2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 10 - F.E. large deformation/general nonlinear analysis

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We developed

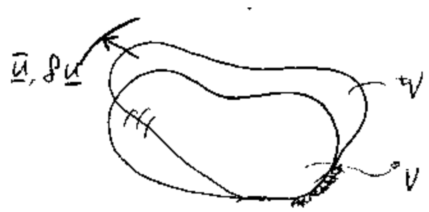
Reading: Ch. 6

$$\int_{^tV} {}^t\tau_{ij}\, {}_t\bar{e}_{ij}\, d^tV = {}^t\mathcal{R} \quad (10.1)$$

$${}_t\bar{e}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial^t x_j} + \frac{\partial \bar{u}_j}{\partial^t x_i}\right) \quad (10.2)$$

$$\int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, d^tV = {}^t\mathcal{R} \quad (10.3)$$

$$\delta_t e_{ij} = \frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial\, {}^t x_j} + \frac{\partial(\delta u_j)}{\partial\, {}^t x_i}\right) \quad (\equiv\ {}_t\bar{e}_{ij}) \quad (10.4)$$



In FEA:

$${}^t\boldsymbol{F} = {}^t\boldsymbol{R} \quad (10.5)$$

In linear analysis

$${}^t\boldsymbol{F} = \boldsymbol{K}\, {}^t\boldsymbol{U} \Rightarrow \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \quad (10.6)$$

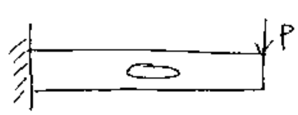
In general nonlinear analysis, we need to iterate. Assume the solution is known "at time t"

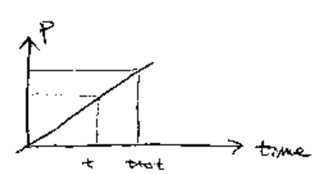
$${}^t\boldsymbol{x} = {}^0\boldsymbol{x} + {}^t\boldsymbol{u} \quad (10.7)$$

Hence ${}^t\boldsymbol{F}$ is known. Then we consider

$${}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \quad (10.8)$$

Consider the loads (applied external loads) to be deformation-independent, e.g.





Then we can write

$$^{t+\Delta t}\boldsymbol{F} = {}^{t}\boldsymbol{F} + \boldsymbol{F} \tag{10.9}$$

$$^{t+\Delta t}\boldsymbol{U} = {}^{t}\boldsymbol{U} + \boldsymbol{U} \tag{10.10}$$

where only ${}^{t}\boldsymbol{F}$ and ${}^{t}\boldsymbol{U}$ are known.

$$\boldsymbol{F} \cong {}^{t}\boldsymbol{K}\Delta\boldsymbol{U}, \quad {}^{t}\boldsymbol{K} = \text{tangent stiffness matrix at time } t \tag{10.11}$$

From (10.8),

$$^{t}\boldsymbol{K}\Delta\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F} \tag{10.12}$$

We use this to obtain an approximation to $\boldsymbol{U}$. We obtain a more accurate solution for $\boldsymbol{U}$ (i.e. ${}^{t+\Delta t}\boldsymbol{F}$) using

$$^{t+\Delta t}\boldsymbol{K}^{(i-1)}\Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \tag{10.13}$$

$$^{t+\Delta t}\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \tag{10.14}$$

Also,

$$^{t+\Delta t}\boldsymbol{F}^{(0)} = {}^{t}\boldsymbol{F} \tag{10.15}$$

$$^{t+\Delta t}\boldsymbol{K}^{(0)} = {}^{t}\boldsymbol{K} \tag{10.16}$$

$$^{t+\Delta t}\boldsymbol{U}^{(0)} = {}^{t}\boldsymbol{U} \tag{10.17}$$

Iterate for $i = 1, 2, 3\ldots$ until convergence. Convergence is reached when

$$\left\|\Delta\boldsymbol{U}^{(i)}\right\|_2 < \epsilon_D \tag{10.18}$$

$$\left\|{}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\right\|_2 < \epsilon_F \tag{10.19}$$

Note:

$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_i (a_i)^2}$$

$$\sum_{i=1,2,3\ldots} \Delta\boldsymbol{U}^{(i)} = \boldsymbol{U}$$

$\Delta\boldsymbol{U}^{(1)}$ in (10.13) is $\Delta\boldsymbol{U}$ in (10.12).

(10.13) is the *full* Newton-Raphson iteration.

