PROFESSOR: So morning. This is my fourth lecture on differential equations, that part of the course. And I haven't said anything about the textbook. That's Differential Equations and Linear Algebra. I wrote it because so many courses like this one want to combine those two topics. They're the two major topics of undergraduate math, after calculus, the two major directions, and the book connects those two directions. And I just wanted to give you the website for lots of things connected with the book. So it's the math website, dela for differential equations and linear algebra.

And so today is differential equations, second order, with a damping term, with a first derivative term. So that in many engineering problems, those coefficients A, B, C would have the meaning of mass, damping, and stiffness. Mass, damping, and stiffness. And physically, we know what the mass comes from. The stiffness comes from a spring, as I drew before. And traditionally we describe the source of damping as a dashpot. I guess in my whole life I've never seen a dashpot, but maybe it's-- think of a piston going up and down within a cylinder of oil or something, with resistance.

OK, so that's the left-hand side. Linear constant coefficients still. We don't have formulas, it's not easy to see what's happening when things are varying, when the equations non-linear, so this is the starting place. OK. With some forcing. So this is damped forced motion. This is the ultimate within linear equations. OK. And now, what's the forcing?

So now, always we solve first for the null solution. With no force, what are the natural motions? And we'll find a formula for those. Then the big one, the special right-hand side is always an exponential. So this is going to be y is going to be the null solution yn, and we'll get a formula for that. This, the exponentials. Always, the response to an exponential is an exponential. That's like the most important fact in getting solutions.

So the response will be some ye to the st at the same frequency-- well, I say frequency. If s is a real number that would be a growth or a decay, but very frequently s is an imaginary number, like last time. And this is comes from a rotation or an oscillation. And then, the complete picture comes from being able to solve it with an impulse. So then y is the impulse response, which I write as g of t.

Let me introduce just that letter g. That stands for growth factor in first order equations stands for Green's function, so that word Green is getting in here, his name. And it represents the impulse response. OK, one, two, three. And then the point of doing this one is that then we can do it, then we can get a formula for any f of t.

So this is the pattern that we followed for first order equations. We followed it for second order equations that didn't have damping. And now we're doing the big one with damping. So what's the damping going to do? What's going to be effect of damping? Say if I had a right side of one, just a unit force? The damping. So I'll still have oscillation-- you'll see this-- I'll still have oscillation, but its amplitude damps out. Like that like everything we know, like something swinging back and forth, but friction is it is eventually damping out that motion. So it'll be oscillating with exponentially decaying amplitude.

Let's find this solution. OK. How do we find the null solution? I've got constant coefficients here. So the nice, the right thing to look at is exponentials. In first order equations it was e to the at. Here, we'll have to see what it is. So I'm going to try a particular-- I'm going to look for the right exponentials. Certain exponentials will be null solution. Can I do it?

So I take that equation and put in y equal e to the st. Remember now, I'm doing 0 here so it's not the same s as the right-hand side. OK. What happens if I put in-- so I'm looking for the null solution. I'll try y equal e to the st. Plug it in. I get m. Two derivatives gives me s squared e to the st, b. One derivative gives me an s, e to the st, and k gives me k e to the st. And that's supposed to equal to 0 for the null solution.

OK. Have we made a good guess at what will work? Yes, because I can cancel e to the st, and I come to the most important equation. The equation that governs everything in this whole lecture is ms squared plus bs plus k equals 0. That's called the characteristic equation, or it has many names. It's obviously the big deal. It's the thing. It's an equation for s, for the special frequencies that solve the null equation with no force.

OK. And how many values of s do we expect? Two. So I expect solutions s1 and s2, and then my null solution is e to the s1t is a null solution. E to the s2t is another one. My equation is linear. I can multiply that by any constant. I can multiply this by any constant. And I can superimpose, i can add, I can combine, because I can do linear algebra here. This is the most important operation in linear algebra, multiply things by constants and add. That's called a linear combination. It's the basic operation in linear algebra and it's a basic operation here, because we're doing linear algebra with functions. Do you see that that's it?

We've got it, except we should really write a formula and draw some pictures to show s1 and s2. What would be the formula for s1 and s2, to the two solutions? We remember that from school, it's the quadratic formula. The two solutions to this. Everybody remember this one? Let's see if I do. Does it start with-- a minus b. And then there's going to be a denominator that I'll remember, which is 2a. No. I said a but I mean m. 2m. And now this is plus or minus-- what goes into here? B squared minus 4ac. The key quantity in this whole business, b squared minus 4ac. Ah, what is it? Mk, thank you. 4mk. Great.

