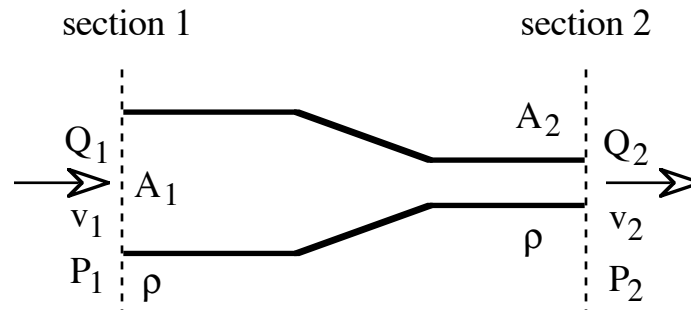


NETWORK MODELS OF BERNOULLI'S EQUATION

The phenomenon described by Bernoulli's equation arises from momentum transport due to mass flow.

EXAMPLE: A PIPE OF VARYING CROSS-SECTION.



Assume:

- **incompressible flow**
- **slug flow**
- **lossless flow**

Mass balance:

$$Q_1 = A_1 v_1 = Q_2 = A_2 v_2$$

Consider kinetic (co-)energy flux at each end:

$$dE_{k,1}^* = \frac{1}{2} \rho A_1 dx_1 (v_1^2)$$

$$\dot{E}_{k,1}^* = \frac{1}{2} \rho A_1 v_1 (v_1^2) = \frac{1}{2} \rho \frac{Q_1^3}{A_1^2}$$

$$\dot{E}_{k,2}^* = \frac{1}{2} \rho A_2 v_2 (v_2^2) = \frac{1}{2} \rho \frac{Q_2^3}{A_2^2}$$

Thus because $Q_1 = Q_2$,

if $A_1 > A_2$ then $\dot{E}_{k,2}^* > \dot{E}_{k,1}^*$

The extra kinetic energy must come from somewhere.

It comes from work done on the fluid.

Power balance:

$$P_1 Q_1 + \frac{1}{2} \rho \frac{Q_1^3}{A_1^2} = P_2 Q_2 + \frac{1}{2} \rho \frac{Q_2^3}{A_2^2}$$

Rearranging:

$$\left(P_1 + \frac{1}{2} \rho \frac{Q_1^2}{A_1^2} \right) Q_1 = \left(P_2 + \frac{1}{2} \rho \frac{Q_2^2}{A_2^2} \right) Q_2$$

$$\left(P_1 + \frac{1}{2} \rho v_1^2 \right) Q_1 = \left(P_2 + \frac{1}{2} \rho v_2^2 \right) Q_2$$

Define:

$$P_{\text{dynamic}} = \frac{1}{2} \rho v_1^2$$

$$P_{\text{hydraulic}} = P_{\text{static}} + P_{\text{dynamic}}$$

Net power flux:

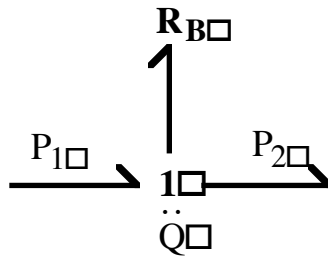
$$P_{\text{hydraulic}} Q$$

NETWORK REPRESENTATION

HOW DO YOU DEPICT THIS PHENOMENON IN A NETWORK (MODEL?)

One possibility is to define a “Bernoulli resistor”

(see Karnopp, D. C. (1972) “Bond Graph Models for Fluid Dynamic Systems.” ASME J. Dyn. Sys. Meas. & Cont. pp. 222-229; Karnopp, D. C, Margolis, D. L. & Rosenberg, R. C. (1990) System Dynamics: A Unified Approach, 2nd. Ed. Wiley Interscience).



The constitutive equation of the “Bernoulli resistor” is defined as

$$P_{\text{Bernoulli}} = \frac{1}{2} \rho (v_2^2 - v_1^2) = \frac{1}{2} \rho \left(\frac{1}{A_2^2} - \frac{1}{A_1^2} \right) Q^2$$

This element is called a “resistor” because it relates a pressure drop to a flow rate.

THIS APPROACH YIELDS THE RIGHT EQUATIONS BUT IT HAS SEVERAL UNSATISFACTORY ASPECTS.

- **The “Bernoulli resistor” does not dissipate free energy.**

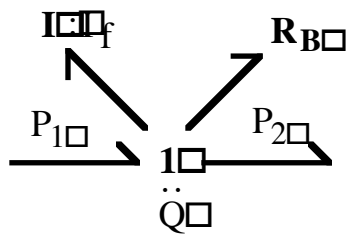
In fact, this “resistor” violates an important constraint on resistor constitutive equations —it may *supply* as well as *absorb* power.

To be fair, that flaw could be rectified by a suitable change of terminology.

- **Hydraulic pressure cannot be represented explicitly. As a result, the bond graph seems to suggest that power flux is $P_{\text{static}} Q$, not $P_{\text{hydraulic}} Q$.**

- The “Bernoulli resistor” is not related to the kinetic energy from which it arises.

