^2} - \left[\frac{\pi}{2}\left(\frac{a}{2}\right)^4 + \pi\left(\frac{a}{2}\right)^2 \overset{\left(\frac{2a}{3}\right)^2}{co'^2}\right] \right\} = \frac{37}{96}\pi a^4 \rho h$$

Problem 1

(i)

$$V = Mg\,y_C = \frac{3}{4}\pi a^2 \rho g h \,(a - \overset{a/6}{OC}\cdot \cos\theta)$$

$$\boxed{V = \frac{3}{4}\rho g h \pi a^3 \left(1 - \frac{\cos\theta}{6}\right)}$$ Potential energy of the cylinder



$$T = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2$$

$$\underline{\omega} = -\dot\theta\, \underline{e}_z$$

$$\underline{v}_C = \cancel{\underline{v}_A}^{\,0} + \underline{\omega} \times \underline{AC} \longrightarrow v_C = \dot\theta (AC) = \dot\theta \sqrt{OC^2 + OA^2 - 2(OC)(OA)\cos\theta}$$

$$\longrightarrow v_C = \dot\theta \sqrt{\left(\frac{a}{6}\right)^2 + a^2 - 2\left(\frac{a}{6}\right)(a)\cos\theta} = a\dot\theta \sqrt{\frac{37}{36} - \frac{\cos\theta}{3}}$$

$$\therefore \quad T = \frac{1}{2}\left(\frac{3}{4}\pi a^2 \rho h\right)\left[a^2\dot\theta^2\left(\frac{37}{36} - \frac{\cos\theta}{3}\right)\right] + \frac{1}{2}\left(\frac{37}{96}\pi a^4 \rho h\right)\dot\theta^2$$

$$\boxed{T = \pi \rho h a^4 \dot\theta^2 \left(\frac{37}{64} - \frac{\cos\theta}{8}\right)}$$ Kinetic energy of the cylinder

(ii) conservative system $\longrightarrow \frac{d}{dt}(T+V) = 0$

Assume θ and $\dot\theta$ are small $\left(\sin\theta \approx \theta, \ \cos\theta \approx 1 - \frac{\theta^2}{2}\right)$:

Keeping up to quadratic term in $\theta, \dot\theta$,

$$T \approx \pi\rho h a^4 \dot\theta^2 \left(\frac{29}{64}\right) \qquad \& \qquad V \approx \frac{3}{4}\rho g h \pi a^3 \left(\frac{5}{6} + \frac{\theta^2}{12}\right)$$

$$\frac{d}{dt}(T+V) = 0 \implies \frac{29}{32} a\ddot\theta + \frac{1}{8} g\,\theta = 0 \longrightarrow \omega_n^2 = \frac{4}{29}\frac{g}{a}$$

$$\therefore \quad \text{natural frequency} \quad \boxed{\omega_n = 2\sqrt{\frac{g}{29a}}}$$

Problem 1

(iii)

$$T + V = \text{Const.} \longrightarrow \frac{3}{4}\rho g h \pi a^3\left(1-\frac{\cos\theta}{6}\right) + \pi\rho h a^4 \dot{\theta}^2\left(\frac{37}{64}-\frac{\cos\theta}{8}\right) = E_0$$

