Hamiltonian Systems: Ideal Oscillators

Consider a system composed of an ideal capacitor and an ideal inertia interacting through a common-flow junction as shown in figure 4.11. A physical example would be a rigid body connected to a spring (so that a point on the spring and a point on the rigid body move at the same velocity) as shown schematically in figure 4.11.

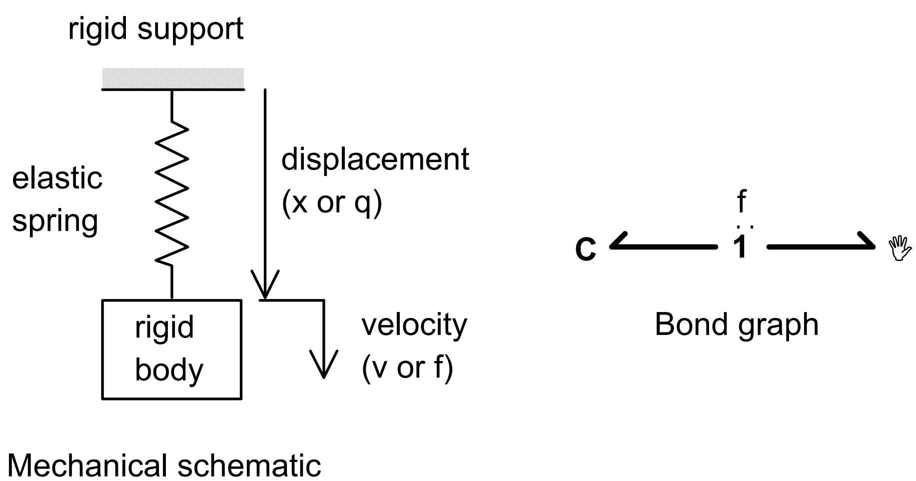

Figure 4.11: Mechanical schematic and corresponding bond graph for an ideal oscillator.

In general notation, the constitutive equations for the capacitor and the inertia (subscripts c and i, respectively) are

$$e_c = \Phi(q_c) \quad (4.55)$$

$$f_i = \Psi(p_i) \quad (4.56)$$

The differential form of their defining equations suggest that displacement and momentum may be used as state variables.

$$dq_c/dt = f_c \quad (4.57)$$

$$dp_i/dt = e_i \quad (4.58)$$

By definition, the flows of the elements connected to the one-junction are identical.

$$f_c = f_i \quad (4.59)$$

From the associated compatibility equation the efforts sum to zero.

$$e_i = -e_c \quad (4.60)$$

By direct substitution (4.56 into 4.59 into 4.57 and 4.55 into 4.60 into 4.58) these equations may be assembled into a pair of first-order differential equations.

$$dq_c/dt = \Psi(p_i) \tag{4.61}$$

$$dp_i/dt = -\Phi(q_c) \tag{4.62}$$

These are state equations for this system. The reason that q_c and p_i are a good choice for state variables (other choices are possible — see below) is that they define the energetic state of the system. In fact, any set of numbers from which we can define the energetic state of a system will suffice as state variables. And the reason for that is because the dynamic behavior of a physical system is due to the movement of energy within the system.

It is informative to rewrite these equations in terms of the total energy, $H(p_i,q_c)$, of the system.

$$H(p_i,q_c) \triangleq E_k(p_i) + E_p(q_c) \tag{4.63}$$

From the definitions of potential and kinetic energy we may write the constitutive equations of the capacitor and inertia as partial derivatives of the total energy.

$$\Psi(p_i) = \partial H/\partial p_i \tag{4.64}$$

$$\Phi(q_c) = \partial H/\partial q_c \tag{4.65}$$

The state equations may then be written as follows.

$$dq_c/dt = \partial H/\partial p_i \tag{4.66}$$

$$dp_i/dt = -\partial H/\partial q_c \tag{4.67}$$

This is the *Hamiltonian form* of the state equations (named after Sir William Rowan Hamilton, the illustrious Dublinman who first formulated it) and the generalized displacement and momentum are known as Hamiltonian or energy state variables. Hamiltonian systems play a fundamental role in modern physics, and have application at all scales, from celestial mechanics to quantum mechanics.

A more compact way to write these equations, which leads to further physical insight, is to write the displacement, q_c, and the momentum, p_i, as a state vector, **r**.

$$\mathbf{r} \triangleq \begin{bmatrix} q_c \\ p_i \end{bmatrix} \tag{4.68}$$

The state equations then become:

$$\frac{d}{dt}\begin{bmatrix} q_c \\ p_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \partial H/\partial q_c \\ \partial H/\partial p_i \end{bmatrix} \tag{4.69}$$

Alternatively:

$$\dot{\mathbf{r}} = -\mathbf{J}\,\partial H/\partial \mathbf{r} \tag{4.70}$$

where

$$\mathbf{J} \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.71}$$

This form of the equations is known as the *symplectic form*. The *symplectic matrix,* **J**, has an interesting property:

$$\mathbf{J}^{-1} = \mathbf{J}^{t} = -\mathbf{J} \tag{4.72}$$

It is therefore one of the square roots of minus unity.

