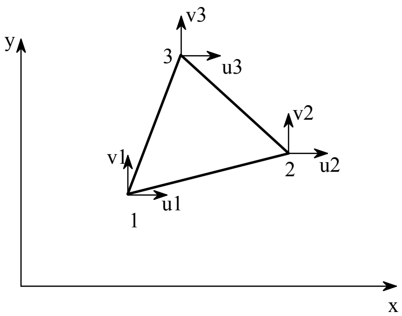
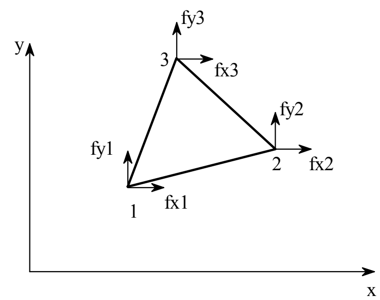
Constant Strain Triangle (CST) Element Hughes 7.3

ORIGIN = 1



$$\delta\_nodes := \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$



$$f := \begin{pmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{pmatrix}$$

nodal displacements

nodal forces

Step I. select suitable displacement function

$$u := C_1 + C_2 \cdot x + C_3 \cdot y \qquad v := C_4 + C_5 \cdot x + C_6 \cdot y \qquad 7.3.3$$

$$\delta(x,y) := \begin{pmatrix} u \\ v \end{pmatrix} \qquad H(x,y) := \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \qquad \delta(x,y) := H(x,y) \cdot C \qquad 7.3.4$$

$$C \rightarrow \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix}$$

$$\delta(x,y) \rightarrow \begin{pmatrix} C_1 + x \cdot C_2 + y \cdot C_3 \\ C_4 + x \cdot C_5 + y \cdot C_6 \end{pmatrix} \qquad 7.3.3$$

Step 2. relate the general displacement within the element to the nodal displacements

nodal displacements:

$$\delta\_nodes := \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \qquad \delta\_nodes = stack(H(x_1, y_1), stack(H(x_2, y_2), H(x_3, y_3))) \cdot C$$

$$A := stack(H(x_1, y_1), stack(H(x_2, y_2), H(x_3, y_3))) \qquad \delta\_nodes = A \cdot C$$

$$A \rightarrow \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \qquad 7.3.5 \qquad \text{and} \qquad C = A^{-1} \cdot \delta\_nodes$$

$$(-x_2 \cdot y_3 + y_1 \cdot x_2 + x_1 \cdot y_3 + x_3 \cdot y_2 - x_3 \cdot y_1 - x_1 \cdot y_2) \cdot A^{-1} \rightarrow \left(\begin{matrix} x_3 \cdot y_2 - x_2 \cdot y_3 & 0 & x_1 \cdot y_3 - x_3 \cdot y_1 & 0 & y_1 \\ -y_2 + y_3 & 0 & -y_3 + y_1 & 0 & \\ -x_3 + x_2 & 0 & x_3 - x_1 & 0 & \\ 0 & x_3 \cdot y_2 - x_2 \cdot y_3 & 0 & x_1 \cdot y_3 - x_3 \cdot y_1 & \\ 0 & -y_2 + y_3 & 0 & -y_3 + y_1 & \\ 0 & -x_3 + x_2 & 0 & x_3 - x_1 & \end{matrix} \right.$$

after a -1 multiplication of each segment =>

$$A^{-1} = \frac{\begin{pmatrix} x_2 \cdot y_3 - x_3 \cdot y_2 & 0 & -x_1 \cdot y_3 + x_3 \cdot y_1 & 0 & x_1 \cdot y_2 - y_1 \cdot x_2 & 0 \\ -y_3 + y_2 & 0 & -y_1 + y_3 & 0 & -y_2 + y_1 & 0 \\ x_3 - x_2 & 0 & x_1 - x_3 & 0 & -x_1 + x_2 & 0 \\ 0 & x_2 \cdot y_3 - x_3 \cdot y_2 & 0 & -x_1 \cdot y_3 + x_3 \cdot y_1 & 0 & x_1 \cdot y_2 - y_1 \cdot x_2 \\ 0 & -y_3 + y_2 & 0 & -y_1 + y_3 & 0 & -y_2 + y_1 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & -x_1 + x_2 \end{pmatrix}}{(x_2 \cdot y_3 - y_1 \cdot x_2 - x_1 \cdot y_3 - x_3 \cdot y_2 + x_3 \cdot y_1 + x_1 \cdot y_2)}$$

$$A_{123} := \frac{1}{2} \cdot \left| \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \right|$$

= area of triangle

$$2 \cdot A_{123} \rightarrow -x_3 \cdot y_2 + x_1 \cdot y_2 + x_2 \cdot y_3 - x_1 \cdot y_3 + x_3 \cdot y_1 - y_1 \cdot x_2$$

denominator above =>

$$A_1 := \begin{pmatrix} x_2 \cdot y_3 - x_3 \cdot y_2 & 0 & -x_1 \cdot y_3 + x_3 \cdot y_1 & 0 & x_1 \cdot y_2 - y_1 \cdot x_2 & 0 \\ -y_3 + y_2 & 0 & -y_1 + y_3 & 0 & -y_2 + y_1 & 0 \\ x_3 - x_2 & 0 & x_1 - x_3 & 0 & -x_1 + x_2 & 0 \\ 0 & x_2 \cdot y_3 - x_3 \cdot y_2 & 0 & -x_1 \cdot y_3 + x_3 \cdot y_1 & 0 & x_1 \cdot y_2 - y_1 \cdot x_2 \\ 0 & -y_3 + y_2 & 0 & -y_1 + y_3 & 0 & -y_2 + y_1 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & -x_1 + x_2 \end{pmatrix}$$

