ENERGY-STORING COUPLING BETWEEN DOMAINS

MULTI-PORT ENERGY STORAGE ELEMENTS

Context: examine limitations of some basic model elements.

EXAMPLE:

open fluid container with deformable walls

$$P = \rho\, g\, h$$

$$h = A\, V$$

$$V = C_f\, P$$

where $C_f = \dfrac{A}{\rho\, g}$

—fluid capacitor

But when squeezed, h (and hence P) may vary with time even though V does not.

Seems to imply $C_f = C_f(t)$

i.e., $C_f = \dfrac{A(t)}{\rho\, g}$

—apparently a "modulated capacitor"

PROBLEM!

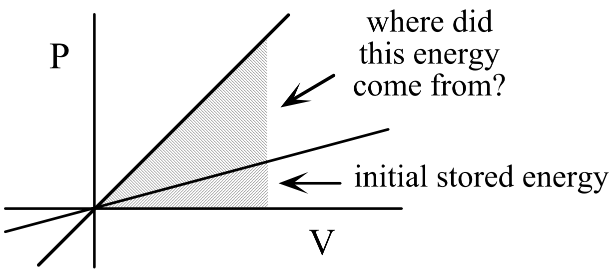
$$E_p = \frac{V^2}{2\ C_f}$$

V is constant, therefore no (pressure) work done

$$dV = 0 \ \therefore\ PdV = 0$$

—yet (stored) energy may change

This would violate the first law (energy conservation)

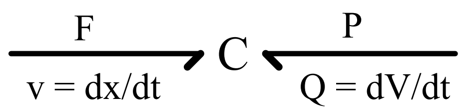


—a BIG problem!

MODULATED ENERGY STORAGE IS PROSCRIBED!

SOLUTION

Identify another power port to keep track of the work done to change the stored energy



introduces a new network element: a *multiport capacitor*

Mathematical foundations:

Power variables:

Each power port must have properly defined conjugate power and energy variables.

Net input power flow is the sum of the products of effort and flow over all ports.

$$P = \sum_i e_i \, f_i$$

In vector notation:

$$\mathbf{e} \triangleq \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix} \qquad \mathbf{f} \triangleq \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$$

$$P = \mathbf{e}^t \, \mathbf{f} = \mathbf{f}^t \, \mathbf{e}$$

Energy variables

Energy variables are defined as in the scalar case as time integrals of the flow and effort vectors respectively.

generalized displacement

$$\mathbf{q} = \int \mathbf{f}\, dt + \mathbf{q}_o$$

generalized momentum

$$\mathbf{p} = \int \mathbf{e}\, dt + \mathbf{p}_o$$

MULTI-PORT CAPACITOR

A "vectorized" or multivariable generalization of the one-port capacitor.

definition

A *multiport capacitor* is defined as an entity for which effort is a single-valued (integrable) function of displacement .

$$\mathbf{e} = \boldsymbol{\Phi}(\mathbf{q})$$

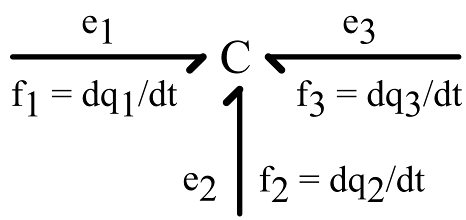
The vector function $\boldsymbol{\Phi}(\cdot)$ is the capacitor constitutive equation.

—a *vector field* (in the mathematical sense of the word).

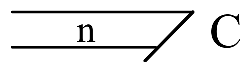
A multi-port capacitor is sometimes called a C-field or capacitive field.

BOND GRAPH NOTATION

By convention, power is defined positive into all ports.



An alternative notation:



n denotes the number of ports.

More on this *multi-bond* notation later.

STORED ENERGY:

determined by integrating the constitutive equation.

$$E_p - E_{po} = \int \mathbf{e}^t\mathbf{f}\, dt = \int \mathbf{e}^t d\mathbf{q} = \int \mathbf{\Phi}(\mathbf{q})^t d\mathbf{q} = E_p(\mathbf{q})$$

potential energy, as it is a function of displacement

—a function of as many displacements as there are ports.