And can I just remark, a little remark about units. B squared has the same units as 4mk. It has to or such a formula would be crazy. So we will see that actually there's something called the damping ratio that involves the ratio of these guys. OK, that's the formula.

But it's not like-- OK it's got a square root, and what's the point of-- the thing we have to remember with this square root is that if it's the square root of a positive number then we have a plus or minus, ordinary real numbers. If this thing is negative, then what? What's up if b squared is smaller than 4mk? So if b squared is-- if there's not much damping, if it's underdamped, b squared would be smaller than 4mk.

And what then? We've got a negative number here. When we take its square root we have an imaginary number. That's oscillation. Under-damping is going to show oscillation. Let me draw this. Let me draw that curve for different choices of m, b, and k. This is a good picture to see. So let me draw first of all one that has b equals 0.

OK. So there's a curve. What I'm drawing is, up on this curve is this ms squared plus bs plus k. Ms squared and bs and k. And here, this one is one with no damping at all. This is just s squared plus 0 s plus 1. That's what that curve is That's the example we saw before. Y double prime plus y. Y double prime plus y equals 0. This is coming from Y double prime plus y equals 0. This is pure oscillation.

OK. Now let me bring in some damping. Now, as I bring in damping the curve will move. And it takes a little patience to see where it moves. Let me have a little damping a little damping, so s squared plus 1 s plus 1. I think the curve-- so this is s here. And it's a parabola. Everything I draw here is a parabola, is just that different parabolas come from different choices of b. So I think that this choice, I think it goes a little bit like that. That's the next one. It goes down a bit.

Now, let me do s squared plus 2s plus 1. Now, let's see, I really should stop for a moment and solve the equation, find the roots for each of these guys. And then I'm going to have an s squared plus 3s plus 1. So we're doing something very straightforward, parabolas. But it shows us the different possibilities. And we could give them names.

OK. So I would call this one undamped. And what are the roots of that equation? S squared plus 0 s plus 1 equals 0. What are the s1 and s2 for that guy. So the roots of s squared plus 1 equals 0 are? Stay with me here. S squared plus 1 equals 0, that's the equation that has roots.

STUDENT: I and minus i

PROFESSOR: I and minus i. I and minus i. So this is undamped. S1 is i and s2 is minus i. That's the pure oscillation. Pure oscillation, that's the case where b is 0 here, I have a square root of a negative number, and it gives me plus or minus 2i, and then the 2s cancel and I get plus or minus i. So I can always go back to this but I'll try to choose numbers that come out nicely.

Now, what happens with some damping? This guy. What are the roots for this one? Well, I better use this formula. Now, I'm always keeping m equal 1 and k equal 1, in all those samples. But now b has increased to 1. So if b is 1, what do I have? I have s is-- the roots are minus 1, plus or minus the square root-- And it's going to be just 2 down below. What's in the square root, the all important square root. When m and k and b are all 1. Just do the calculation with me, so you see it. It's negative?

STUDENT: Three.

PROFESSOR: Negative 3. So what's the point there? The point is, this is going to be square root of 3i or minus i. We have oscillation at a frequency square root of 3, and we have decay from s minus one-half. The real part of this is giving us the drop off. We didn't have any drop off at all in this case. They were pure imaginary. Now the s1 and s2 are whatever I have a minus 1, plus or minus square root of 3i over 2. Those are the roots.

All right, I'm ready for this guy, and it's particularly nice. It's particularly nice. What do you see for s squared plus 2s plus 1? When you see this parabola, now b has moved up to 2. What's up with that one? It's going to be-- let's see-- that looks like a perfect square to me. Right, inside the square root is 0. Right, exactly. Inside the square root, b squared is 4, and 4mk is 4, so I have 0 inside this square root. So now I have s equals minus 1 plus or minus 0 over 2. That's when this was the case, when b moved up to 2.

STUDENT: [INAUDIBLE] minus 2.

PROFESSOR: I'm messing it up?

STUDENT: [INAUDIBLE] equals minus 2, plus or minus 0.

PROFESSOR: Minus 2 plus or minus 0. Thank you. So what do I have? What are the roots of this guy? Negative 1. What's the other one? Negative 1. A double root. It's critical damping. Critical damping-- it's not underdamped. It's not overdamped. It's right on the borderline.

