

## 2.14/2.140 Problem Set 8

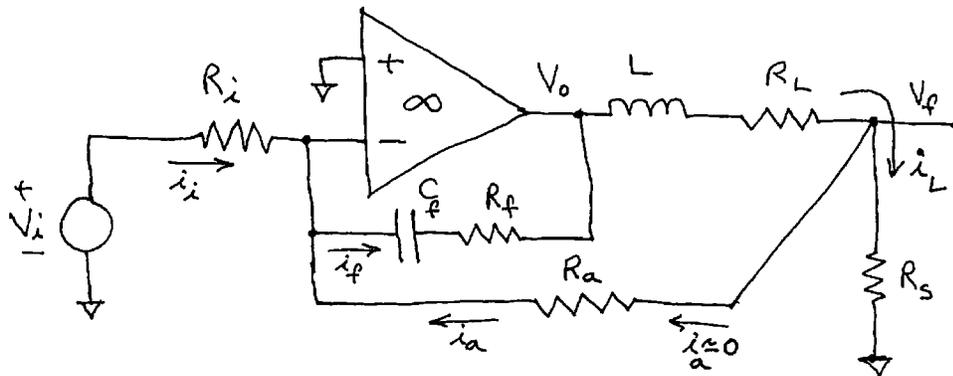
**Assigned:** Fri. Apr. 4, 2014

**Due:** Wed. April 9, 2014, in class

**Reading:** Section 3.5 of Roberge's *Operational Amplifiers*, attached to this problem set.  
This excerpt catalogs relationships for time- and frequency-domain parameters.

The following problems are assigned to both 2.14 and 2.140 students.

**Problem 1** The circuit shown below is adapted from the Lab 5 circuit to include a resistor  $R_f$  which allows adding a zero to the loop transfer function  $L(s)$ . This allows independent adjustment of the loop crossover frequency and phase margin. Note that we assume an infinite gain model for the op-amp. Also, as before we assume that  $R_s \ll R_a$  and thus that  $i_a \approx 0$ .

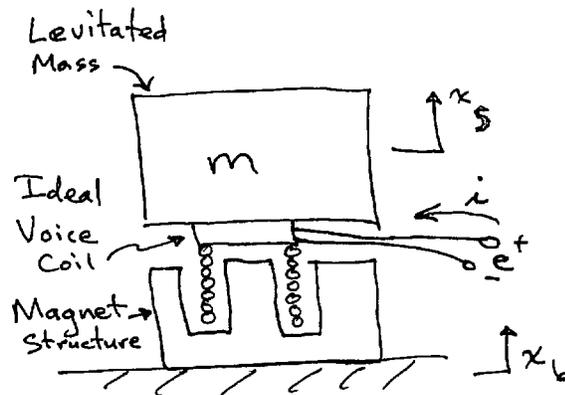


- a) Draw a block diagram which includes  $V_i$ ,  $V_o$ ,  $V_f$ ,  $i_i$ ,  $i_f$ , and  $i_a$ . The entries in the various blocks should be given in terms of the system parameters.
- b) Use the block diagram to write an expression for the loop return ratio  $L(s)$  for the current control loop. Draw a pole zero-diagram for  $L(s)$  with the singularity locations labeled in terms of the system parameters.  
For the remainder of this problem, assume that  $R_a = 1 \text{ k}\Omega$ ,  $R_s = 100 \text{ }\Omega$ ,  $R_L = 100 \text{ }\Omega$ , and  $L = 50 \text{ mH}$ .
- c) Choose  $R_i$  to set the DC gain such that  $i_L = -1 \text{ mA/V} \times V_i$ .
- d) Choose  $R_f$  and  $C_f$  to set the loop crossover at  $\omega_c = 10^4 \text{ rad/sec}$  with phase margin  $\phi_m = 45^\circ$ . Show your reasoning; you should be able to do this design with hand calculations. Use Matlab to create a Bode plot of  $L(s)$  which shows  $\omega_c$  and  $\phi_m$  for your chosen design values.
- e) Use Matlab to compute the transfer functions  $V_o(s)/V_i(s)$  and  $V_f(s)/V_i(s)$ . For this purpose, use of the `series` and `feedback` commands will be helpful. Create Bode plots for these two transfer functions and explain the asymptotic magnitude characteristics in terms of the transfer function poles and zeros.
- f) For the input-output transfer function  $V_f/V_i$  Bode plot, what is the -3 dB frequency  $\omega_h$ ? (See Roberge Figure 3.17) What is the frequency response peaking ratio  $M_p$ ? At what frequency