COUPLING BETWEEN PORTS.

Each effort may depend on any or all displacements.

$$e_i = \Phi_i(q_1,q_2,\ldots q_n) \text{all } i$$

This coupling between ports is constrained.

Mathematically:

Energy stored is a scalar function of vector displacement.

Stored energy is a *scalar potential field.*

The effort vector is the gradient of this potential field.

$$\mathbf{e} = \nabla_{\mathbf{q}} E_p(\mathbf{q})$$

Therefore the constitutive equation, $\mathbf{\Phi}(\cdot)$, must have *zero curl.*

$$\nabla \times \mathbf{e} = \mathbf{0}$$

or

$$\nabla \times \mathbf{\Phi} = \mathbf{0}$$

In terms of the individual efforts and displacements,

$$e_i = \frac{\partial E_p}{\partial q_i} \qquad \text{all i}$$

$$\frac{\partial e_i}{\partial q_j} = \frac{\partial}{\partial q_j}\frac{\partial E_p}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{\partial E_p}{\partial q_j} = \frac{\partial e_j}{\partial q_i} \qquad \text{all i, j.}$$

Coupling between ports must be symmetric.

The dependence of e_i on q_j must be identical to the dependence of e_j on q_i.

This is known as *Maxwell's reciprocity* condition.

Later we will see that *stability* and *passivity* further constrain the capacitor constitutive equation.

EXAMPLE: "CONDENSER" MICROPHONE

HEADS UP!

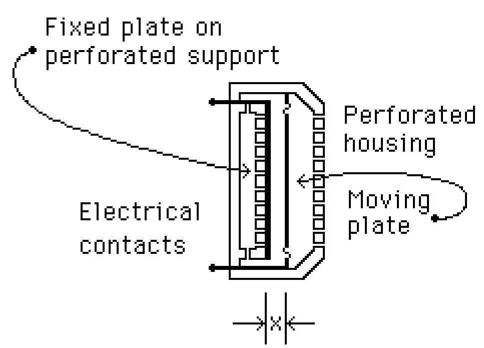
There's an error in what follows — see if you can spot it.

The sketch depicts a simple electro-mechanical transducer, a "condenser" microphone.

—essentially a moving-plate capacitor.

Electrically a capacitor, but capacitance varies with plate separation.

Mechanically, electric charge pulls the plates together.



Schematic of condenser microphone

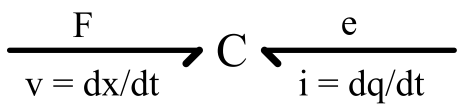
DEVICE CAN STORE ENERGY.

Energy can change in two ways:

mechanical displacement

charge displacement

—a two-port capacitor



like a spring in the mechanical domain

like a capacitor in the electrical domain

Two constitutive equations needed

both relate effort to displacement

$e = e(q, x)$

$F = F(q, x)$

Assuming electrical linearity:

$$e = \frac{q}{C(x)}$$

To find C(x) assume a pair of parallel plates very close together.

(i.e., plates are very large compared to their separation)

(Fringing effects can be handled in a completely analogous way)

$$C = \varepsilon_0 A/x$$

ε_o is a permittivity

A is plate area

One constitutive equation is

$$e = \frac{qx}{\varepsilon A}$$

To find the other constitutive equation we could work out the attraction due to the charges on the plates ...

AN EASIER WAY:

—use the energy equation

$$E_p = E_p(q, x)$$

capacitor effort = gradient of stored energy.

