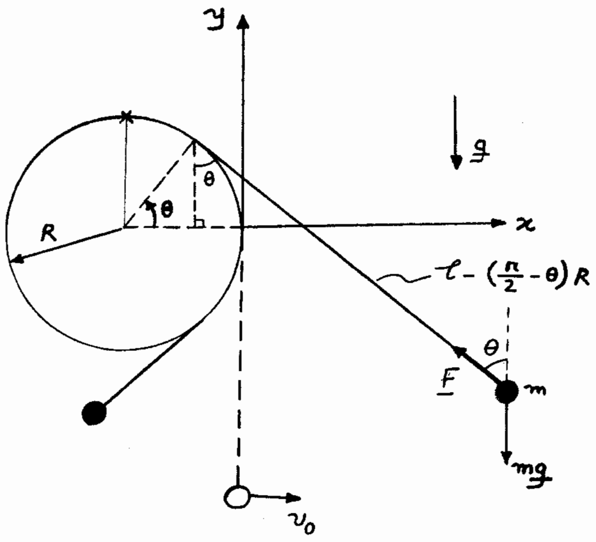
2.032 Dynamics

Fall 2004

Problem Set No. 1

Problem 1



$$\begin{cases} x_m = \left[\ell - \left(\frac{\pi}{2} - \theta\right)R\right]\sin\theta - R(1 - \cos\theta) \\ y_m = -\left[\ell - \left(\frac{\pi}{2} - \theta\right)R\right]\cos\theta + R\sin\theta \end{cases}$$

$$\begin{cases} \dot{x}_m = \left[\ell - \left(\frac{\pi}{2} - \theta\right)R\right]\cos\theta \cdot \dot{\theta} \\ \dot{y}_m = \left[\ell - \left(\frac{\pi}{2} - \theta\right)R\right]\sin\theta \cdot \dot{\theta} \end{cases}$$

$$\underline{v}_m = \dot{x}_m \underline{e}_x + \dot{y}_m \underline{e}_y$$

$$\underline{F} = |\underline{F}|\left(-\sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_y\right)$$

$$\therefore \quad \underline{F} \cdot \underline{v}_m = 0$$

$\Longrightarrow$ $\underline{F}$ does not work $\Longrightarrow$ The only force that does work is gravitational force which is conservative. $\Longrightarrow$ System is conservative

$$\Longrightarrow \quad T + V = \text{const.}$$

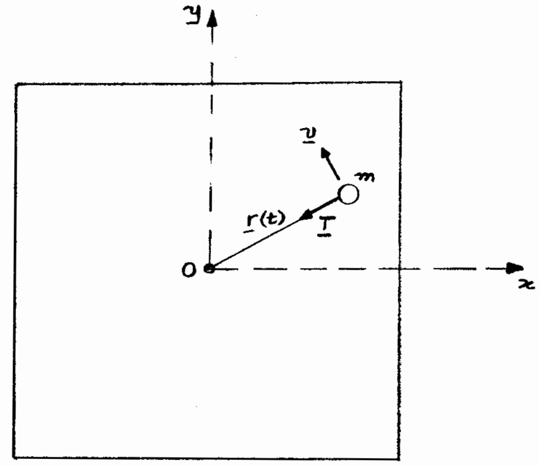
At $\theta = 0$,
$$\begin{cases} T = \frac{1}{2} m v_m^2 = \frac{1}{2} m v_0^2 \\ V = m g y_m(\theta = 0) = -mg\left(\ell - \frac{\pi}{2}R\right) \end{cases}$$

At two extreme deflections, $v_m = 0 \Longrightarrow T = 0$

$$\therefore \quad V\Big|_{\theta = \theta_{max/min}} = (T + V)\Big|_{\theta = 0} = \frac{1}{2} m v_0^2 - mg\left(\ell - \frac{\pi}{2}R\right) = m g y_m\Big|_{\theta = \theta_{max/min}}$$

$$\Longrightarrow \quad y_m\Big|_{\theta = \theta_{max/min}} = \frac{v_0^2}{2g} - \ell + \frac{\pi}{2}R = \left[R\sin\theta - \left[\ell - \left(\frac{\pi}{2} - \theta\right)R\right]\cos\theta\right]_{\theta = \theta_{max}/\theta_{min}}$$