And I see that, when you first saw quadratics, before anybody brought up that awful formula, you would have factored this into s plus 1 squared equals 0, and you would have discovered that s was minus 1 twice. A double root. So the picture there would be-- there's minus 1. Yeah. Everybody recognize that this, we're hitting 0, height 0. We're hitting 0 twice at s equal minus 1.

So this is now the case. This was b equal 0, no damping. This was b equal 1, under-damping. This is b equal 2, critical damping, just on the border. And what do you think the s squared plus 3s plus 1 is going to look like. Again, we can find it. No, let me do the s squared plus-- let me take b equal 3 and find the roots and draw the picture. If you're with, me that picture and these formulas tell you the difference between these four cases.

So what do I get? Minus 3 plus or minus the square root of, 9 minus 4, is 5, over 2. So I have two negative roots. I have decay. I have decay at a fast rate and a slow rate but both are giving decay. So the curve now is coming down here and back up there, and it hits there, and it's got the two roots. These two roots are x and x.

So these are s1 and s2. Let me copy them over here. S1 and s2 are minus 3 plus or minus the square root of 5 over 2. Yeah. Real. Real roots. So this is two real roots. This is a double real root. This is two complex roots. And this is two pure imaginary roots. The four possibilities. The famous four. OK.

And for me, that picture is a-- so this was the b equal 3 curve, more overdamped. It's a little interesting that overdamping has this root that's pretty near 0. So overdamping doesn't mean that you go to 0 real fast. Actually, the one that goes fastest to 0 is the critical damping. Then, as b grows, one root gets closer to 0, so it's like slower decay, and another root is going off to fast decay.

I think you have to know those guys, because the physically that's very important, where's the damping. But we've now found the null solution completely. The null solution completely is-- let me write it again here-- is anything times e to the s1t and anything times e to the s2t. And where do those constants, c1 and c2 get decided? By the?

STUDENT: [INAUDIBLE].

PROFESSOR: Initial conditions. Right. These two conditions determine c1 and c2. OK. Are we good? So the null solution has already separated the different cases that depend on how much damping.

I'm ready for number two. Number two now. OK. So the idea is I now have a forcing term, some frequency e to the st. And I will assume that it's not-- s not equal s1 or s2. That's to make my life easy. Just as last time, the formulas will break down.

And I'll have to put in something different if there's resonance, if the driving force is at the same frequency as a natural frequency. In that case, there's a resonance. And the way you spot a resonance formula is there's an extra factor of t. There's a growth of t. Up here, we're seeing no factors of t. Over there in the exponent, of course. I mean, down below, there would be a t times e to the st. And the book does that carefully. I don't see that it has a place in the very first lecture on this topic.

OK. So this is saying no resonance. All right. What's the solution then? The solution is a multiple of e to the st. The input was e to the st. The response is a multiple of e to st, so that's the frequency response. Capital Y is telling us the response to s. And it's a very, very important function. It's called the transfer function. It's just the key to everything.

I probably say that so often, that this is the key to everything. Well, it's partly because I just have one lecture to do a big part of differential equations, and it's got some key ideas. And the first key idea is s1 and s2. And the second key idea is Y. And let's go for it. All right, so this is the s1 and s2 picture. I'll move that up, and now, in the equation, try Y e to the st. We hope it works. What is this? I'm trying, this is my solution, and it's a particular solution now. I got the null solution. I've moved to a particular

And this is when the force is e to the st. In other words, I'll just say again, if we have an exponential force, try an exponential response. 1803 would call this the exponential response formula. So you could use the word exponential response, very appropriate. You could use the word frequency response. That frequency response is kind of the right word when s is an imaginary number giving an oscillation frequency.

OK, I'm going to plug that in. So into the differential equation, m. Second derivative of that is s squared Y e to the st, right? That's m Y double prime. The beauty of exponentials. Take every derivative, just brings down an s. The next term. What's the next term? Maybe do it with me, do it for me. What happens when I plug in this as Y into b, so there'll be a b Y prime. So what's Y prime?

STUDENT: [INAUDIBLE].

PROFESSOR: s, Y, this constant, e to the st. And then the final term is k times this Y itself, no derivative, so it's just Y e to the st, and that's matching e to the st. That's the force. This is f. F, this is an exponential force. Of course, I could and should have a constant here to give me the units of force. Let me just keep the formula as clean as possible by taking units, so that's one.

OK what do I do?

STUDENT: [INAUDIBLE].