(by definition)

energy is the same in all domains.

compute energy in the electrical domain

(easy)

gradient with respect to plate separation = force

(also easy)

Stored electrical energy:

$$E_{electrical} = \frac{1}{2}\ C\ e^2 = \frac{1}{2}\frac{\varepsilon A}{x}\ e^2$$

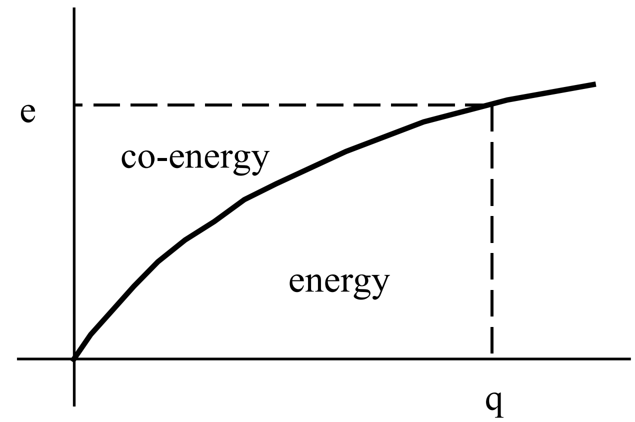
Gradient:

$$F = \frac{\partial E_{electrical}}{\partial x} = -\frac{1}{2}\frac{\varepsilon A}{x^2}\ e^2$$

Why the minus?

—a sign error!

ENERGY AND CO-ENERGY



There are two ways to integrate the capacitor constitutive equation.

—only one of them is energy

—the other is co-energy

energy:

$$E_p(\mathbf{q}) \triangleq \int \mathbf{e}^t d\mathbf{q}$$

$\frac{q^2}{2\,C}$ is electrical energy

co-energy:

$$E_p^*(\mathbf{q}) \triangleq \int \mathbf{q}^t d\mathbf{e}$$

$\frac{1}{2}\,C\,e^2$ is electrical *co-energy*

THE ERROR WAS TO CONFUSE ENERGY WITH CO-ENERGY

Stored electrical energy:

$$E_p = \frac{q^2 x}{2\varepsilon A}$$

gradient of energy with respect to plate separation:

$$F = \frac{\partial E_p}{\partial x} = \frac{q^2}{2\varepsilon A}$$

Sign error corrected, but ...

this equation implies *force is independent of plate separation.*

IS THAT PHYSICALLY REASONABLE?

Shouldn't electrostatic attraction weaken as plate separation increases?

Would the plates pull together just as hard if they were infinitely far apart?

A PARADOX?

Cross-check:

are the two constitutive equations reciprocal (symmetric)?

partial derivatives

$$\frac{\partial F}{\partial q} = \frac{q}{\varepsilon A}$$

$$\frac{\partial e}{\partial x} = \frac{q}{\varepsilon A}$$

As required, the constitutive equations are reciprocal.

WHAT'S WRONG?

A “PARADOX” RESOLVED:

A clue: the electrical constitutive equation

$$e = \frac{qx}{\varepsilon A}$$

voltage drop increases with plate separation.

for a fixed charge, infinite separation requires infinite voltage.

—NOT THE USUAL ARRANGEMENT

real devices cannot sustain arbitrarily large voltages.

Change "boundary conditions" to input voltage:

$$q = \frac{e\varepsilon A}{x}$$

$$F = \frac{1}{2\varepsilon A}\left(\frac{e\varepsilon A}{x}\right)^2 = \frac{e^2\varepsilon A}{2x^2}$$

For fixed voltage, force between plates declines sharply with separation.

—much more plausible

Mechanically, a spring

—albeit a highly nonlinear one.

KEY POINT:

BOUNDARY CONDITIONS PROFOUNDLY INFLUENCE BEHAVIOR

CAUSAL ASSIGNMENT

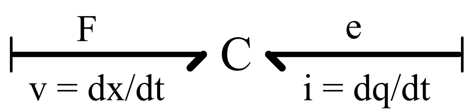
Different "boundary conditions" correspond to different causal assignments.

displacement in, effort out on both ports

$$e = e(q, x) = \frac{qx}{\varepsilon A}$$

$$F = F(q, x) = \frac{q^2}{2\varepsilon A}$$

—Integral causality



Energy function:

$$E_p = E_p(q,x) = \frac{q^2 x}{2\varepsilon A}$$

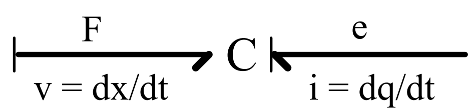
Expressing force as a function of voltage and displacement is equivalent to changing the electrical boundary conditions.

voltage in, charge out on the electrical port.

$$q = e(e, x) = \frac{e\varepsilon A}{x}$$

$$F = F(e, x) = \frac{q^2}{2\varepsilon A}$$

—differential causality on the electrical port.