$\omega_p$  does this occur? What are the closed-loop damping ratio  $\zeta$  and natural frequency  $\omega_n$  of the dominant pair in  $V_f/V_i$ ? How do these values compare with those predicted via the approximations  $\zeta \approx \phi_m/100$  and  $\omega_n \approx \omega_c$ ?

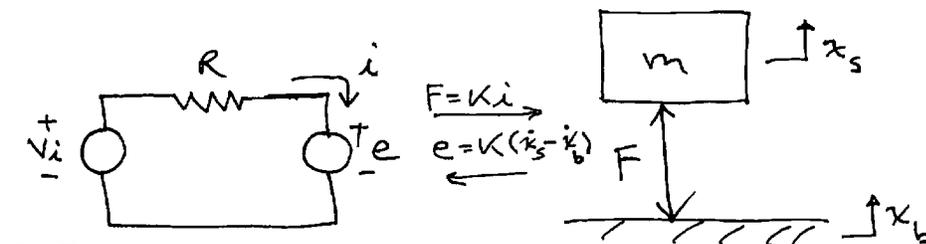
- g) Let  $V_i(t) = u_s(t)$ , a unit step. Use Matlab to compute  $V_o(t)$  and  $V_f(t)$  assuming zero initial conditions. Explain the character of the two responses in terms of the poles and zeros of the respective transfer functions. With respect to the response  $V_f(t)$  what is the 10–90% rise time  $t_r$ ? (See Roberge Figure 3.17). What is the ratiometric peak overshoot  $P_0$ ? What rise time would be predicted from  $t_r = 2.2/\omega_h$  (Roberge equation 3.57) on the basis of  $\omega_h$  determined in part e) above? How does this compare with the actual step response rise time? The relationship  $t_r = 2.2/\omega_h$  is extremely useful.

**Problem 2** This problem considers a one degree of freedom magnetic levitation system using a Lorentz actuator as shown below



In this figure, the motion of the levitated mass is given by  $x_s$  relative to inertial space. The supporting base which carries the magnet structure moves relative to inertial space as  $x_b$ . This base motion can be used to model floor vibration, which is important to consider in precision applications. We assume no gravity in this problem. The objective of the levitation system is to control  $x_s$  to follow a desired trajectory without interference from base vibrations. In this problem, we consider the effect of using voltage- or current-control for the actuator drive relative to the effect of base vibrations.

The Lorentz actuator is considered to be ideal, and thus to have no inductance or resistance, with coil current  $i$  and back emf voltage  $e$ . The actuator is driven by a voltage source  $V_i$  through a series resistance  $R$ . The coupled electromechanical system can be represented as shown below



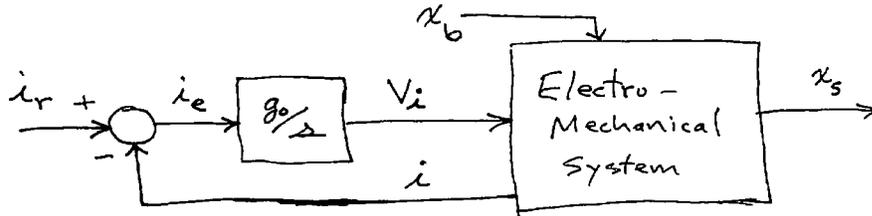
- a) **Voltage Drive** Draw a block diagram which includes  $V_i$ ,  $i$ ,  $e$ ,  $F$ ,  $x_s$ ,  $\dot{x}_s$ ,  $\ddot{x}_s$  and  $x_b$ .