PROFESSOR: I cancel. The nice part of 287 is canceling e to the st. That's the most fun you get. And now, I have Y's on the left. So Y is-- can I see what Y is? Y is this 1, divided by the coefficient of Y. And what's the coefficient of Y? We know it. We've seen that coefficient. Y on the left is multiplied by ms squared plus bs plus k ms squared, again, ms squared, bs, and k. Multiplying Y, I divide by that, so I put it down here. Ms squared plus bs plus k. And because s is not one of the roots, that's not 0, so we're golden.

That problem took five minutes. The null solution took half an hour. The exponential response is clear. And you can see what it would be. And let's give it a name. This is the transfer function. Widely used name. Other names could be given but that's the best.

So it's a function of s. It's a function of s. And so, again, when f is e to the st, it sort of transfers the input into the output. That's the way I think of the transfer function. Here is the input. The output is just multiplied by the transfer function. And the transfer function is just that nice expression. Just that nice expression. So we are golden for a frequency, for a linear equation with an exponential forcing function.

What would be another example? I'm using mass, dashpot, spring here. If this was in electrical engineering, what three things would I be using instead of mass, dashpot, spring. The three guys would be? What would correspond to the dashpot? So can I just draw here a little-- let me put on a low voltage and put on something that does this, and then something like this. And then there's something like this. And give me a break, tell me what these things are. This guy is?

STUDENT: Inductor.

PROFESSOR: An inductor. This guy is a?

STUDENT: Resistor.

PROFESSOR: Resistor. Now that's the one that's like damping. This resistor here is like damping. Like the damping term, or maybe 1/b. I'm not getting its units right because I haven't got any equation here at all. Resisting is-- there's friction in that resistor. It burns up heat. And similarly, the dashpot slows things down. And then this guy is a--

STUDENT: [INAUDIBLE].

PROFESSOR: Capacitor, right. In other words, you can do the mechanical application and the electrical application with exactly the same ideas, just a change of letter, and of course, different units, but same problem. OK, so that's a comment that you've seen before. What else do I want to comment on? Because this example was really so straightforward.

I think what I want to mention, and this is important, is that this is the central starting point for the Laplace transform. So I can't do Laplace transforms all today by any means, and so Professor Fry will talk about the Laplace transform next week. But what is the point of the Laplace transform? The point of the Laplace transform is to get your money's worth out of the simple formula for exponentials.

Having an exponential there turns the whole differential equation problem into an algebra problem. We just have quadratic equations. We just have a division by a quadratic. That's the great thing about the Laplace transform. It turns the t domain, the time domain, where we have exponentials, into the s domain, the exponent domain, the frequency domain, where we just have quadratics. And then first order equation's just linear.

And we can even get from second order of the quadratic to linear, because I can factor that guy. If I factor into s minus s1 times s minus s2, I've two linear pieces. And that's the first step in the Laplace transform, in the algebra. So all of the algebra in the Laplace transform is this algebra for e to the st.

And then the job of the Laplace transform-- and this is the tricky part. So let me even take a little board space on that. So this is like a heads up for next week. So the Laplace transform.

OK. So for any forcing function, f of t. That's the thing that we're going to take the Laplace transform of and the response and its response, y of t. OK. So here's the idea of transforms in general. I choose some terrific functions like exponentials. So I want to convert my problem to exponentials. e to the st for all s, s between let's say 0 and infinity.

So what do I do? I take my function, and I figure out how much of every exponential is in it. That's the Laplace transform. I take my function f of t, and I go to what you'll see next week, its Laplace transform. Let me call it capital F of s, or it's sometimes written the Laplace transform of F of s. Something like that. I'm not going to do that. I'm not going there.

So F of s is like the amount of a particular exponential in my function. If my function is just the sum of two exponentials, then the Laplace transform just is a big bump on one and a big bump on the other. But most functions, like some forcing function, has some e to the st is for all for a whole range of s. So I figure out how much of this is.

OK now second step. Using linearity, I can solve the problem for when the right hand side is just that. Then solve for right hand side, F of s, e to the st. Solve for each frequency, each s separately. And what does that mean? That means just dividing by this. So this was the easy step. So this is step one, is take the Laplace transform. The Laplace transform tells you how much of each exponential is in it. Now step two is a cinch. Step two is a cinch. I just multiply by the transfer function. I divide by this, bs plus k.

And now, what's step three. Step three, I have the solution for each separate exponential, but I've got a whole lots of exponentials, so I have to do an inverse Laplace transform, add up, figure out what function has this Laplace transform. And that's often the place where the algebra gets harder. In principle, we can do it for any F of t. We can take its Laplace transform, we can solve for each frequency in the transform, we can assemble them all together. That's the inverse Laplace transform, so this is the inverse Laplace to get the solution y of t.