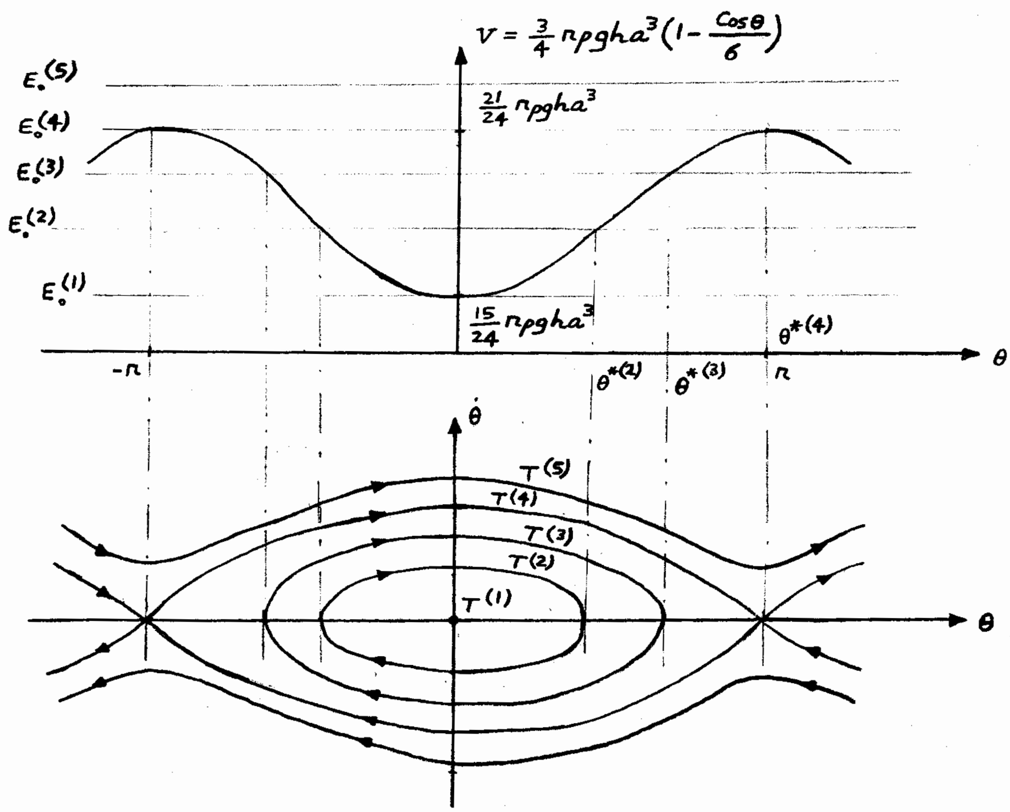
$$\longrightarrow \quad \dot{\theta} = \pm\sqrt{\frac{E_0 - \frac{3}{4}\rho g h \pi a^3\left(1-\frac{\cos\theta}{6}\right)}{\underbrace{\pi\rho h a^4\left(\frac{37}{64}-\frac{\cos\theta}{8}\right)}_{>0}}}$$

a real solution exists for $\dot{\theta}$ if $E_0 \geqslant \frac{3}{4}\pi\rho g h a^3\left(1-\frac{\cos\theta}{6}\right)$.

For $E_0^{(n)} \geqslant \frac{15}{24}\pi\rho g h a^3$, if $E_0^{(n)} = \frac{3}{4}\pi\rho g h a^3\left(1-\frac{\cos\theta^{*(n)}}{6}\right)$, the behavior of $\dot{\theta}$ in the neighborhood of $\theta^{*(n)}$, $\theta^{*(n)}+\theta$ (θ small):

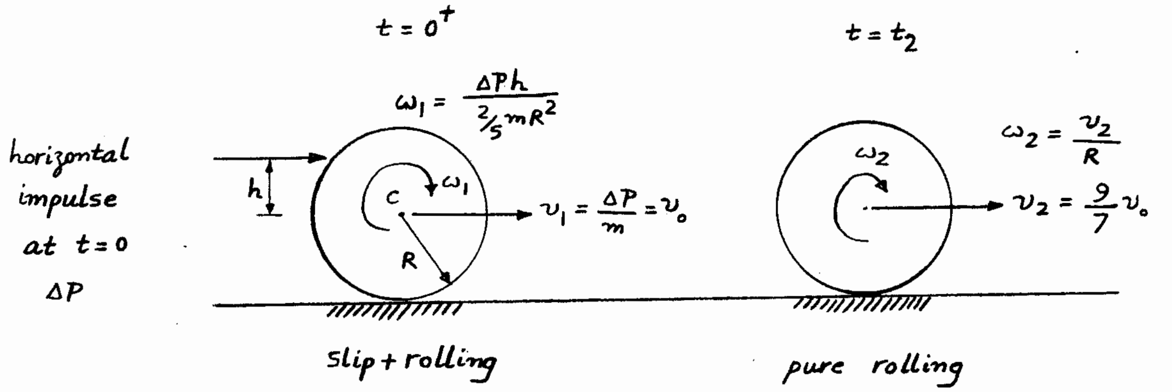
$$\begin{cases} \dot{\theta} \sim \sqrt{|\theta|} \\ \text{if } \sin\theta^{*(n)} = 0 \quad \left(\theta^{*(n)} = \pm\pi\right), \quad \dot{\theta} \sim \pm\theta \end{cases}, \qquad \theta \text{ close to } \theta^{*(n)}$$

Qualitative plots of trajectories on the $(\theta, \dot{\theta})$ phase plane:



$\theta = 0$ is a stable equilibrium point and $\theta = \pm\pi$ are unstable equilibrium points.