Use this block diagram to calculate the transfer function  $X_s(s)/X_b(s)$  in terms of the system parameters. You may assume  $V_i = 0$  to aid in block diagram reduction, but note that this is not strictly necessary, given that the system model is linear.

On the basis of this transfer function, what equivalent mechanical element is connected between the base and the levitated mass? How does this result in transmission of base vibration into the levitated mass?

Hand sketch a Bode plot for  $X_s(s)/X_b(s)$  with values labeled in terms of the system parameters. Indicate how the base vibration transmission is shown in the Bode plot.

- b) **Current Drive** We now close a current control loop on the coil by driving  $V_i$  on the basis of measurement of coil current  $i$ . In this context, we assume that  $i$  is directly measured, and that  $V_i$  is a dependent voltage source whose value is set by the current controller, *i.e.*, a power amplifier. In this simple model, we will assume that the coil has no inductance, and thus that the current can be controlled by a pure integral controller  $G_c(s) = g_0/s$ . Draw a modified block diagram which includes the same signals as in part a), but adds the current reference  $i_r$ , current error  $i_e$ , and controller transfer function  $G_c(s) = g_0/s$ .



- c) We will assume that  $g_0$  is large enough that the current-loop dynamics are very fast relative to the mechanical subsystem dynamics. Under this assumption, we assert that the current loop can be designed with the mass motion set to zero,  $x_s = 0$ . Calculate the open loop plant transfer function  $I(s)/V_i(s)$ , and determine its high-frequency asymptotic behavior. Sketch a Bode plot and show the high-frequency region.

Use this high-frequency model to write an approximate expression for the current-loop crossover frequency and phase margin as they depend upon  $g_0$ , and thus argue that the current loop will be stable for this controller and for a wide range of gains  $g_0$ .

- d) Use your modified block diagram to calculate the transfer function  $X_s(s)/X_b(s)$  in terms of the system parameters with the current control loop active. You may assume  $i_r = 0$  to aid in block diagram reduction, but note that this is not strictly necessary, given that the system model is linear.

On the basis of this transfer function, what equivalent mechanical element is connected between the base and the levitated mass? How does this this element vary with current loop gain  $g_0$ ?

Hand sketch a Bode plot for  $X_s(s)/X_b(s)$  with values labeled in terms of the system parameters, and with several overlaid plots for varying  $g_0$ . How does the current loop bandwidth affect the ability of base vibrations to affect the levitated platform motion  $x_s$ ?

quency scale in Fig. 3.14*a*. These factors are combined to yield the Bode plot for the complete transfer function in Fig. 3.14*b*. The same information is presented in gain-phase form in Fig. 3.15.

### 3.5 RELATIONSHIPS BETWEEN TRANSIENT RESPONSE AND FREQUENCY RESPONSE

It is clear that either the impulse response (or the response to any other transient input) of a linear system or its frequency response completely characterize the system. In many cases experimental measurements on a closed-loop system are most easily made by applying a transient input. We may, however, be interested in certain aspects of the frequency response of the system such as its *bandwidth* defined as the frequency where its gain drops to 0.707 of the midfrequency value.