So this is really the Laplace transform of the solution, and we have to get back to the solution itself. Can I just let you think about those ideas? I'm not up to describing the algebra here. The point is the Laplace transform takes this into separate exponentials. Each of those right hand sides is simple algebra, divide by that. And then you have the job of okay, what function has this Laplace transform, to go backwards. And that's usually the hard part.

So people make tables of Laplace transforms. Everybody remembers the Laplace transforms of a few functions. You could say that those few functions are the golden functions of differential equations. The golden functions of differential equations are the ones where you know their Laplace transform and you can go back and forth easily.

And what are those golden functions? Well, you might guess exponentials, simple polynomials, 1 t, t squared. You can do Laplace transform for those. What else do you think would be nice? Cosine and sine. And you could multiply those together. We could deal with t times cosine t. So a little bit different. But having said that, the reality is I've given you the whole list of nice functions. Those functions show up in every simple method for solving equations.

There's a method called undetermined coefficients and what does it amount to? I'm sorry, I don't have time to say it this morning, but it can come later. Undetermined. It just means if the right hand side is one of these nice guys-- shall I write down again the golden functions? e to the st is like the platinum function. And then some golden functions are like t and t squared and so on. Of course, when we get that, we're close to cosine omega t, and sine omega t. All these guys, their Laplace transforms are nice. We can deal with them completely. Or multiply any of those together. And when the right hand side is one of these golden functions, you can write down the answer.

We've focused on this one because it's the platinum one. And we did these two too, because they come from s equal i omega. And then these guys are a little bit of juice. But that's it. I'm sorry the list isn't longer. It'd be nice to have--and of course, people for centuries have worked with the next hardest functions. You know, the silver functions. Famous functions have names like Bessel's function or Legendre's function. Others where you can get pretty far. Those are the best.

Then you have the famous ones where you get pretty far, in the web has the Laplace transforms, and then you get the general function, F of t. Oh, could you get anywhere with delta of t? Oh, yes. Does it belong on the list of golden functions? Yes, it does. I almost forgot it, and it's like-- I'll call it-- Yeah, delta of t. Yeah, that's a beauty. That's a beauty.

The Laplace transform of delta of t happens to come out 1 over s. You can't ask for more than that. Or maybe it's one. Yeah, it's probably one. Yeah, the Laplace transform of that is one. Yeah. It's got all exponentials in sort of, you could say, equal amounts.

OK. So that's some thoughts about that about Laplace transforms, just sort of the big picture that takes the differential equation, turns it into an algebra problem, and then at the end, you have to get back, and that's the part that's not always doable.

OK. So what's left for today is this guy. Now this one now. Have I got to that point? So this will be the final ideas in this course, this four unit core elective.

What is the impulse response when there's damping? What's the impulse response when there's damping? OK. OK. So that means that I would really like to solve m y double prime, b y prime, k Y equals an impulse, delta of t. Because if I can solve this, then I can solve everything. And I can solve this. It turns out to be easy. Now why is it easy? You might think, my gosh, we've got second order equations here, we've got a delta function there, where do we go? And so my advice is go this way.

The solution to that is the same. This with y of 0 equals 0 and y prime of 0 equal 0. So you have a spring, so again we have a spring with a mass. And that spring is in a damper. So can I just, without knowing what I'm doing, draw a damper around it.

So the idea is I'm striking that mass at time 0. Striking that mass at time 0. What happens? What happens immediately? And then I don't touch it again. I strike it and that's it. I've set off, and what have I-- so my point is this has the same solution as m y double prime plus b y prime. And Ky equals 0. Nothing happens beyond time 0.

But what are y of 0-- Well, let me give this a letter g, just to emphasize it's special and deserves its own name. So now, what is the starting position and the starting velocity for this picture? So I'm saying that the y here is the same as the g there.

And really if you see it physically then you see it best. So again, at the instant t equals 0, I'm hitting that mass with a unit impulse.

So what is the position of that mass, instantly after I've hit it?

STUDENT: 0.

PROFESSOR: Still 0. Good for you. Good for you. And what is the velocity of that mass instantly after I've hit it?

STUDENT: 1.

PROFESSOR: 1 is essentially the right answer. 1 is the right answer. But the way I've set it up is, there is an m there. If I put an m there, which would be nicer, then the answer would be 1. But the way I've written it here, I have an m there, and I haven't fixed the units. So that turns out to give me a 1/m there. I could explain why it's a 1/m, but let me just for the moment say it's my fault. It's because I didn't get any units right that we have 1/m. No big deal.