Problem 2

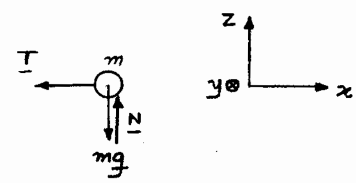


Use linear momentum in the z direction:

$$m\overset{\nearrow 0}{\underline{a}}\big|_z = N - mg = 0 \implies N = mg$$

Unknown string force $\underline{T}$ always passes through the fixed point O :

$$\underline{M}_0 = \overset{\nearrow 0}{\underline{r} \times \underline{T}} + \overset{\nearrow 0}{\underline{r} \times (\underline{N} + m\underline{g})} = 0$$



Use angular momentum principle w.r.t. O :

$$\dot{\underline{H}}_0 + \overset{\nearrow 0}{\underline{v}_0 \times \underline{P}} = \underline{M}_0 = 0 \longrightarrow \dot{\underline{H}}_0 = 0$$

$\longrightarrow$ $\underline{H}_0$ is conserved. (*)

At t_0, mass m moves along a circle and we gradually pull the string so the radial component of the velocity is negligible.

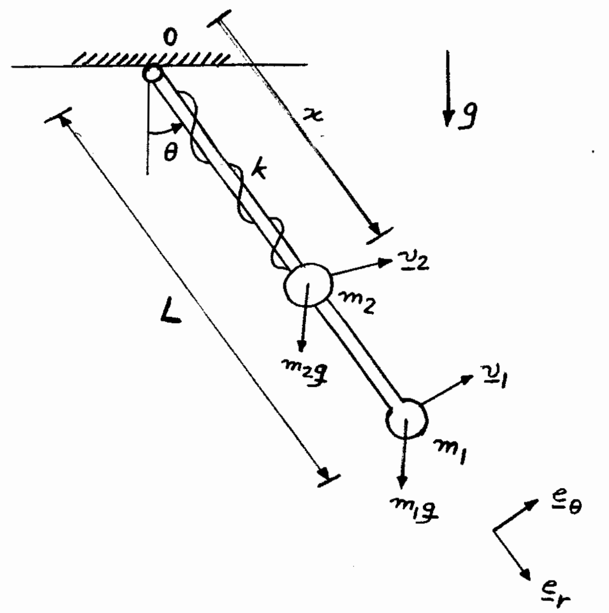
$$\underline{H}_0 = \underline{r} \times \underline{P} = \underline{r} \times (m\underline{v}) = m r v \, \underline{e}_z \qquad (**)$$

$$(*) \ \& \ (**) \implies m\, r(t_0)\, v(t_0) = m\, r(t_1)\, v(t_1)$$

$$L_0 v_0 = \frac{L_0}{2} v(t_1) \implies \boxed{v(t_1) = 2v_0}$$

Kinetic Energy increases because work is done by the pulling force.

Problem 3



$\# \text{ DOF} = 2 \times 2 - 1 - 1 = 2$

a) One needs two coordinates θ and x to describe the motion of the system.

θ and x are a complete and independent set of generalized coordinates.

b) We need to find two equations:

First, angular momentum about point O for the system:

$$\underline{M}_0 = \dot{\underline{H}}_0 + \cancel{\underline{v}_0 \times \underline{P}}^{\,0} = \dot{\underline{H}}_0$$

$$\underline{M}_0 = -(m_1 g L \sin\theta + m_2 g x \sin\theta)\, \underline{e}_z$$

$$\underline{v}_1 = L\dot\theta\, \underline{e}_\theta \qquad \& \qquad \underline{v}_2 = \dot x\, \underline{e}_r + x\dot\theta\, \underline{e}_\theta$$

$$\underline{H}_0 = \sum_{i=1}^{2} \underline{r}_i \times \underline{P}_i = (L\, \underline{e}_r)\times(m_1 L\dot\theta\, \underline{e}_\theta) + (x\, \underline{e}_r)\times[m_2(\dot x\, \underline{e}_r + x\dot\theta\, \underline{e}_\theta)]$$

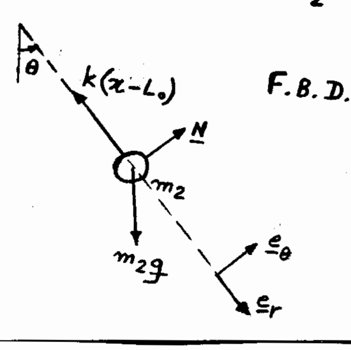
$$\Longrightarrow \underline{H}_0 = (m_1 L^2\dot\theta + m_2 x^2 \dot\theta)\, \underline{e}_z$$

$$\therefore \quad -(m_1 g L \sin\theta + m_2 g x \sin\theta) = (m_1 L^2 + m_2 x^2)\ddot\theta + 2 m_2 x \dot x \dot\theta$$

$$\Longrightarrow \quad \boxed{(m_1 L^2 + m_2 x^2)\ddot\theta + 2 m_2 x \dot x \dot\theta + (m_1 L + m_2 x) g \sin\theta = 0}$$

To find the second equation, apply linear momentum in the radial direction for m_2:

m_2 is free to slide along the rod so the force $\underline{N}$ has no component along the radial direction.