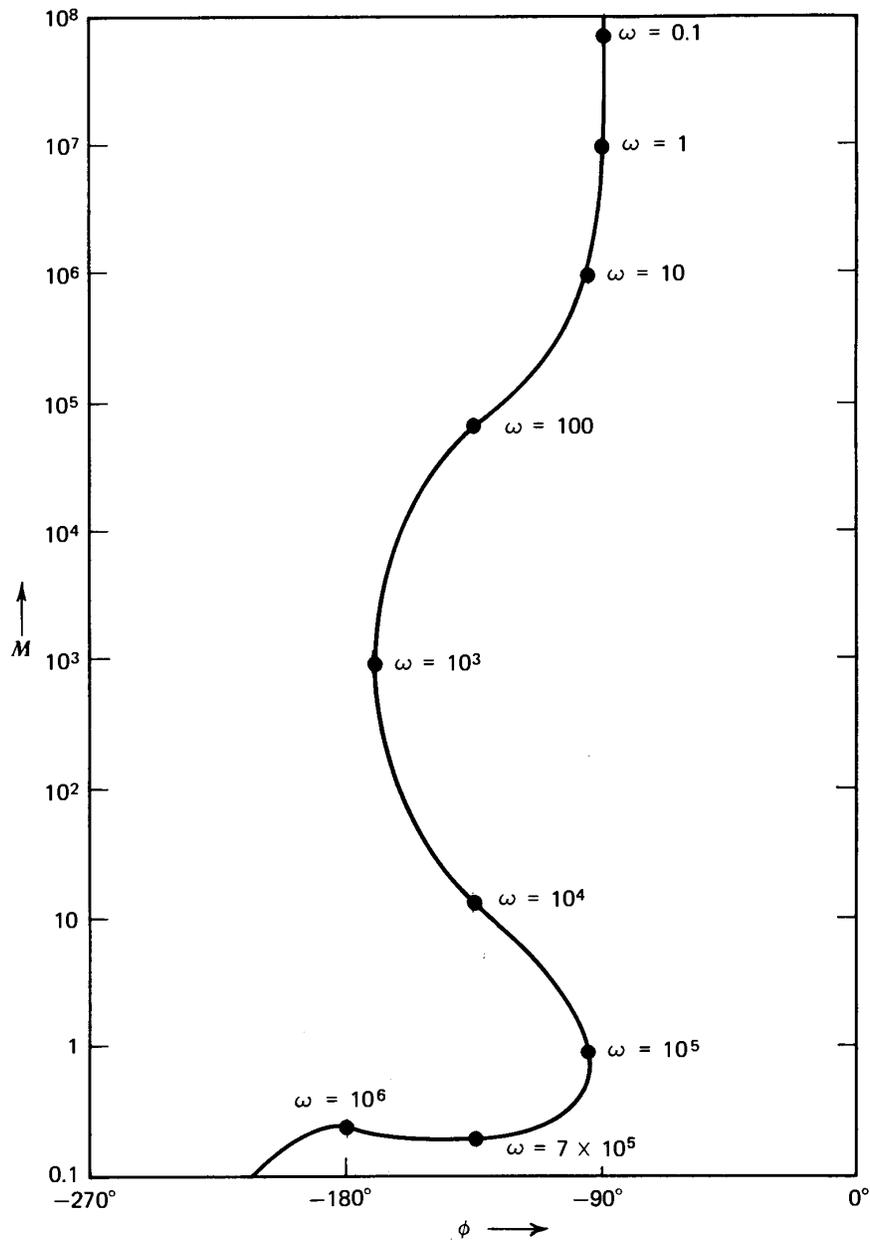
Since either the transient response or the frequency response completely characterize the system, it should be possible to determine performance in one domain from measurements made in the other. Unfortunately, since the measured transient response does not provide an equation for this response, Laplace techniques cannot be used directly unless the time response is first approximated analytically as a function of time. This section lists several approximate relationships between transient response and frequency response that can be used to estimate one performance measure from the other. The approximations are based on the properties of first- and second-order systems.

It is assumed that the feedback path for the system under study is frequency independent and has a magnitude of unity. A system with a frequency-independent feedback path  $f_0$  can be manipulated as shown in Fig. 3.16 to yield a scaled, unity-feedback system. The approximations given are valid for the transfer function  $V_a/V_i$ , and  $V_o$  can be determined by scaling values for  $V_a$  by  $1/f_0$ .