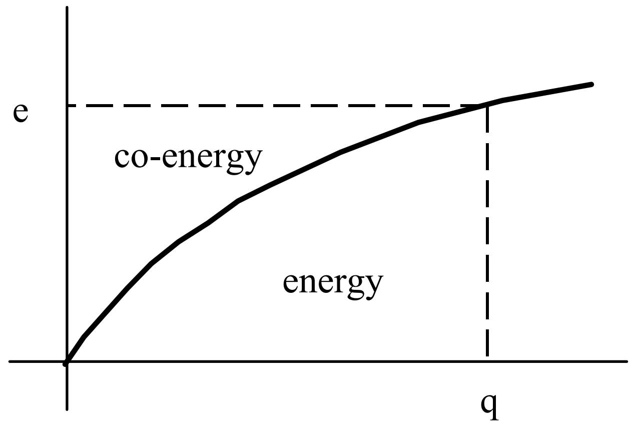


Co-energy function:

$$E_p^* = E_p^*(e,x) = \frac{e^2\varepsilon A}{2x}$$

CO-ENERGY AND LEGENDRE TRANSFORMATIONS

Energy and co-energy are related by a Legendre transformation



The rectangle eq is the sum of energy and co-energy.

$$eq = E_p(q) + E_p^*(e)$$

Re-arranging:

$$E_p^*(e) = eq - E_p(q)$$

This is the negative of a Legendre transformation

$$L\{E_p(q)\} = E_p(q) - \left(\frac{\partial E_p}{\partial q}\right) q = -E^*(e)$$

Commonly used in thermodynamics

SINGLE-PORT CAPACITOR:

Constitutive equation

$$e = \Phi(q)$$

Energy equation

$$E_p = E_p(q) = \int \Phi(q)dq$$

$$e = \partial E_p(q)/\partial q$$

Co-energy equation

$$E_p^*(e) = \int \Phi^{-1}(e)de$$

Legendre transformation

$$L\{E_p(q)\}_e = E_p(q) - eq = -E_p^*(e)$$

thus

$$E_p^*(e) = eq - E_p(q)$$

Partial differential with respect to e

$$\partial E_p^*(e)/\partial e = q$$

Note the *positive* sign.

TWO-PORT CAPACITOR:

Constitutive equations

$$e_1 = \Phi(q_1,q_2)$$

$$e_2 = \Phi(q_1,q_2)$$

Energy equation

$$E_p = E_p(q_1,q_2)$$

$$e_1 = \partial E_p/\partial q_1$$

$$e_2 = \partial E_p/\partial q_2$$

Co-energy equations: *three* possibilities

$$E_p^* = E_p^*(e_1,q_2)$$

$$E_p^* = E_p^*(q_1,e_2)$$

$$E_p^* = E_p^*(e_1,e_2)$$

Legendre transformation applied to port 1

$$E_p^*(e_1,q_2) = e_1 q_1 - E_p(q_1,q_2)$$

Partial differential with respect to e_1

$$\partial E_p^*(e_1,q_2)/\partial e_1 = q_1$$

Partial differential with respect to q_2

$$\partial E_p^*(e_1,q_2)/\partial q_2 = - \partial E_p(q_1,q_2)/\partial q_2$$

$$\partial E_p^*(e_1,q_2)/\partial q_2 = - e_2$$

Note the *negative* sign.