To model the power required to accelerate or decelerate the fluid, we may add a fluid inertia



the fluid inertia appears to be independent of the dynamic pressure effects

but they are different aspects of the same phenomenon

—kinetic energy stored in the fluid

AN ALTERNATIVE (AND SUPERIOR) APPROACH:

Carefully analyze the kinetic energy stored in the pipe.

The co-energy is

$$E_k^* = E_k^*(Q,m) = \frac{1}{2} I_f Q^2$$

where I_f depends on the specific geometry but is proportional to m .

The corresponding “pressure momentum”, Γ , is defined by

$$\Gamma = \left. \frac{\partial E_k^*}{\partial Q} \right|_m = I_f Q$$

The kinetic energy may be found using a (negative) Legendre transform.

$$E_k(\Gamma,m) = E_k^*(Q,m) - \Gamma Q$$

This kinetic energy storage element has *three* ports.

One is due to the power required to accelerate the fluid.

It is associated with the change of pressure momentum, Γ , and the conjugate (equilibrium-determining) variable is the volumetric flow rate, Q .

The other two ports are due to the mass flows at sections 1 and 2.

The corresponding conjugate variables are found from the gradient of the kinetic energy with respect to mass.

In this case the kinetic energy and co-energy are numerically equal.

$$\left. \frac{\partial E_k}{\partial m} \right|_{\Gamma} = \left. \frac{\partial E_k^*}{\partial m} \right|_Q$$

It is easier to work with the kinetic co-energy.

The effort corresponding to the mass flow, \dot{m}_i , is the kinetic (co-)energy flux per unit mass.

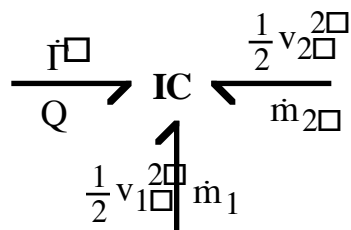
$$\mu_i^{\text{kin}} = \frac{1}{2} v_i^2 = \frac{1}{\rho} P_{\text{dynamic},i}$$

As the total mass of fluid in the pipe is proportional to the total volume of the pipe

and the corresponding effort is proportional to the dynamic pressure,

the mass flow ports have the character of a capacitor while the momentum port has the character of an inertia.

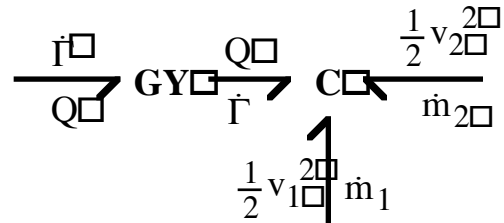
This may be represented as an IC-type storage element as follows.



AN ALTERNATIVE REPRESENTATION

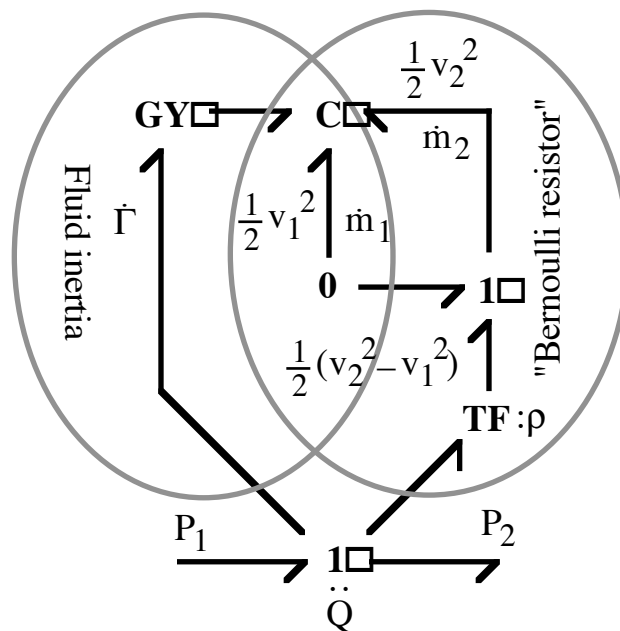
(perhaps less ambiguous)

a three-port capacitor with a unit gyrator on one port.



As the fluid is assumed incompressible, we need to add a constraint that the mass flow in equals the mass flow out.

This may be represented by a junction structure as follows.



The transformer relates mass flow rate, \dot{m} , to volumetric flow rate, Q .

$$\dot{m} = \rho Q$$

It also relates the “Bernoulli pressure” to the difference of the kinetic efforts.

$$P_{\text{Bernoulli}} = \frac{1}{2} \rho (v_2^2 - v_1^2)$$

The zero and one junctions represent the incompressibility constraint.

