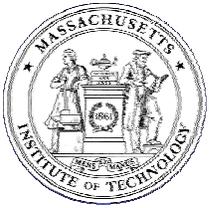


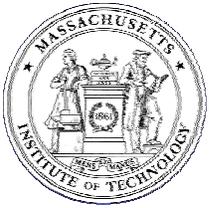
Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation
- Broadband Modeling



Broadband Modeling

- Fourier Synthesis
- Time-domain Methods
- Numerical Examples
- Doppler Shift in Ocean Waveguides
 - Numerical Examples



Reference: Kuperman, W. A. and Henrick Schmidt. "Spectral and modal representations of the Doppler-shifted field in ocean waveguides." *The Journal of the Acoustical Society of America* 96, no.1 (July 1994): 386-395.

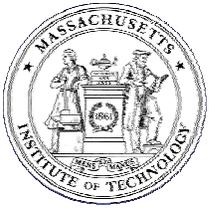
Doppler Shift in a Waveguide

Wave Equation - Moving Source

$$\nabla^2 \psi(\mathbf{r}, z, t) - \frac{1}{c^2} \frac{\partial^2 \psi(\mathbf{r}, z, t)}{\partial t^2} = -\delta(\mathbf{r} - \mathbf{v}_s t) \delta(z - z_s) e^{-i\Omega t} .$$

Inhomogeneous Helmholtz Equation

$$[\nabla^2 + k_\omega^2] \psi(\mathbf{r}, z, \omega) = -\delta(z - z_s) \int \delta(\mathbf{r} - \mathbf{v}_s t) e^{i(\omega - \Omega)t} dt ,$$



Wavenumber Integral Representation

2D Fourier Transform

$$\psi(\mathbf{r}, z; \omega) = \int \psi(\mathbf{k}_r, z; \omega) e^{i\mathbf{k}_r \cdot \mathbf{r}} d^2\mathbf{k}_r,$$

$$\psi(\mathbf{k}_r, z; \omega) = \frac{1}{(2\pi)^2} \int \psi(\mathbf{r}, z; \omega) e^{-i\mathbf{k}_r \cdot \mathbf{r}} d^2\mathbf{r},$$

Depth-separated Wave Equation

$$\begin{aligned} \frac{d^2\psi(\mathbf{k}_r, z; \omega)}{dz} + [k_\omega^2 - k_r^2] \psi(\mathbf{k}_r, z; \omega) &= -\frac{\delta(z - z_s)}{(2\pi)^2} \int e^{i(\omega - \Omega - \mathbf{k}_r \cdot \mathbf{v}_s)t} dt \\ &= -\frac{\delta(z - z_s)}{2\pi} \delta(\omega - \Omega - \mathbf{k}_r \cdot \mathbf{v}_s), \end{aligned}$$

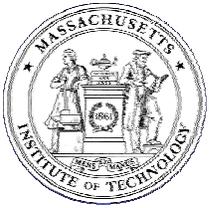
Integral Identities

$$\int \delta(\mathbf{r} - \mathbf{v}_s t) e^{-i\mathbf{k}_r \cdot \mathbf{r}} d^2\mathbf{r} = e^{-\mathbf{k}_r \cdot \mathbf{v}_s t},$$

$$\frac{1}{2\pi} \int e^{i(\omega - \Omega - \mathbf{k}_r \cdot \mathbf{v}_s)t} dt = \delta(\omega - \Omega - \mathbf{k}_r \cdot \mathbf{v}_s).$$

Frequency-Wavenumber Solution

$$\psi(\mathbf{k}_r, z; \omega) = \delta(\omega - \Omega - \mathbf{k}_r \cdot \mathbf{v}_s) g(k_r, z; \omega),$$



Time-domain Solution

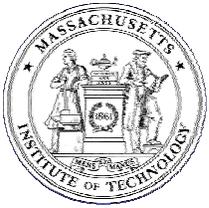
$$\psi(\mathbf{r}, z, t) = \frac{1}{2\pi} \int e^{-i\omega t} d\omega \int \psi(\mathbf{k}_r, z, \omega) e^{i\mathbf{k}_r \cdot \mathbf{r}} d^2\mathbf{k}_r,$$

$$\psi(\mathbf{r}, z, t) = \frac{1}{2\pi} \int g(k_r, z; \Omega + \mathbf{k}_r \cdot \mathbf{v}_s) e^{-i[(\Omega + \mathbf{k}_r \cdot \mathbf{v}_s)t - \mathbf{k}_r \cdot \mathbf{r}]} d^2\mathbf{k}_r.$$

Doppler Frequency Shift

Time-wavenumber
coupling

$$\omega = \Omega + \mathbf{k}_r \cdot \mathbf{v}_s.$$



Moving Receiver

$$\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, t) = \frac{1}{2\pi} \int g(k_r, z; \Omega + \mathbf{k}_r \cdot \mathbf{v}_s) e^{-i[\Omega + \mathbf{k}_r \cdot (\mathbf{v}_s - \mathbf{v}_r)]t - \mathbf{k}_r \cdot \mathbf{r}_0} d^2 \mathbf{k}_r .$$

Broadband Source

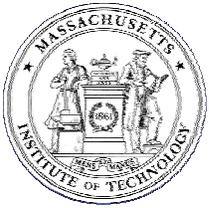
$$\begin{aligned} \psi(\mathbf{r}_0 + \mathbf{v}_r t, z, t) &= \frac{1}{4\pi^2} \int d\Omega S(\Omega) \int d^2 \mathbf{k}_r \\ &\times G(k_r, z; \Omega + \mathbf{k}_r \cdot \mathbf{v}_s) e^{-i[(\Omega + \mathbf{k}_r \cdot (\mathbf{v}_s - \mathbf{v}_r))t - \mathbf{k}_r \cdot \mathbf{r}_0]} . \end{aligned}$$

