MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

# 2.051 Introduction to Heat Transfer

## Equation Sheet (Fall 2015)

### STEADY HEAT TRANSFER:

| Mode of Heat Transfer | Equation | |
|---|---|---|
| Conduction | $\vec{q} = -k\vec{\nabla}T$ | Fourier's Law |
| Convection | $\dot{Q} = hA(T_s - T_\infty)$ | Newton's law of cooling |
| Radiation | $\dot{Q}_{12} = \varepsilon_1 \sigma A(T_1^4 - T_2^4)$ where $\sigma = 5.67\times10^{-8}$ W/(m$^2$K$^4$), $\varepsilon_1$ = emissivity of object (1) | Radiation heat transfer from a small grey object (1) to a large isothermal environment (2) |

### THERMAL RESISTANCE:

| Mode of Heat Transfer | | Resistance [K/W] |
|---|---|---|
| Conduction | Slab | $R_{cond} = L/(kA)$ |
| | Cylindrical (radial direction) | $R_{cond} = \dfrac{\ln(r_{out}/r_{in})}{2\pi kL}$ |
| | Spherical (radial direction) | $R_{cond} = \left(\frac{1}{r_{in}} - \frac{1}{r_{out}}\right)\Big/(4\pi k)$ |
| Convection | | $R_{conv} = 1/(hA)$ |
| Radiation | | $R_{rad} = 1/(h_r A)$ where $h_r = 4\varepsilon_1 \sigma T_{avg}{}^3$ when $T_1 \approx T_2$ |

Steady state heat equation with energy generation:

Planar coordinate: $\dfrac{\partial}{\partial x}\left(k\dfrac{\partial T}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right) + \dot{q}_{gen} = 0$

Cylindrical coordinate: $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(kr\dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial \phi}\left(k\dfrac{\partial T}{\partial \phi}\right) + \dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right) + \dot{q}_{gen} = 0$

### FIN EQUATIONS

Fin Equation with convection alone.

$\dfrac{d^2\theta}{dx^2} - m^2\theta(x) = 0$ General solution: $\theta(x) = Ae^{-mx} + Be^{mx}$

| Tip condition (x=L) | Temperature distribution $\theta/\theta_b$ | Fin heat transfer rate $\dot{Q}_f$ | Fin Resistance $R_{fin}$ |
|---|---|---|---|
| Convection heat transfer: $h\theta(L) = -k\left.\dfrac{d\theta}{dx}\right\|_{x=L}$ | $\dfrac{\cosh[m(L-x)] + \frac{h}{mk}\sinh[m(L-x)]}{\cosh[mL] + \frac{h}{mk}\sinh[mL]}$ | $M\dfrac{\sinh[mL] + \frac{h}{mk}\cosh[mL]}{\cosh[mL] + \frac{h}{mk}\sinh[mL]}$ | $\dfrac{1}{\sqrt{hPkA_c}}\dfrac{\cosh[mL] + \frac{h}{mk}\sinh[mL]}{\sinh[mL] + \frac{h}{mk}\cosh[mL]}$ |
| Adiabatic: $\left.\dfrac{d\theta}{dx}\right\|_{x=L} = 0$ | $\dfrac{\cosh[m(L-x)]}{\cosh[mL]}$ | $M\tanh[mL]$ | $\dfrac{1}{\sqrt{hPkA_c}\tanh[mL]}$ |
| Prescribed temperature: $\theta(L) = \theta_L$ | $\dfrac{\frac{\theta_L}{\theta_b}\sinh[mx] + \sinh[m(L-x)]}{\sinh[mL]}$ | $M\dfrac{\left(\cosh[mL] - \frac{\theta_L}{\theta_b}\right)}{\sinh[mL]}$ | $\dfrac{1}{\sqrt{hPkA_c}}\dfrac{\sinh[mL]}{\left(\cosh[mL] - \frac{\theta_L}{\theta_b}\right)}$ |
| Infinite fin $(L\to\infty)$: $\theta(L) = 0$ | $e^{-mx}$ | $M$ | $\dfrac{1}{\sqrt{hPkA_c}}$ |

$\theta \equiv T - T_\infty$; $\theta_b = \theta(0) = T_b - T_\infty$; $m^2 = hP/kA_c$; $M = \theta_b\sqrt{hPkA_c}$

**HYPERBOLIC OPERATORS:**

| Function: f(x) | Definition | Derivative: $\frac{d}{dx}[f(x)]$ |
|---|---|---|
| $\sinh(x)$ | $\left(e^x - e^{-x}\right)/2$ | $\cosh(x)$ |
| $\cosh(x)$ | $\left(e^x + e^{-x}\right)/2$ | $\sinh(x)$ |
| $\tanh(x)$ | $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ | $1 - \tanh^2(x)$ |

**UNSTEADY HEAT TRANSFER**

**Heat diffusion equation:** $\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}_{gen}}{\rho c}$

where $\alpha = k/(\rho c)$ and $\dot{q}_{gen}$ is the heat generation rate per unit volume [W/m$^3$]

*Biot number:* $\mathrm{Bi} = \frac{\Delta T_{\text{internal}}}{\Delta T_{\text{external}}} = \frac{R_{\text{internal}}}{R_{\text{external}}} = \frac{hL_c}{k}$ where, possibly, $L_c = V/A_s$

*Fourier number:* $Fo = \frac{\alpha t}{L^2}$

**Lumped Parameter Model (for Bi<<1):**

General lumped equation: $\rho c V \frac{\partial T}{\partial t} = -hA\left(T - T_\infty\right) + \dot{Q}_{gen}$

Solution: $T = T_\infty + \frac{\dot{Q}_{gen}}{hA} + \left[T_i - T_\infty - \frac{\dot{Q}_{gen}}{hA}\right] e^{-t/\tau}$ where $\tau = \frac{\rho V c}{hA}$

**Semi-Infinite Solid**:

| Boundary Condition | Temperature distribution | Heat Flux at Surface |
|---|---|---|
| Constant Surface Temperature: $T(x=0,t) = T_s$ | $\frac{T(x,t) - T_i}{T_s - T_i} = \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)$ | $q\big\vert_{x=0} = \frac{k\left(T_s - T_i\right)}{\sqrt{\pi \alpha t}}$ |
| Constant Surface Heat Flux: $q(x=0) = q_0$ | $T(x,t) - T_i = \frac{q_0}{k}\left(\frac{4\alpha t}{\pi}\right)^{1/2} e^{-x^2/(4\alpha t)} - x \cdot \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)$ | |
| Surface Convection: $-k\frac{\partial T}{\partial x}\Big\vert_{x=0} = h\left(T_\infty - T(x=0,t)\right)$ | $\frac{T(x,t) - T_i}{T_\infty - T_i} = \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$ | |

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