APPLY TO THE CONDENSER MICROPHONE

Energy:

$$E_p = \frac{q^2 x}{2\varepsilon A}$$

Legendre transform:

$$L\left\{\frac{q^2 x}{2\varepsilon A}\right\} = E_p(q) - eq = \frac{q^2 x}{2\varepsilon A} - eq = -E_p^*(e)$$

Substitute

$$q = \frac{e\varepsilon A}{x}$$

Co-energy:

$$E_p^*(e) = -\left(\frac{e\varepsilon A}{x}\right)^2 \frac{x}{2\varepsilon A} + e\left(\frac{e\varepsilon A}{x}\right) = \frac{e^2 \varepsilon A}{2x}$$

This is the “electrical energy” we had used previously. It is actually a *co-energy*. Thus

$$F = -\frac{\partial E_p{}^*}{\partial x} = \frac{e^2 \varepsilon A}{2x^2}$$

Note:

mechanical force is the *negative* gradient of electrical co-energy with respect to displacement.

—That fixes our sign error.

Comment:

In this simple (electrically linear) example, co-energy may as easily be determined without the Legendre transform by substitution for q in the energy equation.

REMARKS:

Even with the idealizing assumptions above

(no electrical saturation, no "fringe effects" in the electrostatic field)

the multi-port constitutive equations are

profoundly nonlinear

$$e = e(q, x) = \frac{qx}{\varepsilon A}$$

$$F = F(q, x) = \frac{e^2 \varepsilon A}{2x^2}$$

fundamentally coupled

$$\frac{\partial F}{\partial q} = \frac{\partial e}{\partial x} = \frac{q}{\varepsilon A} \neq 0 \text{ if } q \neq 0$$

That is, except when the stored charge is identically zero,

—the electrical domain affects the mechanical domain and

—the mechanical domain affects the electrical domain.

The condenser microphone is not well modeled by one-port energy storage elements in either the mechanical or the electrical domains.

Because of inter-domain coupling, this device serves is both

a sensor (a microphone)

or

an actuator (a speaker)

—It is commonly used for both purposes.

It is an example of an energy-storing transducer.

Energy may be stored or removed from either domain

Thus energy and power may be transferred between domains.

INTRINSIC STABILITY

Review the multi-port capacitor definition

$$\mathbf{e} = \Phi(\mathbf{q})$$

such that

$$E_p - E_{po} = \int \mathbf{\Phi}(\mathbf{q})^t d\mathbf{q} = E_p(\mathbf{q})$$

That is, $\mathbf{q}$ uniquely determines $\mathbf{e}$ and hence E_p

The converse is not required — $\mathbf{q}$ need not be a well-defined function of $\mathbf{e}$.

The constitutive equation(s) may be recovered by differentiation

$\mathbf{e} = \nabla_{\mathbf{q}} E_p(\mathbf{q})$

In other notation,

$$e_i = \frac{\partial E_p}{\partial q_i} \quad \text{all i}$$

The constitutive equations may be coupled

$$e_i = \Phi_i(q_1, q_2, \ldots q_n) \text{all i}$$

MAXWELL'S RECIPROCITY CONSTRAINT

The coupling is constrained such that

$$\nabla \times \mathbf{e} = \mathbf{0}$$

In other notation,

$$\frac{\partial e_i}{\partial q_j} = \frac{\partial}{\partial q_j}\frac{\partial E_p}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{\partial E_p}{\partial q_j} = \frac{\partial e_j}{\partial q_i} \qquad \text{all i, j.}$$

Define *inverse capacitance*

$$C^{-1} = \left[\frac{\partial \mathbf{e}}{\partial \mathbf{q}}\right] = \left[\frac{\partial^2 E_p}{\partial q_i \partial q_j}\right]$$

The inverse capacitance must be a *symmetric* matrix.

STABILITY

A physically observable multi-port capacitor must also be *intrinsically stable.*

A further constraint on the constitutive equations.

Mathematically:

Intrinsically stable if C^{-1} positive definite

—sufficient condition, not necessary

EXAMPLE: CONDENSER MICROPHONE (REVISITED)

Energy

$$E_p(q, x) = \frac{q^2 x}{2\varepsilon A}$$

Inverse capacitance

$$C^{-1} = \left[\frac{\partial \mathbf{e}}{\partial \mathbf{q}}\right] = \left[\frac{\partial^2 E_p}{\partial q_i \partial q_j}\right] = \begin{bmatrix} 0 & q/\varepsilon A \\ q/\varepsilon A & x/\varepsilon A \end{bmatrix}$$

Stability

$$\text{determinant } C^{-1} = -\left(\frac{q}{\varepsilon A}\right)^2$$

—Unstable for non-zero charge

PHYSICALLY REASONABLE

— STABILITY REQUIRES SOMETHING TO OPPOSE THE ATTRACTIVE ELECTROSTATIC FORCE.