$$\dot{m}_2 = -\dot{m}_1$$

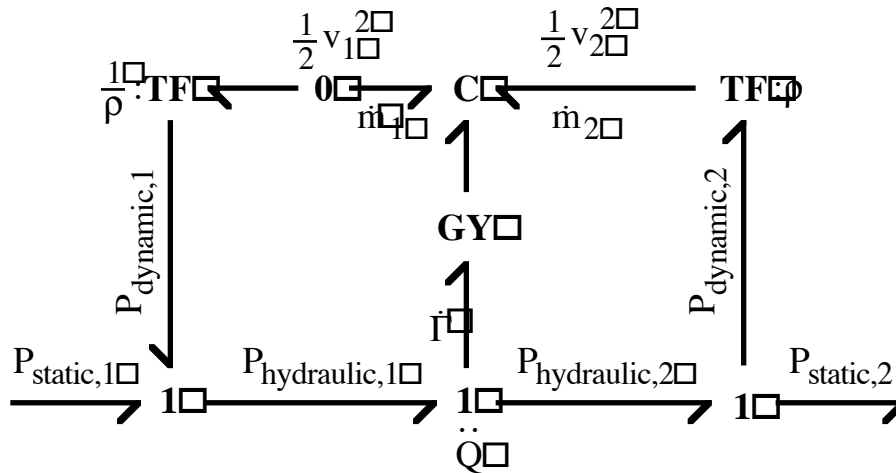
The pattern of power orientations has been chosen to resemble that used with the “Bernoulli resistor” above.

With this orientation, the left and right sides of the energy-storage element correspond to the fluid inertia and “Bernoulli resistor” as indicated.

**The multiport representation has two merits
the lossless assumption is self-evident
the fundamental relation between the two phenomena
—fluid inertia and “dynamic pressure”—
has been represented explicitly.**

Hydraulic pressure has not been represented explicitly.

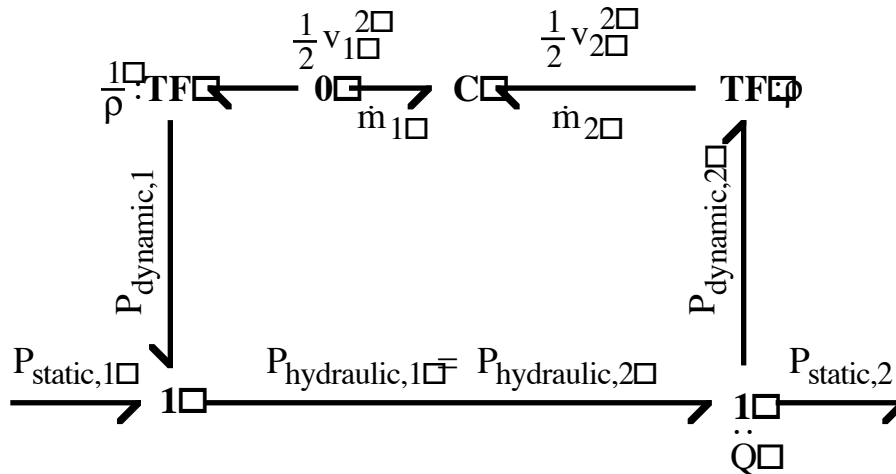
Revise the bond graph as follows.



This representation clearly shows that the rate of change of pressure momentum is driven by the difference in *hydraulic* pressures.

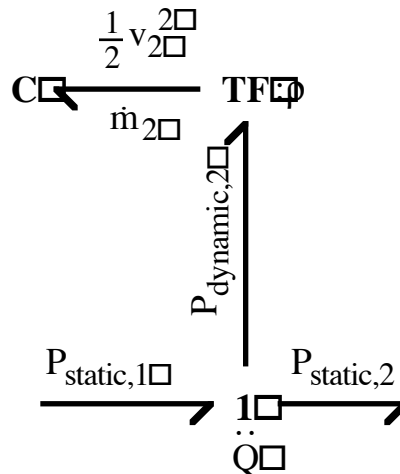
$$\frac{d\Gamma}{dt} = P_{hydraulic,1} - P_{hydraulic,2}$$

If we assume steady flow, $\frac{d\Gamma}{dt} = 0$ and hence $\Delta P_{\text{hydraulic}} = 0$ and we may eliminate the momentum port as follows.



This representation clarifies the transition from static fluid storage to steady fluid flow.

For example, it is common to assume that the kinetic energy flux at the inlet section may be neglected, resulting in the following model.



This is an energetically correct model of an orifice flow “resistor”.

If the exit kinetic energy flux is discarded, the result is a loss of available energy as in a dissipator.

However, as indicated by the capacitor, that energy has been stored and can be recovered, which is what happens in a diffuser.

REMARK:

One curious aspect of this depiction of Bernoulli's equation is the difficulty of identifying the net power flow.

For example, if we consider “vertical sections” of the graphs, reading from left to right we first encounter P_{static} Q , then $P_{\text{hydraulic}}$ Q .

This is related to the fact that the kinetic energy storage is fundamentally coupled to the flow phenomenon.

It also shows that “simple” pipe flow may not be as simple as it looks.

For example, incompressible flow through a simple pipe “tee” junction appears on first examination to exhibit the behavior of a three-port common-flow (type one) junction and, furthermore, to be equivalent to a pipe “wye” junction.

However, even at modest flow rates, that is far from correct.

In fact, these components (especially the “wye” junction) can be used as primitive amplifiers and switching elements.