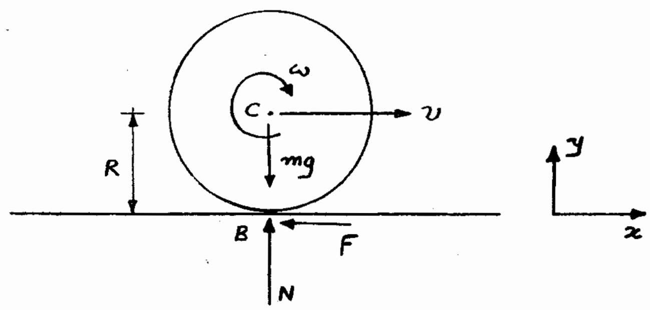
Problem 2



horizontal impulse at $t=0$ ΔP, $\quad \Delta P$ = change in linear momentum $= m v_1 = m v_0$

$\Delta P . h$ = change in angular momentum about C $= I\omega_1 = \frac{2}{5} m R^2 \omega_1$

Apply angular momentum principle about (moving) contact point B:



$$\underline{M}_B = \dot{\underline{H}}_B + \underline{v}_B \times \underline{P}$$

$$\left.\begin{array}{l} \underline{P} = m\underline{v} \\ \underline{v}_B \parallel \underline{v} \end{array}\right\} \Longrightarrow \underline{v}_B \times \underline{P} = \underline{0}$$

$$\underline{M}_B = 0$$

$\therefore \quad \dot{\underline{H}}_B = 0 \longrightarrow \underline{H}_B$ is conserved. $\longrightarrow \underline{H}_B \big|_{t=0^+} = \underline{H}_B \big|_{t=t_2}$

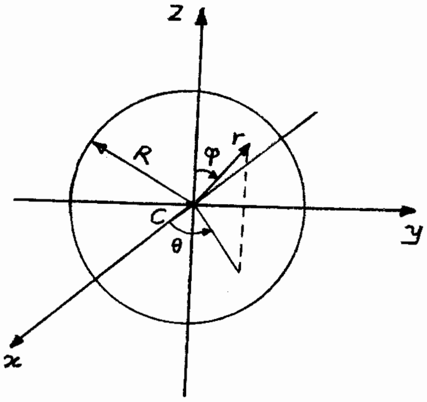
$$\underline{H}_B = \underline{H}_C + \underline{r}_{BC} \times \underline{P} = \left(\frac{2}{5} m R^2 \omega + m R v\right)(-\underline{e}_z)$$

$$\left.\begin{array}{ll} t = 0^+, & H_B \big|_{t=0^+} = \frac{2}{5} m R^2 \omega_1 + m R v_1 = \Delta P . h + m R v_0 = m v_0 (h + R) \\ t = t_2, & H_B \big|_{t=t_2} = \frac{2}{5} m R^2 \omega_2 + m R v_2 = \left(\frac{2}{5} m R + m R\right) \frac{9}{7} v_0 = \frac{9}{5} v_0 m R \end{array}\right\} \Longrightarrow h + R = \frac{9}{5} R$$

$$\therefore \quad \boxed{h = \frac{4}{5} R}$$

Problem 3

(a) a sphere of radius R,



It is convenient to use spherical coordinate system :

$$x = r\sin\varphi\cos\theta$$

$$y = r\sin\varphi\sin\theta$$

$$z = r\cos\varphi$$

$$dV = r^2\sin\varphi\, dr\, d\theta\, d\varphi$$

$$M = \frac{4}{3}\pi\rho R^3$$

Because of symmetry : $I_{xx} = I_{yy} = I_{zz}$ & $I_{xy} = I_{yz} = I_{zx} = 0$

$$I_{zz} = \int \rho\, dV (x^2+y^2) = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{R} dr\, \rho r^2 \sin\varphi\, (r^2\sin^2\varphi)$$

$$= 2\pi\rho \int_0^{\pi} d\varphi \int_0^{R} dr\, r^4 \sin^3\varphi = 2\pi\rho\frac{R^5}{5}\int_0^{\pi}\sin^3\varphi\, d\varphi$$

$$\sin^3\varphi = \frac{1}{4}(3\sin\varphi - \sin 3\varphi)$$

$$= 2\pi\rho\frac{R^5}{5}\left[-\frac{3\cos\varphi}{4} + \frac{\cos 3\varphi}{12}\right]_0^{\pi} = \frac{8\pi}{15}\rho R^5 = \frac{2}{5}MR^2$$