But now, what's good here? What's good? This was a problem with a mysterious delta function. This is a problem with 0. And the only price we're paying is the impulse gave the mass a little velocity. And you can imagine that the velocity gives it is 1/m because the strike didn't tell us about that.

So what I'm saying is we can solve that equation for g. We can find g, and in fact, we have. So this really brings the lecture full circle. What do I have here? I have a null solution. So this g here is a null solution. So what form does it have? What can you tell me about g of t? And it's the same as y of t. So y is the same as g, and what's the form for it? Yes? Tell me? I'm taking it back to the very beginning of the lecture where I solved it with a 0.

STUDENT: [INAUDIBLE].

PROFESSOR: C1, thanks.

STUDENT: [INAUDIBLE].

PROFESSOR: e to the? C1, e to the s1, t, the two s's, s1 and s2, the special two s's, the special roots of the key equation. That equation gives us s1 and s2, by this formula, or in a homework or an exam problem, we hope that it would come out easily. So this is that part of it. What's the rest of it? C2 e to the s2 t. The impulse response is the particular null solution that starts with a shot, starts with an impulse, starts with a strike.

OK. So I just have to find c1 and c2 here. All right? We've come back to the basic problem in differential equations. We've got the solution. We've got two constants. We've got two equations. We just plug that function into that equation. It gives us one fact about c1, c2. This gives us the second fact. We solve them. Why don't I just write down the answer? It turns out to be e to the s1 t, and the other guy will come in with an opposite sign e to the s2t over s1 minus s2. I think that gives us the c's. Oh, there'd be an m. There'd be an m times this, because of my messing with units.

So can we just check? At t equals 0, what do I get out of this?

STUDENT: 0.

PROFESSOR: 0. What I'm supposed to. What's the derivative at t equals 0? The derivative at t equals 0, this derivative is going to bring down an s1. The derivative here will bring down an s2. At t equals 0 the exponentials will all be one so I'll just have the s1 minus the s2. It'll cancel that and it'll be 1 over m. So this is the neat formula for the impulse response. That's the neat formula for the impulse response.

And then why-- can I use this little corner of the board? Why do I want that impulse response? What can I use it for? It gives me the answer, not just for the impulse but for everything. The particular solution is for any forces, force, I multiply by whatever the-- let me write the formula and I'll show you what it says. Yes, there is the formula. I'm sorry it's squeezed.

But really, the goal here was simply to get a handle on what is the response to any f. And again, I look at that this way. F of s is the input at time s. G is the growth factor over the remaining time up until time t. So Y at time t, I take all the inputs up to time t, And each input gets multiplied by its growth factor. It was e to the a, t minus s in the first order equation. Now we've got two exponentials. But that's the solution of the general problem.

So we have now in one lecture completed a solution to the second order constant coefficient differential equation. Right. Yeah. By finding the impulse response. Yes?

STUDENT: [INAUDIBLE] would we still be able to [INAUDIBLE] if s1 is equal to s2?

PROFESSOR: Ah, if s1 equals s2. That's the case where formulas need a patch. They need a patch. If s1 equals s2, what do you think happens? If s1 equal s2, everybody sees, I have 0/0. And so this is like a technical question that I wasn't going to ask myself. You asked it. You're responsible. What do we do for 0/0? What did you learn in calculus?

Who's the crazy guy who figured out how to deal with 0/0? In a way, calculus is all about 0/0, right? Delta y over delta x, they're both headed to 0. And suppose you have-- let me take the most famous example of 0/0. It's like sine x over x, as x goes to 0. So x going to 0. Sine x goes to 0, x goes to 0. What's the answer, by the way? What happens to that ratio as x goes to 0? This is maybe the most famous example. Sine x over x, when x gets very small, is close to?

STUDENT: 1.

PROFESSOR: 1, thanks. And now just help me out with the name of the guy. It's a crazy spelling name, and do you remember? L'Hopital. L'Hopital. Everybody despises him. Probably hi friends despised him. But anyway, L'Hopital says in a situation like that, when you're going to 0/0, you're allowed to do something a little strange. You're allowed to take the derivative of the top, so it has the same limit. Instead of looking at this 0/0, you can take the derivative of the top, cos x, divided by the derivative of the bottom, 1. And now you can let x go to 0, and you get the right answer.