Frequency-domain solution

$$\begin{aligned} \psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) &= \int dt e^{i\omega t} \psi(\mathbf{r}_0 + \mathbf{v}_r t, z, t) \\ &= \frac{1}{4\pi^2} \int d\Omega S(\Omega) \int d^2 \mathbf{k}_r e^{i\mathbf{k}_r \cdot \mathbf{r}_0} G(k_r, z; \Omega + \mathbf{k}_r \cdot \mathbf{v}_s) \\ &\quad \times \int dt e^{-i(\Omega - \omega + \mathbf{k}_r \cdot (\mathbf{v}_s - \mathbf{v}_r))t} \\ &= \frac{1}{2\pi} \int d^2 \mathbf{k}_r e^{i\mathbf{k}_r \cdot \mathbf{r}_0} \int d\Omega S(\Omega) \\ &\quad \times G(k_r, z; \Omega + \mathbf{k}_r \cdot \mathbf{v}_s) \delta(\Omega - \omega + \mathbf{k}_r \cdot (\mathbf{v}_s - \mathbf{v}_r)) \\ &= \frac{1}{2\pi} \int d^2 \mathbf{k}_r e^{i\mathbf{k}_r \cdot \mathbf{r}_0} S(\Omega_k) G(k_r, z; \omega + \mathbf{k}_r \cdot \mathbf{v}_r) , \end{aligned}$$

Doppler-shifted Source Frequency

$$\Omega_k = \omega - \mathbf{k}_r \cdot (\mathbf{v}_s - \mathbf{v}_r) .$$



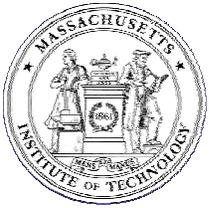
Far-field Approximation

$$\begin{aligned} \psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) & \\ & \simeq \int_0^\infty dk_r k_r J_0(k_r r_0) \\ & \quad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos \theta_r) \\ & = \frac{1}{2} \int_{-\infty}^\infty dk_r k_r H_0^{(1)}(k_r r_0) \\ & \quad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos \theta_r), \end{aligned}$$

$$\Omega_k = \omega - k_r (v_s \cos \theta_s - v_r \cos \theta_r).$$

Temporal doppler

Spatial doppler



Normal Mode Representation

Depth-dependent Green's Function

$$G(k_r, z; \omega) \simeq \frac{1}{2\pi\rho(z_s)} \sum \frac{\Psi_n(z)\Psi_n(z_s)}{k_r^2 - k_n^2},$$

Doppler Shifted Modal Wavenumbers

$$k_n^* \simeq k_n \left(1 + v_r \cos \theta_r \frac{dk_n}{d\omega} \right) = k_n \left(1 + \frac{v_r}{v_{ng}} \cos \theta_r \right),$$

Modal Field

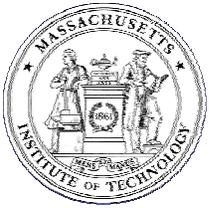
$$\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) \simeq \frac{i}{4\rho(z_s)} \sum_n S(\Omega_n) \times \Psi_n(z)\Psi_n(z_s) H_0^{(1)} \left(k_n r_0 \left(1 + \frac{v_r}{v_{ng}} \cos \theta_r \right) \right),$$

Temporal doppler

Spatial doppler

$$\Omega_n = \omega - k_n(v_s \cos \theta_s - v_r \cos \theta_r)$$

$$= \omega \left(1 - \frac{v_s}{v_{np}} \cos \theta_s + \frac{v_r}{v_{np}} \cos \theta_r \right),$$



Adiabatic Approximation

$$\psi(r, z, \omega) \simeq$$

$$\frac{iS(\omega)e^{-i\pi/4}}{\rho(z_s)\sqrt{8\pi}} \sum_n \Psi_n(z) \Psi_n(z_s) \frac{e^{i \int_0^r k_n(r') dr'}}{\sqrt{\int_0^r k_n(r') dr'}} .$$

$$\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) \simeq$$

$$\frac{ie^{-i\pi/4}}{\rho(z_s)\sqrt{8\pi}} \sum_n S(\Omega_n^*) \Psi_n(z) \Psi_n(z_s) \frac{e^{i \int_0^{r_n^*} k_n(r') dr'}}{\sqrt{\int_0^{r_n^*} k_n(r') dr'}} .$$

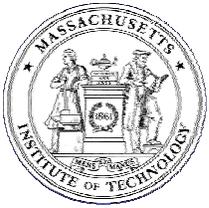
Temporal doppler

$$\Omega_n^* = \omega \left(1 - \frac{v_s}{v_{np}(0)} \cos \theta_s + \frac{v_r}{v_{np}(r_0)} \cos \theta_r \right) ,$$

Doppler-perturbed Ranges

Spatial doppler

$$\longrightarrow r_n^* = r_0 \left(1 + \frac{v_r}{v_{ng}(r_0)} \cos \theta_r \right) .$$



Shallow Water Waveguide Moving Source and Receiver

Far-field Approximation

$$\begin{aligned}\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) &\simeq \int_0^\infty dk_r k_r J_0(k_r r_0) \\ &\quad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos \theta_r) \\ &= \frac{1}{2} \int_{-\infty}^\infty dk_r k_r H_0^{(1)}(k_r r_0) \\ &\quad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos \theta_r),\end{aligned}$$

$$\Omega_k = \omega - k_r \underbrace{(v_s \cos \theta_s - v_r \cos \theta_r)}_{= 0}.$$