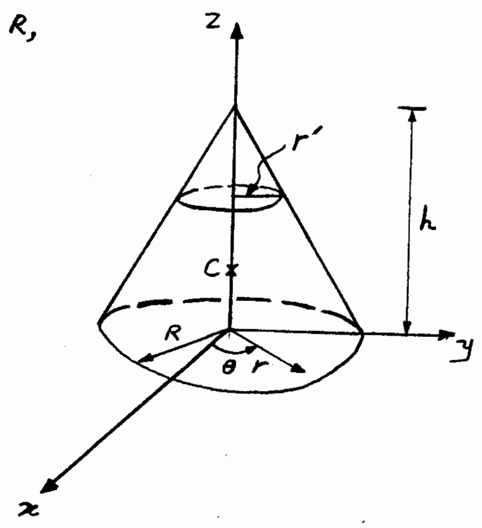
$$I_{xx} = I_{yy} = \frac{2}{5}MR^2$$

$$\therefore \quad \underline{\underline{I}}_C = MR^2\begin{bmatrix} \frac{2}{5} & 0 & 0 \\ & \frac{2}{5} & 0 \\ & & \frac{2}{5} \end{bmatrix}$$ centroidal moments of inertia

clearly, $Cxyz$ are principal axes at C and I_{xx}, I_{yy} & I_{zz} are principal centroidal moments of inertia. Due to symmetry, any axis passing through C is a principal axis with principal moment $\frac{2}{5}MR^2$.

Problem 3

(b) a circular cone of height h and base radius R,



It is convenient to use cylindrical coordinate system:

$$x = r\cos\theta$$

$$y = r\sin\theta \qquad dV = r\,dr\,d\theta\,dz$$

$$z = z$$

$$r' = \left(1-\frac{z}{h}\right)R$$

$$M = \frac{1}{3}\rho\pi R^2 h$$

Because of symmetry: $\begin{cases} x_C = y_C = 0 \\ I_{xy} = I_{yz} = I_{zx} = 0 \end{cases}$ & $I_{xx} = I_{yy}$

$$z_C = \frac{\int_0^h \rho\pi r'^2 dz\, z}{\frac{1}{3}\rho\pi R^2 h} = \frac{\rho\pi\int_0^h \left(1-\frac{z}{h}\right)^2 R^2 z\, dz}{\frac{1}{3}\rho\pi R^2 h} = \frac{h}{4}$$

$$I_{zz} = \int \rho\, dV\,(x^2+y^2) = \rho\int_0^{2\pi} d\theta \int_0^h dz \int_0^{r'} r\,dr\,(r^2)$$

$$= 2\pi\rho\int_0^h dz\int_0^{\left(1-\frac{z}{h}\right)R} r^3 dr = 2\pi\rho\frac{R^4}{4}\int_0^h dz\left(1-\frac{z}{h}\right)^4 = \frac{\pi}{10}\rho R^4 h = \frac{3}{10}MR^2$$

$$I_{xx} = \int \rho\, dV\,(y^2+z^2) = \rho\int_0^{2\pi} d\theta\int_0^h dz\int_0^{r'} r\,dr\,(r^2\sin^2\theta + z^2)$$

$$= \rho\int_0^{2\pi} d\theta\int_0^h dz\int_0^{\left(1-\frac{z}{h}\right)R}(r^3\sin^2\theta + rz^2)\,dr = \rho\int_0^{2\pi} d\theta\int_0^h dz\left[\frac{\left(1-\frac{z}{h}\right)^4 R^4}{4}\sin^2\theta + \frac{\left(1-\frac{z}{h}\right)^2 R^2 z^2}{2}\right]$$

$$= \rho\int_0^{2\pi} d\theta\left(\frac{R^4 h\sin^2\theta}{20} + \frac{h^3R^2}{60}\right) = \rho\frac{\pi R^4 h}{20} + \rho\frac{\pi h^3 R^2}{30} = \frac{3}{20}MR^2 + M\frac{h^2}{10}$$

$$I_{yy} = \frac{3}{20}MR^2 + \frac{Mh^2}{10}$$

Problem 3

Using parallel-axes theorem $\left(a = b = 0 , \quad c = z_C = \frac{h}{4} \right)$:

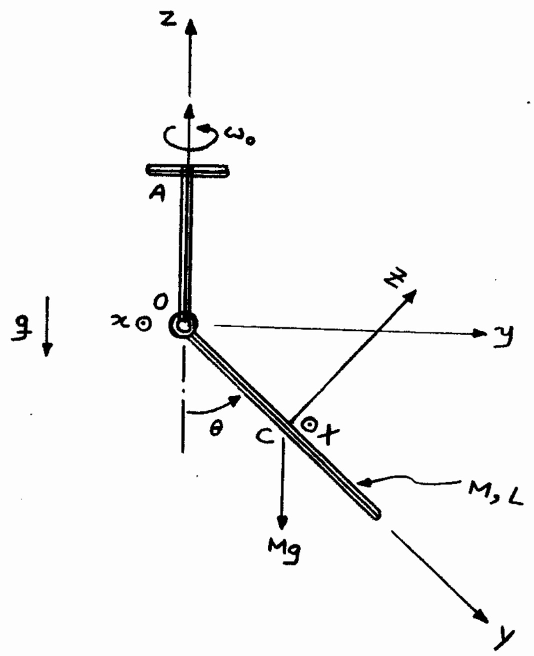
$$\underline{\underline{I}}_C = \begin{bmatrix} \frac{3}{20}MR^2 + \frac{1}{10}Mh^2 & 0 & 0 \\ & \frac{3}{20}MR^2 + \frac{1}{10}Mh^2 & 0 \\ & & \frac{3}{10}MR^2 \end{bmatrix} - M \begin{bmatrix} \left(\frac{h}{4}\right)^2 & 0 & 0 \\ & \left(\frac{h}{4}\right)^2 & 0 \\ & & 0 \end{bmatrix}$$

$$\underline{\underline{I}}_C = \begin{bmatrix} \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right) & 0 & 0 \\ & \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right) & 0 \\ & & \frac{3}{10}MR^2 \end{bmatrix}$$

principal centroidal moments of inertia

Because of symmetry, any axis passing through C and parallel to xy plane is a principal axis with principal moment $\frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right)$.

Problem 4



xyz coordinate system rotates about z with ω_0 so that the rod is always in yz plane.

XYZ coordinate system is fixed to the rod.

$$\begin{cases} \underline{e}_Z = \cos\theta\, \underline{e}_y + \sin\theta\, \underline{e}_z \\ \underline{e}_z = -\cos\theta\, \underline{e}_Y + \sin\theta\, \underline{e}_Z \end{cases} \qquad \underline{e}_X = \underline{e}_x$$

$$\underline{v}_O = \underline{0} \longrightarrow \underline{M}_O = \dot{\underline{H}}_O$$

$$\underline{H}_O = \underline{H}_C + \underline{P} \times \underline{r}_{CO} \qquad \underline{H}_C = \underline{\underline{I}}_C\, \underline{\omega}_{rod} \qquad \underline{P} = M \underline{v}_C \qquad \underline{r}_{CO} = -\frac{L}{2}\underline{e}_Y$$

$$\underline{\omega}_{rod} = \omega_0 \underline{e}_z + \dot{\theta}\underline{e}_x = \omega_0\left(-\cos\theta\, \underline{e}_Y + \sin\theta\, \underline{e}_Z\right) + \dot{\theta}\underline{e}_X$$

$$\underline{r}_C = \frac{L}{2}\sin\theta\, \underline{e}_y - \frac{L}{2}\cos\theta\, \underline{e}_z$$

$$\underline{v}_C = \frac{L}{2}\dot{\theta}\cos\theta\, \underline{e}_y + \frac{L}{2}\dot{\theta}\sin\theta\, \underline{e}_z + \left[\omega_0 \underline{e}_z \times \left(\frac{L}{2}\sin\theta\, \underline{e}_y - \frac{L}{2}\cos\theta\, \underline{e}_z\right)\right]$$

$$= -\omega_0 \frac{L}{2}\sin\theta\, \underline{e}_x + \frac{L}{2}\dot{\theta}\left(\cos\theta\, \underline{e}_y + \sin\theta\, \underline{e}_z\right) = -\omega_0\frac{L}{2}\sin\theta\, \underline{e}_X + \frac{L}{2}\dot{\theta}\, \underline{e}_Z$$