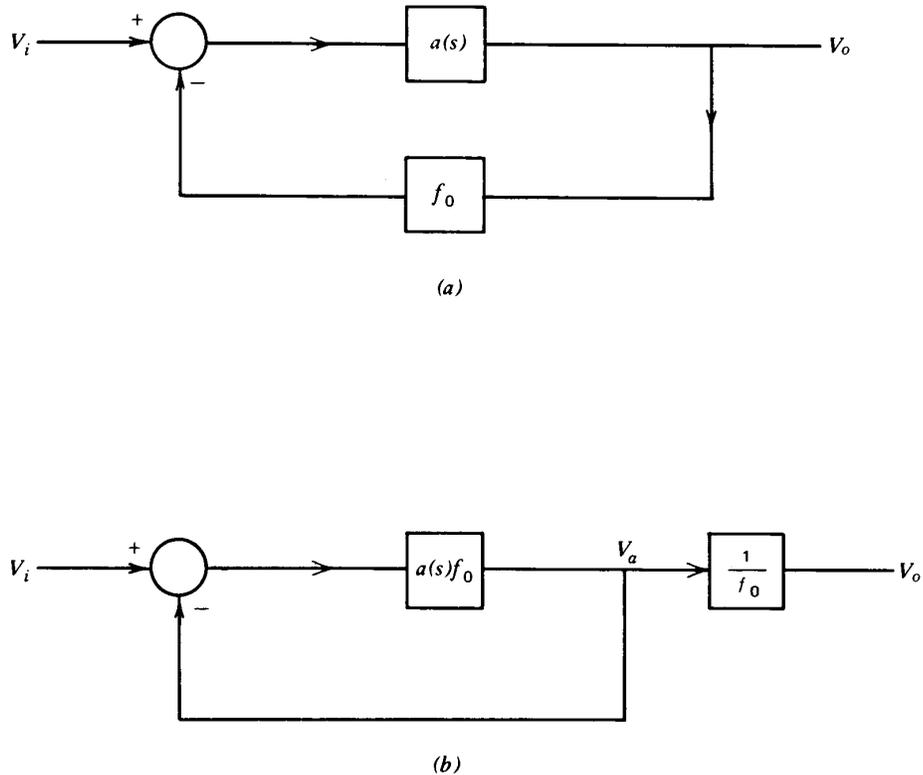
It is also assumed that the magnitude of the d-c loop transmission is very large so that the closed-loop gain is nearly one at d-c. It is further assumed that the singularity closest to the origin in the  $s$  plane is either a pole or a complex pair of poles, and that the number of poles of the function exceeds the number of zeros. If these assumptions are satisfied, many practical systems have time domain-frequency domain relationships similar to those of first- or second-order systems.

The parameters we shall use to describe the transient response and the frequency response of a system include the following.

(a) Rise time  $t_r$ . The time required for the step response to go from 10 to 90% of final value.



**Figure 3.15** Gain phase plot of  $\frac{10^7(10^{-4}s + 1)}{s(0.1s + 1)(s^2/10^{12} + 2(0.2)s/10^6 + 1)}$ .



**Figure 3.16** System topology for approximate relationships. (a) System with frequency-independent feedback path. (b) System represented in scaled, unity-feedback form.

- (b) The maximum value of the step response  $P_0$ .
  - (c) The time at which  $P_0$  occurs  $t_p$ .
  - (d) Settling time  $t_s$ . The time after which the system step response remains within 2% of final value.
  - (e) The error coefficient  $e_1$ . (See Section 3.6.) This coefficient is equal to the time delay between the output and the input when the system has reached steady-state conditions with a ramp as its input.
  - (f) The bandwidth in radians per second  $\omega_h$  or hertz  $f_h$  ( $f_h = \omega_h/2\pi$ ). The frequency at which the response of the system is 0.707 of its low-frequency value.
  - (g) The maximum magnitude of the frequency response  $M_p$ .
  - (h) The frequency at which  $M_p$  occurs  $\omega_p$ .
- These definitions are illustrated in Fig. 3.17.