Problem 3

$$\underline{v}_2 = \dot{x}\,\underline{e}_r + x\dot{\theta}\,\underline{e}_\theta$$

$$\underline{a}_2 = \frac{d\underline{v}_2}{dt} = \ddot{x}\,\underline{e}_r + \dot{x}\underbrace{\frac{d\underline{e}_r}{dt}}_{\dot{\theta}\,\underline{e}_\theta} + (x\ddot{\theta} + \dot{x}\dot{\theta})\,\underline{e}_\theta + x\dot{\theta}\underbrace{\frac{d\underline{e}_\theta}{dt}}_{-\dot{\theta}\,\underline{e}_r}$$

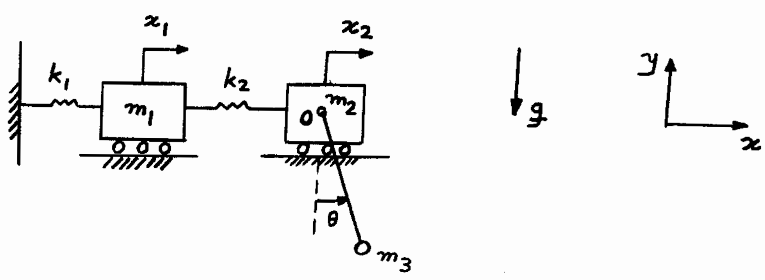
so r-component of $\underline{a}_2$, $\underline{a}_2\big|_r = \ddot{x} - x\dot{\theta}^2$

$$\therefore \quad m_2 g\cos\theta - k(x - L_0) = m_2\ddot{x} - m_2 x\dot{\theta}^2$$

$$\Longrightarrow \quad \boxed{m_2\ddot{x} - m_2 x\dot{\theta}^2 - m_2 g\cos\theta + k(x - L_0) = 0}$$

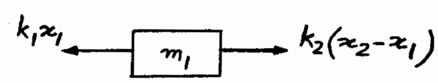
Problem 4

pendulum length = L



Linear momentum in x direction for m_1 :

F.B.D.

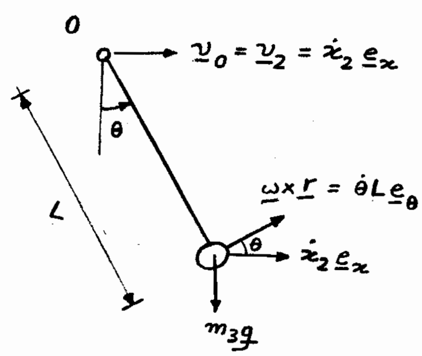


$$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1$$

$$\Longrightarrow \quad \boxed{m_1\ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2}$$

Angular momentum about point O :

F.B.D.



$$\underline{M}_0 = \dot{\underline{H}}_0 + \underline{v}_0 \times \underline{P}$$

$$\underline{M}_0 = -m_3 g L \sin\theta \; \underline{e}_z$$

$$\underline{v}_0 = \underline{v}_2 = \dot{x}_2 \underline{e}_x$$

$$\underline{v}_3 = (\dot{x}_2 + L\dot{\theta}\cos\theta)\,\underline{e}_x + (L\dot{\theta}\sin\theta)\,\underline{e}_y$$

$$\underline{H}_0 = \underline{r}\times\underline{P} = \underline{r}\times m_3\underline{v}_3 = m_3\left(L\sin\theta\,\underline{e}_x - L\cos\theta\,\underline{e}_y\right)\times\left[(\dot{x}_2 + L\dot{\theta}\cos\theta)\,\underline{e}_x + (L\dot{\theta}\sin\theta)\,\underline{e}_y\right]$$

$$= m_3\left(L^2\dot{\theta} + \dot{x}_2 L\cos\theta\right)\underline{e}_z$$

$$\underline{v}_0\times\underline{P} = \underline{v}_0\times m_3\underline{v}_3 = m_3 L\dot{x}_2\dot{\theta}\sin\theta \;\underline{e}_z$$

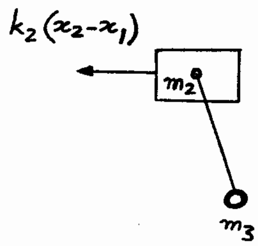
$$\therefore \quad -m_3 g L\sin\theta = m_3\left(L^2\ddot{\theta} + \ddot{x}_2 L\cos\theta \cancel{- \dot{x}_2 L\sin\theta\,\dot{\theta}}\right) + \cancel{m_3 L\dot{x}_2\dot{\theta}\sin\theta}$$

$$\Longrightarrow \quad \boxed{L\ddot{\theta} + \ddot{x}_2\cos\theta + g\sin\theta = 0}$$

Problem 4

Linear momentum in x direction for m_2 & m_3 : $\left(\underline{F}^{ext} = m_2 \underline{a}_2 + m_3 \underline{a}_3\right)$

F.B.D.