$$\mathbf{J}\,\mathbf{J} = -\mathbf{1} \tag{4.73}$$

Experience with simple differential equations will have taught you to associate the square root of minus one with oscillatory behavior, and this case is no exception. The oscillatory character of Hamiltonian systems is easiest to see if we consider a linear case by replacing the pure capacitor and inertia with an ideal capacitor and inertia, for example a Hookeian spring of stiffness k and a Newtonian rigid body of mass m (e.g. as shown in figure 4.11). The state equations are then as follows:

$$\frac{d}{dt}\begin{bmatrix} q_c \\ p_i \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -k & 0 \end{bmatrix}\begin{bmatrix} q_c \\ p_i \end{bmatrix} \tag{4.74}$$

Double-differentiating q_c and substituting for $\dot{p}_i$ yields the familiar second-order differential equation of a simple harmonic oscillator.

$$\ddot{q}_c + \omega^2 q_c = 0 \tag{4.75}$$

where

$$\omega \triangleq \sqrt{k/m} \tag{4.76}$$

The same result can be derived directly from the first-order symplectic form, and that yields some further insight and understanding of how the oscillatory behavior arises. To simplify matters[1], let us suppose that k = m = 1. The state equations become:

$$\frac{d}{dt}\begin{bmatrix} q_c \\ p_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} q_c \\ p_i \end{bmatrix} \tag{4.77}$$

or

$$\dot{\mathbf{r}} = -\mathbf{J}\,\mathbf{r} \tag{4.78}$$

Equation 4.77 shows that the rate of change of the state vector is always orthogonal to the state vector — their inner product is zero.

$$\mathbf{r}^t\,\dot{\mathbf{r}} = -\mathbf{r}^t\,\mathbf{J}\,\mathbf{r} = 0 \tag{4.79}$$

Now remember that the behavior of a state-determined system can be represented geometrically as the motion of a point in an abstract state-space. Geometrically, the tangent to the path of the point representing the state is always at right angles to the line joining that point to the origin, as shown in figure 4.12. If the rate of change is non-zero, that is only possible if the state trajectory is a circle.

A state trajectory which closes on itself represents periodic behavior. A circular state trajectory means that the magnitude of the state vector is constant. From equation 4.77 the magnitude of the rate vector is proportional to the magnitude of the state vector, hence the circle is traversed at constant speed. As a result, q_c executes a sinusoidal oscillation.

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[1] More fundamentally, we could reason that because the fundamental form of the system behavior should not depend on an arbitrary choice of the units for time or space, we may choose them so that k and m are numerically equal to unity.

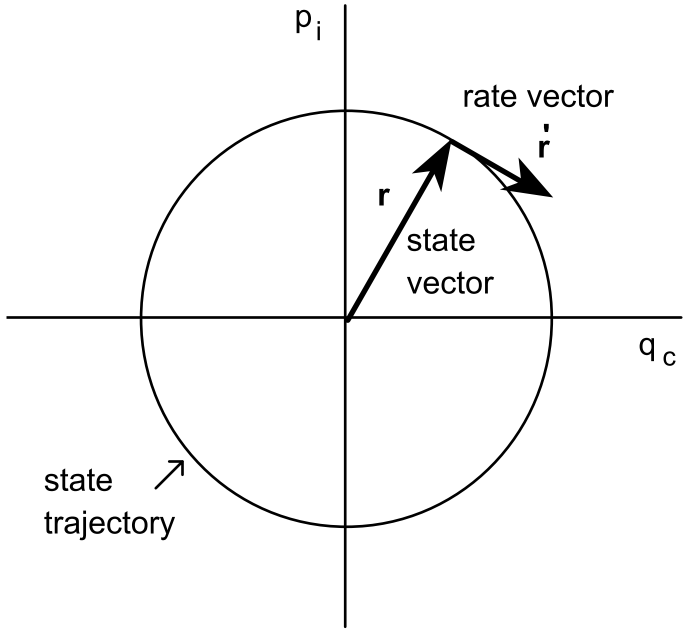

Figure 4.12: Geometric representation of the behavior of equation 4.77. The orthogonality of the rate and state vectors means that the state trajectory must be circular.

This result can be derived formally by integrating the matrix differential equations but the value of these manipulations is to illustrate that the oscillatory behavior comes from the symplectic form of the equations, rather than from any particular choice of parameter values. Consequently, we expect oscillatory behavior in a nonlinear system with this symplectic form, and this is in fact the case. Indeed, one of the reasons why Hamiltonian systems are so ubiquitous is because they are fundamentally oscillators, and oscillatory phenomena are found at all scales, from stellar objects to subatomic particles.

If we return to the derivation of the equations, we can see that in this case the symplectic form is due to the interaction between two energy storage elements of *dual* type through a one-junction. As we will see shortly, interaction between two energy storage elements of the same type through a junction structure composed of zero- and one-junctions does not lead to the symplectic structure and therefore does not lead to oscillatory behavior.