for display

$$A\_inv := \frac{1}{2 \cdot A_{123}} \cdot A_1 \qquad C := A\_inv \cdot \delta\_nodes$$

not required but displacement in the element can be evaluated as follows: $\delta(x,y) := H(x,y) \cdot A\_inv \cdot \delta\_nodes$

too big to show but does compute

Step III. express the internal deformation (strain) in terms of the nodal displacements

$$\varepsilon(x,y) = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{pmatrix} \qquad \varepsilon(x,y) := \begin{pmatrix} \frac{d}{dx}u \\ \frac{d}{dy}v \\ \frac{d}{dx}v + \frac{d}{dy}u \end{pmatrix} \qquad \varepsilon(x,y) \rightarrow \begin{pmatrix} C_2 \\ C_6 \\ C_5 + C_3 \end{pmatrix}$$ (7.3.9 & 10)

where: $\varepsilon(x,y) = G \cdot C$ $\quad G := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$ G is constant (7.3.11)

now using 7.3.6 $C = A^{-1} \cdot \delta\_nodes$

$\varepsilon(x,y) = G \cdot A^{-1} \cdot \delta\_nodes$ and we can define

$B(x,y) := G \cdot A\_inv$ => $\varepsilon(x,y) := B(x,y) \cdot \delta\_nodes$

$$(x_2 \cdot y_3 - y_2 \cdot x_3 - x_1 \cdot y_3 + y_1 \cdot x_3 + x_1 \cdot y_2 - y_1 \cdot x_2) \cdot B(x,y) \rightarrow \begin{pmatrix} -y_3 + y_2 & 0 & -y_1 + y_3 & 0 & -y_2 + y_1 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & -x_1 + x_2 \\ x_3 - x_2 & -y_3 + y_2 & x_1 - x_3 & -y_1 + y_3 & -x_1 + x_2 & -y_2 + y_1 \end{pmatrix}$$

factors out $(x_2 \cdot y_3 - y_2 \cdot x_3 - x_1 \cdot y_3 + y_1 \cdot x_3 + x_1 \cdot y_2 - y_1 \cdot x_2)$ which as above is 2 x area triangle

=>

$$B := \frac{1}{2 \cdot A_{123}} \cdot \begin{pmatrix} -y_3 + y_2 & 0 & -y_1 + y_3 & 0 & -y_2 + y_1 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & -x_1 + x_2 \\ x_3 - x_2 & -y_3 + y_2 & x_1 - x_3 & -y_1 + y_3 & -x_1 + x_2 & -y_2 + y_1 \end{pmatrix}$$ (7.3.14)

$B = strain\_coefficient\_matrix$

Step IV. express the internal force (stress) in terms of the nodal displacement using the element's law of elastic behavior

$$\varepsilon := \frac{1}{E} \cdot \begin{bmatrix} \sigma_x - \nu \cdot \sigma_y \\ -\nu \cdot \sigma_x + \sigma_y \\ 2 \cdot (1 + \nu) \cdot \tau \end{bmatrix} \qquad \sigma(x,y) := \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau \end{pmatrix}$$

$$\varepsilon := \frac{1}{E} \cdot \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2 \cdot (1 + \nu) \end{bmatrix} \cdot \sigma \qquad \sigma := \left[\frac{1}{E} \cdot \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2 \cdot (1 + \nu) \end{bmatrix} \right]^{-1} \cdot \varepsilon$$

$$D := \left[\frac{1}{E}\cdot\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2\cdot(1+\nu) \end{bmatrix}\right]^{-1}$$

$$\frac{\left(-1+\nu^2\right)}{-E}\cdot D \rightarrow \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{-1}{2\cdot(1+\nu)}\cdot\left(-1+\nu^2\right) \end{bmatrix}$$

=>

$$D := \frac{E}{1-\nu^2}\cdot\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$$

for display $\frac{-1+\nu^2}{-E}$ is used and lower right term can be factored

$\sigma := D\cdot\varepsilon$ and ... $\sigma := D\cdot B\cdot\delta\_nodes$

using $\varepsilon := B\cdot\delta\_nodes$ from above

as with displacement above the stress distribution within the element could be calculated

Step V. obtain the element stiffness matrix by relating the nodal forces to nodal displacements

this step is derived separately in the text using the energy method
the approach is identical to what we did with the general derivation of element stiffeness (applied to flex only beam)
assuming virtual displacement and virtual strain
equating the work done to potential energy in the element (hence integral over volume)
using the relationships above
and as B, D and E have no dependence on x can come outside the integral
cancelling the common virtual displacement leads to:

$$k_e := \int B^T\cdot D(E,\nu)\cdot B\, dvol$$

as in the text: the volume

$$k_e := t\cdot A_{123}\cdot B^T\cdot D\cdot B$$

$$\frac{D}{\left[\frac{E}{\left(1-\nu^2\right)}\right]} \rightarrow \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}-\frac{1}{2}\cdot\nu \end{pmatrix}$$