For a first-order system with  $V_o(s)/V_i(s) = 1/(\tau s + 1)$ , the relationships are

$$t_r = 2.2\tau = \frac{2.2}{\omega_h} = \frac{0.35}{f_h} \quad (3.51)$$

$$P_0 = M_p = 1 \quad (3.52)$$

$$t_p = \infty \quad (3.53)$$

$$t_s = 4\tau \quad (3.54)$$

$$e_1 = \tau \quad (3.55)$$

$$\omega_p = 0 \quad (3.56)$$

For a second-order system with  $V_o(s)/V_i(s) = 1/(s^2/\omega_n^2 + 2\zeta s/\omega_n + 1)$  and  $\theta \triangleq \cos^{-1}\zeta$  (see Fig. 3.7) the relationships are

$$t_r \simeq \frac{2.2}{\omega_h} = \frac{0.35}{f_h} \quad (3.57)$$

$$P_0 = 1 + \exp \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = 1 + e^{-\pi/\tan\theta} \quad (3.58)$$

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n \sin \theta} \quad (3.59)$$

$$t_s \simeq \frac{4}{\zeta\omega_n} = \frac{4}{\omega_n \cos \theta} \quad (3.60)$$

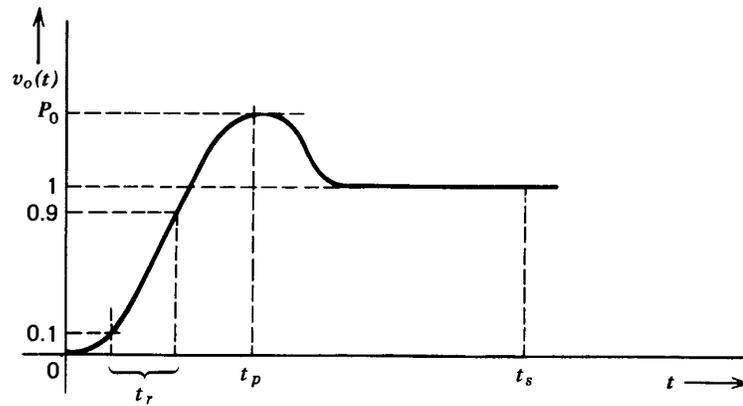
$$e_1 = \frac{2\zeta}{\omega_n} = \frac{2 \cos \theta}{\omega_n} \quad (3.61)$$

$$M_p = \frac{1}{2\zeta \sqrt{1-\zeta^2}} = \frac{1}{\sin 2\theta} \quad \zeta < 0.707, \theta > 45^\circ \quad (3.62)$$

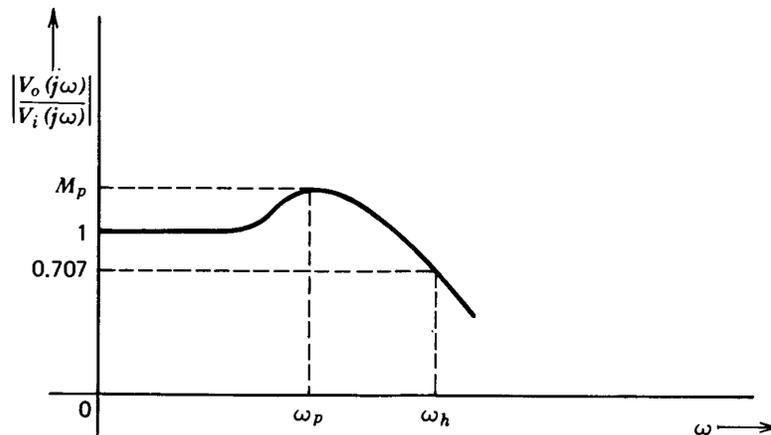
$$\omega_p = \omega_n \sqrt{1-2\zeta^2} = \omega_n \sqrt{-\cos 2\theta} \quad \zeta < 0.707, \theta > 45^\circ \quad (3.63)$$

$$\omega_h = \omega_n \left(1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right)^{1/2} \quad (3.64)$$