Include elasticity of the supporting structure

Assume elastic linearity (for simplicity)

$$E_{elastic} = \frac{1}{2}\, k\, (x - x_o)^2$$

Energy (revised)

$$E_p(q, x) = \frac{q^2 x}{2\varepsilon A} + \frac{1}{2}\, k\, (x - x_o)^2$$

Constitutive equation

$$\mathbf{e} = \begin{bmatrix} F \\ e \end{bmatrix} = \begin{bmatrix} \dfrac{q^2}{2\varepsilon A} + k\,(x - x_o) \\ \dfrac{qx}{\varepsilon A} \end{bmatrix}$$

Inverse capacitance (revised)

$$C^{-1} = \left[\frac{\partial \mathbf{e}}{\partial \mathbf{q}}\right] = \left[\frac{\partial^2 E_p}{\partial q_i \partial q_j}\right] = \begin{bmatrix} k & q/\varepsilon A \\ q/\varepsilon A & x/\varepsilon A \end{bmatrix}$$

Stability (revised)

$$\text{determinant } C^{-1} = \frac{kx}{\varepsilon A} - \left(\frac{q}{\varepsilon A}\right)^2$$

Sufficient condition for stability:

$$k > 0$$

$$\frac{x}{\varepsilon A} > 0$$

$$k > \frac{q^2}{x \varepsilon A}$$

PHYSICAL INTERPRETATION

With charge as an input, electrostatic force is independent of displacement.

$$F_{electrostatic} = F(q, x) = \frac{q^2}{2\varepsilon A}$$

Electrostatic force will pull the capacitor plates together until equilibrium is reached.

$$F_{total} = \frac{q^2}{2\varepsilon A} + k\,(x - x_o) = 0$$

$$x_{equilibrium} = x_o - \frac{q^2}{2\varepsilon Ak}$$

This establishes a minimum value for x_o if $x_{equilibrium}$ is to be positive.

Intuitively, stability about that equilibrium point should only require non-zero mechanical stiffness.

Why does the mechanical stiffness have to be any larger?

CONSIDER EACH SUFFICIENT CONDITION IN TURN

If charge remains constant ($\Delta q = 0$)

a displacement from equilibrium of Δx

requires an applied force change of $k \, \Delta x$

$k > 0$ means that

increasing displacement requires increasing applied force

—provided charge remains constant

If displacement remains constant ($\Delta x = 0$)

a displacement from equilibrium of Δq

requires an applied voltage change of $\frac{x}{\varepsilon A} \, \Delta q$

$\frac{x}{\varepsilon A} > 0$ means that

increasing charge requires increasing applied voltage

—provided displacement remains constant

HOWEVER

If displacement may also change ($\Delta x \neq 0$)

Δq also increases electrostatic force by $\frac{q}{\varepsilon A}\ \Delta q$

that *decreases* displacement by $\Delta x = -\frac{q}{\varepsilon A}\frac{1}{k}\ \Delta q$

that, in turn, *decreases* voltage by $\Delta e = -\frac{q}{\varepsilon A}\frac{1}{k}\frac{q}{\varepsilon A}\ \Delta q$

The net voltage increase is $\left(\frac{x}{\varepsilon A} - \frac{q}{\varepsilon A}\frac{1}{k}\frac{q}{\varepsilon A}\right)\Delta q$

If increasing charge is to require increasing applied voltage, then

$$\frac{x}{\varepsilon A} > \frac{q}{\varepsilon A}\frac{1}{k}\frac{q}{\varepsilon A}$$

manipulating:

$k > \frac{q^2}{x\varepsilon A}$ **means that**

increasing charge requires increasing applied voltage

—when both charge and displacement are free to change