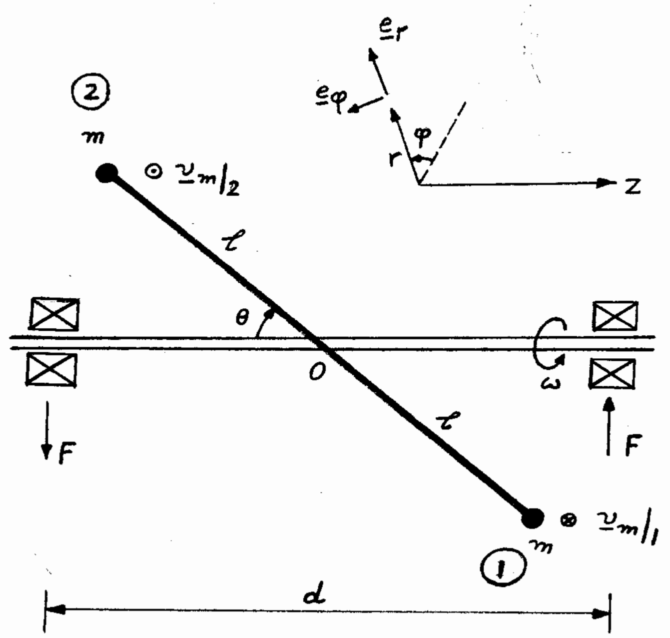
$$\underline{a}_2 = \ddot{x}_2 \underline{e}_x$$

$$\underline{a}_3\big|_x = \ddot{x}_2 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta$$

$$\therefore \quad -k_2(x_2-x_1) = m_2\ddot{x}_2 + m_3\left(\ddot{x}_2 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right)$$

$$\Rightarrow \quad (m_2+m_3)\ddot{x}_2 + m_3 L\ddot{\theta}\cos\theta - m_3 L\dot{\theta}^2\sin\theta + k_2(x_2-x_1) = 0$$

Problem 5



$$\underline{v}_m|_1 = \underline{v}_m|_2 = \underline{\omega} \times \underline{r}$$

$$= \omega \ell \sin\theta \ \underline{e}_\varphi$$

Apply angular momentum about point 0: $(\underline{v}_0 = 0)$

$$\underline{H}_0 = \underline{r} \times \underline{P} = (\ell \cos\theta \ \underline{e}_z + \ell \sin\theta \ \underline{e}_r) \times m\omega\ell \sin\theta \ \underline{e}_\varphi$$

$$+ (-\ell \cos\theta \ \underline{e}_z + \ell \sin\theta \ \underline{e}_r) \times m\omega\ell \sin\theta \ \underline{e}_\varphi$$

$$= -m\omega\ell^2 \sin\theta \cdot \cos\theta \ \underline{e}_r + m\omega\ell^2 \sin^2\theta \ \underline{e}_z$$

$$+ m\omega\ell^2 \sin\theta \cos\theta \ \underline{e}_r + m\omega\ell^2 \sin^2\theta \ \underline{e}_z$$

$$\dot{\underline{H}}_0 = -m\omega^2\ell^2 \sin\theta \cos\theta \ \underline{e}_\varphi|_1 + m\omega^2\ell^2 \sin\theta \cos\theta \ \underline{e}_\varphi|_2$$

(Note that $\dfrac{d\underline{e}_r}{dt} = \omega \underline{e}_\varphi$; $\underline{e}_\varphi|_1$ and $\underline{e}_\varphi|_2$ are in opposite directions)

$$M_0 = F d$$

$$\underline{M}_0 = \dot{\underline{H}}_0 \implies F d = 2m\omega^2\ell^2 \sin\theta \cos\theta = m\omega^2\ell^2 \sin 2\theta$$

$$\implies F = \frac{m\omega^2\ell^2}{d} \sin 2\theta \implies F_{max} = \frac{m\omega^2\ell^2}{d} \text{ for the case of } \theta = 45^\circ$$

Note that force F changes direction but is always in the plane of the shaft and the rods.