So this gave a 0/0. Unclear. Fuzzy. This gives-- what's the right answer then? Just tell me again. When x goes to 0 this becomes?

STUDENT: 1.

PROFESSOR: 1. Right. So that's L'Hopital. So that's what I would have to do here. I would take the derivative of this, and the derivative of this-- the only sort of tricky part is it's the s derivative. It's s1 going to s2. Let me just tell you the result. Since you asked. A factor t comes out. It's t e to the s1, or s1 is the same as s2, divided by the m. It actually looks simpler. There's only one. This is in the case s1 equal s2. So s1 is the same as s2. I just chose s1. Yeah. L'Hopital gives a simpler answer. And it's got this suspicious and recognizable factor t. That came from L'Hopital. OK, I won't do that stiff.

So let me say again, we've now done the second order constant coefficient equation I do just have 10 minutes of something to make it better. And that is that the famous quadratic formula for s, for s1 and s2 is not beautiful. It's correct. It's correct. But it's a little bit of a mess. You've got three things, b and m and k playing around.

And we saw in this picture, we saw all the differences. I guess in this example I kept m1 and I kept k1, and I increased b. I could do other examples where I increase the k, I make it stiffer and stiffer. All these examples. And engineers have worked for 100 years to see, out of this formula what are the important parameters, what are the important numbers, and hopefully, where possible dimensionless.

So I just want to- the final minutes would be-- back to high school-- playing with this formula, to get better numbers in there. May I do that? I just think, because then you'll see-- I learned this, actually, so this is like something math professors have no reason to do. Look at that. That's the formula. End of story.

But the Web, 1803 website, has a class in which Professor Miller from the math department was teaching this subject, doing these, exactly these, but also Professor Vandiver from Engineering was putting in his suggestion of what are the good parameters? What are the parameters that engineers look at? So that would be my final comment, and I won't do it as well as Professor Vandiver did. But can I just take that-- let me erase these two special examples, and look at this question. Again, the book will do it.

So one nice-- b/2m is a pretty natural parameter to use. Let me introduce that as one of them. I'm going to, by taking ratios like b/2m, let me call that p. Let me call b/2m. So that's a ratio of damping to mass. And then this has got to come out simpler. What does that come out? If you'll allow me, I'm going to open the book so I don't write the wrong thing here. This is in the book, on page 99. The title is Better Formulas for s1 and s2. Better Formulas for s1 and s2.

And here's my first better formula. You can see that I get a minus b plus or minus the square root of something. And that something will turn out to be p squared. And then it'll be a minus something. And that something will turn out to be the natural frequency squared. Isn't that nice? So what's the natural frequency? Somehow, the natural frequency's coming in from this and this?

And just remind me what that second parameter is, omega n squared, the natural frequency of oscillation with no damping. Tell me again what that is, because that was the fundamental ratio from last time. It's central to all of engineering. It's?

STUDENT: k/m.

PROFESSOR: k/m. Thanks, k/m. So I believe-- and maybe Professor Fry could make this assignment a homework question, which is just algebra question. Everybody sees that I have a minus p here. And with a little care you get p squared, which is quite nice. Which is quite nice. And so we see that the decision between overdamping-- remember now? Overdamping is when you damped so much that this became negative and you got an imaginary number in there. Underdamping, it's still positive. Overdamping, it's negative.

And so really that separation between overdamping and underdamping is the ratio of p to omega n. P to omega n is the damping ratio. I think. There may be a factor too. Let me try to-- everybody sees that's the battle between these, if you accept that formula, and if it's in the book it's got to be right. OK.

And so the damping ratio is just that. That's the damping ratio. Now that's called zeta. The Greek letter zeta. I'm not Greek and not good writing zeta. So I have unilaterally decided to use a capital zeta, which is a Z. Zeta is the Greek letter for Z. I could try to write it but you wouldn't be impressed. So it's that damping ratio.

So now what does this mean? Z smaller than 1, z equal to 1, c greater than 1? Tell me what those-- obviously, when it's smaller than 1, p is smaller than omega n. Yeah, so what's going on here? Which one is underdamping, which one is critical damping, which one is overdamping? Because there's no difficult stuff here. We're coasting in the last minutes here by just choosing words and notation that have turned out over a century to be more revealing than b squared minus 4mk, and this Z.

So z less than 1 will be what? Z less than 1 will be p smaller than omega n. p smaller than this. What's the story on that case then?

STUDENT: [INAUDIBLE].

