Continuous

Random Variables

Probability Density Function (pdf)

Introduce sample space

interval $[a,b]$: $a \le x \le b$. $\quad a < x < b$

Define pdf $f_X(x)$

$f_X(x) \ge 0$ for all x, $a \le x \le b$;

$\int_a^b f_X(x)\,dx = 1$.

Then

$P(x \le X \le x + dx) = f_X(x)\,dx$

and

$P(a' \le X \le b') = \int_{a'}^{b'} f_X(x)\,dx$

for $a \le a' \le b' \le b$.

Note

$P(a \le X \le b) = \int_a^b f_X(x)\,dx = 1$.

Cumulative Distribution Function (cdf)

Define

$F_X(x) \equiv \int_a^x f_X(x')\,dx' \quad (= P(X \le x))$.

Then

$F_X(a) = 0$, $F_X(b) = 1$;

$F_X(x)$ non-decreasing with x;

$P(a' \le x \le b') = F_X(b') - F_X(a')$.

Expectation

$$\mathbb{E}(g(X)) = \int_a^b f_X(x)\, g(x)\, dx .$$

Hence

$$\mu \,(= \mu_X) \equiv \mathbb{E}(X) = \int_a^b f_X(x) \cdot x \, dx , \qquad \text{mean;}$$

$$\sigma^2 (= \sigma_X^2) \equiv \mathbb{E}((X-\mu)^2) = \int_a^b f_X(x)\,(x-\mu)^2\, dx \qquad \text{variance;}$$

$$\sigma \,(= \sigma_X) \equiv \sqrt{\sigma^2} \qquad \text{standard deviation.}$$

Note

$$X \sim f_X(\cdot),\ \mu_X,\ \sigma_X^2,\ \sigma_X$$

$$\mathbb{E}_X\Big(\underbrace{\frac{X-\mu_X}{\sigma_X}}_{Y}\Big) = \int_a^b f_X(x) \cdot \left(\frac{x-\mu_X}{\sigma_X}\right) dx$$

$$= \frac{1}{\sigma_X}\int_a^b f_X(x)\cdot x\, dx - \frac{1}{\sigma_X}\int_a^b f_X(x)\cdot \mu_X\, dx$$

$$= 0 \ ; \qquad \mu_Y = 0$$

$$\mathbb{E}\left(\left(\frac{X-\mu_X}{\sigma_X} - 0\right)^2\right) = \frac{1}{\sigma_X^2}\int_a^b f_X(x)\,(x-\mu_X)^2\, dx$$

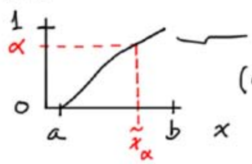
$$= 1 \qquad \sigma_Y^2 = \sigma_Y = 1$$

Quantiles:

$$X \sim f_X(x)$$

The α quantile of X, $\tilde{x}_\alpha$, satisfies

$$F_X(\tilde{x}_\alpha) = \alpha$$



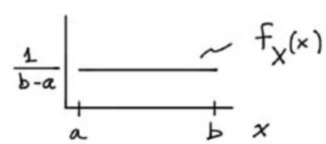
such that

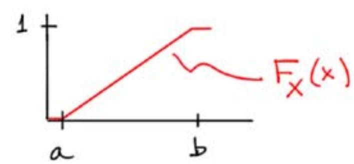
$$P(X \le \tilde{x}_\alpha) = \alpha, \quad P(X \ge \tilde{x}_\alpha) = 1-\alpha .$$

Note $\tilde{x}_{\alpha=\frac{1}{2}}$ is the median of X.

Example: uniform distribution over $[a,b]$

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b \qquad (\ge 0)$$





$$\int_a^b f_X(x)\,dx = \frac{b-a}{b-a} = 1 = F_X(b) - F_X(a)$$

$$\boxed{P(a' \le X \le b') = \frac{b'-a'}{b-a}}$$

Note

$$\mu_X = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{1}{12}(b^2 - a^2) \quad , \quad \sigma_X = \sqrt{\sigma_X^2}$$

and

$$\tilde{x}_{\alpha=\frac{1}{2}} = \frac{a+b}{2} \qquad \text{(median)} .$$

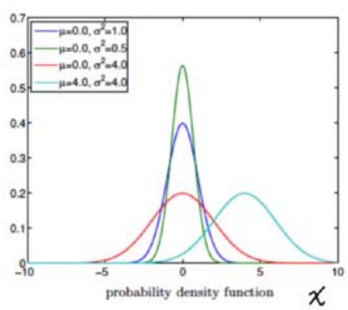
(later: bivariate case)

the Normal Distribution
(Gaussian)

pdf: $\quad X \sim f_X^{\text{normal}}(x; \mu, \sigma^2) \quad$ or $\quad X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X^{\text{normal}}(x; \mu, \sigma^2) =$$

$$\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

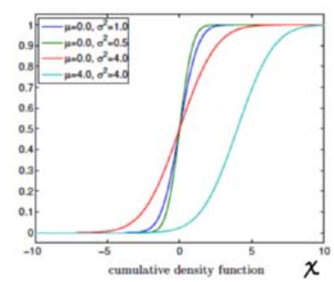


$$\mu_X = \mu \; ; \quad \sigma_X^2 = \sigma^2 \; , \quad \sigma_X = \sigma$$

(also median, mode)

cdf:

$$F_X^{\text{normal}}(x) = \int_{-\infty}^{x} f_X^{\text{normal}}(x')\, dx'$$



quantiles: $\tilde{x}_{0.841} = \mu + \sigma, \; \tilde{x}_{0.977} = \mu + 2\sigma, \; \tilde{x}_{0.9985} = \mu + 3\sigma$;

$\tilde{x}_{0.975} = \mu + 1.96\sigma \Rightarrow P(|X - \mu| \leq 1.96\sigma) = .05$.

Standard normal variable

$\mathcal{N}(0,1)$, with $\mu = 0$, $\sigma^2 = 1$

If

$$Z \sim f_Z^{\text{normal}}(z; \underset{\mu}{0}, \underset{\sigma^2}{1})$$

then Z is a standard normal random variable.

Further notation:

cdf for standard normal

$$F_Z(z) = F_Z^{\text{normal}}(z; \mu = 0, \sigma^2 = 1) \equiv \Phi(z);$$

$$\tilde{z}_\alpha : \Phi(\tilde{z}_\alpha = \alpha) \qquad \tilde{z}_{0.975} \approx 1.96 .$$

some Useful Transformations and Random Variate Generation

Between uniforms

Let U be $\sim$ uniform over $[0,1]$,

$$f_U(u) = 1, \quad 0 \le u \le 1 .$$

Let $X = a + (b-a)U$. $\qquad X = g(U)$

Then X is $\sim$ uniform over $[a,b]$,

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b .$$

(pseudo-) random variates:

```
u = rand(1,n) ;
X = a + (b-a) * u;
```

Between normals

Let Z be a standard normal r.v.,

$$Z \sim \mathcal{N}(0,1) \qquad \mu = 0,\ \sigma^2 = \sigma = 1 .$$

Let $X = \sigma Z + \mu$. $\qquad X = g(Z)$

Then X is a normal r.v.,

$$X \sim \mathcal{N}(\mu, \sigma^2) \qquad \text{scale then shift} .$$

(Backwards: $Z = \dfrac{X - \mu}{\sigma}$: mean zero, variance unity.)

(pseudo-) random variates:

```
z = randn(1,n);
x = mu + sigma * z;
```

MATLAB summary

randi	rand	randn
uniform discrete integer, contiguous	uniform continuous [0,1]	standard normal mean = 0 std dev = 1

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