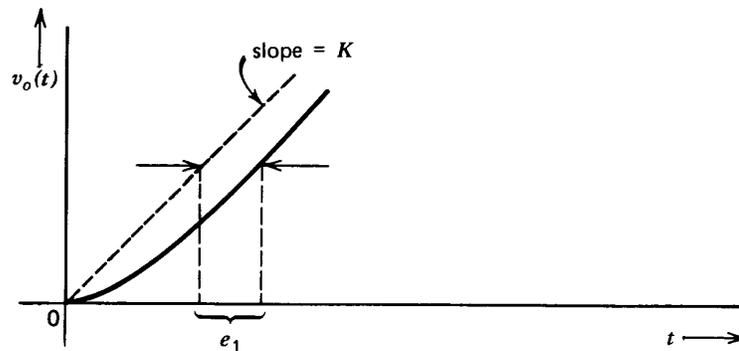
If a system step response or frequency response is similar to that of an approximating system (see Figs. 3.6, 3.8, 3.11, and 3.12) measurements of  $t_r$ ,  $P_0$ , and  $t_p$  permit estimation of  $\omega_h$ ,  $\omega_p$ , and  $M_p$  or vice versa. The steady-state error in response to a unit ramp can be estimated from either set of measurements.



(a)



(b)



(c)

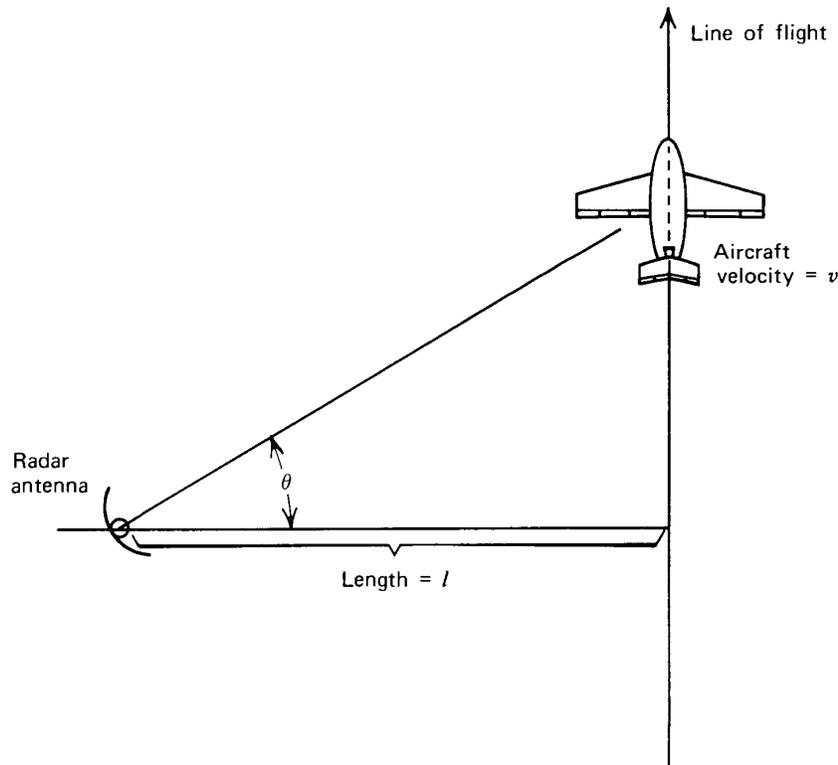
**Figure 3.17** Parameters used to describe transient and frequency responses. (a) Unit-step response. (b) Frequency response. (c) Ramp response.

One final comment concerning the quality of the relationship between 0.707 bandwidth and 10 to 90% step risetime (Eqns. 3.51 and 3.57) is in order. For virtually any system that satisfies the original assumptions, independent of the order or relative stability of the system, the product  $t_r f_h$  is within a few percent of 0.35. This relationship is so accurate that it really isn't worth measuring  $f_h$  if the step response can be more easily determined.

### 3.6 ERROR COEFFICIENTS

The response of a linear system to certain types of transient inputs may be difficult or impossible to determine by Laplace techniques, either because the transform of the transient is cumbersome to evaluate or because the transient violates the conditions necessary for its transform to exist. For example, consider the angle that a radar antenna makes with a fixed reference while tracking an aircraft, as shown in Fig. 3.18. The pointing angle determined from the geometry is

$$\theta = \tan^{-1} \left[ \frac{v}{l} t \right] \quad (3.65)$$



**Figure 3.18** Radar antenna tracking an airplane.

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