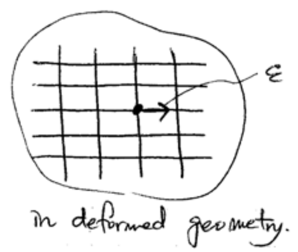
How we could (in principle) calculate ${}^{t}\boldsymbol{K}$

Process

- Increase the displacement ${}^{t}U_i$ by ϵ, with no increment for all ${}^{t}U_j$, $j \neq i$
- calculate ${}^{t+\epsilon}\boldsymbol{F}$
- the i-th column in ${}^{t}\boldsymbol{K} = \left({}^{t+\epsilon}\boldsymbol{F} - {}^{t}\boldsymbol{F}\right)/\epsilon = \frac{\partial^{t}\boldsymbol{F}}{\partial^{t}U_i}$.

So, perform this process for $i = 1, 2, 3, \ldots, n$, where n is the total number of degrees of freedom. Pictorially,

$$ {}^{t}\boldsymbol{K} = \begin{bmatrix} \vdots & \vdots & \\ \vdots & \vdots & \cdots \\ \vdots & \vdots & \end{bmatrix} $$



A **general** difficulty: we cannot "simply" increment Cauchy stresses.

- ${}^{t+\Delta t}\tau_{ij}$ referred to area at time $t + \Delta t$
- ${}^{t}\tau_{ij}$ referred to area at time t.

We define a new stress measure, *2nd Piola - Kirchhoff stress*, ${}^{t+\Delta t}_{0}S_{ij}$, where 0 in the leading subscript refers to original configuration. Then,

$$ {}^{t+\Delta t}_{0}S_{ij} = {}^{t}_{0}S_{ij} + {}_{0}S_{ij} \tag{10.20} $$

The strain measure energy-conjugate to the 2nd P-K stress ${}^{t}_{0}S_{ij}$ is the Green-Lagrange strain ${}^{t}_{0}\epsilon_{ij}$

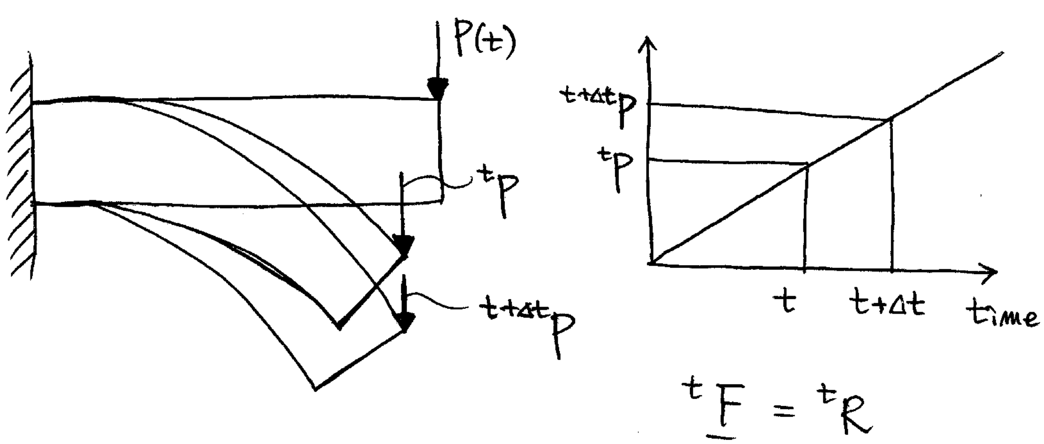
Then,

$$ \int_{{}^{0}V} {}^{t}_{0}S_{ij}\ \delta\, {}^{t}_{0}\epsilon_{ij}\ d^{0}V = {}^{t}\mathcal{R} \tag{10.21} $$

Also,

$$ \int_{{}^{0}V} {}^{t+\Delta t}_{0}S_{ij}\ \delta\, {}^{t+\Delta t}_{0}\epsilon_{ij}\ d^{0}V = {}^{t+\Delta t}\mathcal{R} \tag{10.22} $$

Example



$$ {}^{t}\boldsymbol{F} = {}^{t}\boldsymbol{R} \tag{10.23} $$

$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \tag{10.24} $$

(every time it is in equilibrium)

(10.13) and (10.14) give:

$i = 1$,

$$ {}^{t+\Delta t}\boldsymbol{K}^{(0)}\,\Delta\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(0)} \equiv \mathrm{fn}({}^{t}\boldsymbol{U}) \tag{10.25} $$

$$ {}^{t+\Delta t}\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{U}^{(0)} + \Delta\boldsymbol{U}^{(1)} \tag{10.26} $$

$i = 2$,

$$ {}^{t+\Delta t}\boldsymbol{K}^{(1)}\,\Delta\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(1)} \tag{10.27} $$

$$ {}^{t+\Delta t}\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{U}^{(1)} + \Delta\boldsymbol{U}^{(2)} \tag{10.28} $$

2.094 Finite Element Analysis of Solids and Fluids II
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