To find $\underline{\underline{I}}_C$ for the rod,

$$I_{XX} = \int (Y^2 + Z^2)\, dm = \int_{-\frac{L}{2}}^{L/2} (Y^2)\frac{M}{L}\, dY = M\frac{L^2}{12}$$

$$I_{YY} = \int (X^2 + Z^2)\, dm \simeq 0 \qquad (X \simeq 0,\quad Z \simeq 0)$$

$$I_{ZZ} = \int (Y^2 + X^2)\, dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} (Y^2)\frac{M}{L}\, dY = M\frac{L^2}{12}$$

$$I_{XY} = I_{YZ} = I_{ZX} = 0$$

$$\therefore \quad \underline{\underline{I}}_C = \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix} \frac{ML^2}{12}$$

Problem 4

$$\underline{H}_c = \underline{\underline{I}}_c \, \underline{\omega}_{rod} = \frac{ML^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix}\begin{Bmatrix} \dot\theta \\ -\omega_0 \cos\theta \\ \omega_0 \sin\theta \end{Bmatrix} = \frac{ML^2}{12}\dot\theta \, \underline{e}_X + \frac{ML^2}{12}\omega_0 \sin\theta \, \underline{e}_Z$$

$$\underline{H}_0 = \underline{H}_c + \underline{P} \times \underline{r}_{co} = \frac{ML^2}{12}\left(\dot\theta \underline{e}_X + \omega_0 \sin\theta \, \underline{e}_Z\right) + M\frac{L}{2}\left(-\omega_0 \sin\theta \, \underline{e}_X + \dot\theta \underline{e}_Z\right) \times \left(-\frac{L}{2}\underline{e}_y\right)$$

$$= \frac{ML^2}{3}\left(\dot\theta \underline{e}_X + \omega_0 \sin\theta \, \underline{e}_Z\right)$$

$$= \frac{ML^2}{3}\left(\dot\theta \underline{e}_x + \omega_0 \sin\theta \cos\theta \, \underline{e}_y + \omega_0 \sin^2\theta \, \underline{e}_z\right)$$

$$\dot{\underline{H}}_0 = \frac{ML^2}{3}\left(\ddot\theta \underline{e}_x + \omega_0 \dot\theta \cos 2\theta \, \underline{e}_y + \omega_0 \dot\theta \sin 2\theta \, \underline{e}_Z\right) + \omega_0 \underline{e}_Z \times \underline{H}_0$$

$$= \frac{ML^2}{3}\left(\ddot\theta - \omega_0^2 \sin\theta \cos\theta\right)\underline{e}_x + \frac{ML^2}{3}\left(\omega_0 \dot\theta \cos 2\theta + \omega_0 \dot\theta\right)\underline{e}_y + \frac{ML^2}{3}\omega_0 \dot\theta \sin 2\theta \, \underline{e}_z$$

$$\underline{M}_0 = -Mg\frac{L}{2}\sin\theta \, \underline{e}_x + M_y \underline{e}_y + M_z \underline{e}_z$$

$$\underline{M}_0 = \dot{\underline{H}}_0 \longrightarrow \quad -Mg\frac{L}{2}\sin\theta = M\frac{L^2}{3}\left(\ddot\theta - \omega_0^2 \sin\theta \cos\theta\right)$$

$$\Longrightarrow \quad \boxed{L\ddot\theta + \left(\frac{3g}{2} - L\omega_0^2 \cos\theta\right)\sin\theta = 0} \qquad \text{equation of motion for } \theta(t)$$

(b) For stationary angle θ, $\dot\theta = \ddot\theta = 0$

$$\ddot\theta = 0 \Longrightarrow \left(\frac{3g}{2} - L\omega_0^2 \cos\theta\right)\sin\theta = 0 \Longrightarrow \begin{cases} \theta_0 = 0 \\ \theta_0 = \cos^{-1}\left(\dfrac{3g}{2L\omega_0^2}\right), & \dfrac{3g}{2L\omega_0^2} < 1 \end{cases}$$

$$\text{Stationary angles } \theta_0 : \begin{cases} \theta_0 = 0, & \omega_0^2 < \dfrac{3g}{2L} \\ \theta_0 = \cos^{-1}\left(\dfrac{3g}{2L\omega_0^2}\right) \; \& \; \theta_0 = 0, & \omega_0^2 > \dfrac{3g}{2L} \end{cases}$$

It can be shown that $\theta_0 = 0$ is unstable when $\omega_0^2 > \dfrac{3g}{2L}$.