PROFESSOR: That is underdamping, I guess. p small has to go. p is like b, the damping. And small damping is underdamping. So this is underdamp. Underdamp. And that's the case, in which we're going to have some imaginary stuff. We're going to have some oscillation with the decay coming from there.

Now, what about z equal to 1? z equal to 1 means p equals omega m, so that equals that so it's a big 0 in there. What case is that?

STUDENT: [INAUDIBLE].

PROFESSOR: Critical damping. It's this case in that picture. It's that case with a double 0, equal s's. Formulas that have to take account of that, this is critical.

And then finally, z greater than 1 is what? Overdamped. Overdamping. Z bigger than 1 means p is big. P big means b is big, damping is big, it's overdamped. Overdamped. So we've got it down to one parameter, the damping ratio, to tell us these things. Rather than previously we had to say is b squared smaller than 4mk? Is it equal to 4mk? Is it bigger than 4mk? Now we've got those words down to a single number z.

And let me just write next to us here that the z turns out to be the ratio of the damping to-- I think it's right-- it's the damping divided by the square root of 4mk, I think. Can I just put a question mark there. You couldn't mess around with the letters p and z. But to get some variation from some other, but the point is, you see how much cleaner that is compared to this? You're directly comparing that number to that number. And that ratio is that number. Yeah. So all the formulas come out nicely. Yeah, the formulas come out nicely.

And I guess what we see here-- final comment-- what we see here is what is the frequency of underdamped oscillations. So I want to be in this underdamping case where there is oscillation. There is an imaginary number coming out of that. But there's also a real number.

Is the frequency of underdamped oscillation the same as omega m, the natural frequency? No. The frequency of that number-- so final comment, let me put it just here. I would like this whole thing to be i, to give me oscillation, times omega d, the damped frequency. And let me just say what that is. So omega d, the damped frequency, squared, is this omega natural frequency squared minus the p squared. If I had longer and we didn't have blackboards already full of formulas, I could-- it's the thing whose square root we're taking here.

So this is minus p plus or minus i, omega damped. Omega damped is this square root. There, we succeeded to fit in the better ratios, the good quantities to look at. So again, the good quantities to look at are p, z, the damping ratio, omega d the damped frequency.

I think in a first lecture you could say, well, we already had correct formulas, we should just leave it there, and that's absolutely true, but anyway, this is what-- these are the letters people have introduced to make the formulas easier to understand in an engineering problem.

OK. I'm all done except for questions. Yes? Don't ask me about resonance again. Yes, OK. Yes?

STUDENT: In the case of where we have the delta function, what is the velocity [INAUDIBLE]?

PROFESSOR: What is the what?

STUDENT: Why is the velocity equal to [INAUDIBLE]?

PROFESSOR: A-ha. Okay. You're right on the ball. The question is where did this come from. Where did that come from? Can I tell you? So if I integrate everything here, if I take the integral of everything, between 0, a little bit left of 0-- can I call that 0 minus? Just a little bit left of 0. This is crazy. No math professor or whatever should ever do this. To a little bit right of 0, just a real short time. So what am I going to call a little bit right of 0? 0 plus. OK now what is that integral? Between a little left of 0 and a little right of 0, you know what the integral of the delta function is. It is?

STUDENT: 1.

PROFESSOR: 1. Good. Now, what are these ridiculous things? Well, y is not changing in this tiny, tiny time. So this is something, it's not getting big. I'm integrating it over this tiny little time. It's nothing. Forget it. Similarly here, y prime, the velocity is not climbing to infinity. There's no-- and I'm just integrating over this infinitesimal little time. Nothing here.

So this term has to be responsible for the 1. And now you can tell me the integral of m y double prime. What's the integral of m y prime?

STUDENT: m y prime

PROFESSOR: m y prime. So m y prime plus, at 0 plus, minus m y prime at 0 minus. But at 0 minus, it's 0. You see what's happening? And on the right side I'm getting a 1. This is-- no person who had any skill with a blackboard would allow this to happen. But that happened.

OK. So these lower order terms are typical of math. Lower order terms in the limit, forget them. This is the top term, and it has to have something there, because it has to balance the 1. And what it has is the jump in y prime. So this is the instant jump in y prime in velocity, is times m gives 1. So the instant jump jumped us from 0 to 1 over m. That's where I came from. Well, that was a good question, and a kind of crazy answer but there it is.

OK, so we've got a mention of the Laplace transform as the algebra tool that works when you're staying with exponentials and nice functions. And you'll see more of that. So it's a frequently used tool to turn problems into algebra.